Proc. Indian Acad. Sci. (Math. Sci.) Vol. 117, No. 1, February 2007, pp. 1–12.

Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules

AMIR KHOSRAVI and BEHROOZ KHOSRAVI*

Faculty of Mathematical Sciences and Computer Engineering, University for Teacher Education, 599 Taleghani Ave., Tehran 15614, Iran
*Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424, Hafez Ave., Tehran 15914, Iran
E-mail: khosravi_amir@yahoo.com

MS received 17 November 2005; revised 9 December 2005

Abstract. In this article, we study tensor product of Hilbert C^*-modules and Hilbert spaces. We show that if E is a Hilbert A-module and F is a Hilbert B-module, then tensor product of frames (orthonormal bases) for E and F produce frames (orthonormal bases) for Hilbert $A \otimes B$-module $E \otimes F$, and we get more results.

For Hilbert spaces H and K, we study tensor product of frames of subspaces for H and K, tensor product of resolutions of the identities of H and K, and tensor product of frame representations for H and K.

Keywords. Frame; frame operator; tensor product; Hilbert C^*-module.

1. Introduction

Gabor [12], in 1946 introduced a technique for signal processing which eventually led to wavelet theory. Later in 1952, Duffin and Schaeffer [7] in the context of nonharmonic Fourier series introduced frame theory for Hilbert spaces. In 1986, Daubechies, Grassman and Meyer [6] showed that Duffin and Schaeffer's definition was an abstraction of Gabor's concept. Frames are used in signal processing, image processing, data compression, sampling theory, migrating the effect of losses in packet-based communication systems and improving the robustness of data transmission. Since tensor product is useful in the approximation of multi-variate functions of combinations of univariate ones, Khosravi and Asgari [15] introduced frames in tensor product of Hilbert spaces. Meanwhile, the notion of frames in Hilbert C^*-modules was introduced and some of their properties were investigated [9–11,14,16]. In this article, we study the frames and bases in tensor product of Hilbert C^*-modules which were introduced in [16] and we generalize the techniques of [15] to C^*-modules.

In §2, we briefly recall the definitions and basic properties of Hilbert C^*-modules. In §3, we investigate tensor product of Hilbert C^*-modules, which is introduced in [16] and we show that tensor product of frames for Hilbert C^*-modules E and F, present frames for $E \otimes F$, and tensor product of their frame operators is the frame operator of the tensor product of frames. We also show that tensor product of frames of subspaces produce a frame of subspaces for their tensor product. In §4, we study resolution of the identity and prove that tensor product of any resolutions of H and K, is a resolution of the identity for $H \otimes K$. In §5, we study the frame representation and we show that tensor product of frame vectors is a frame vector. Also we show that tensor product of analysis operators (resp. decomposition operators) is an analysis operator (resp. a decomposition operator).

Throughout this paper, $\mathbb{N}$ and $\mathbb{C}$ will denote the set of natural numbers and the set of complex numbers, respectively. A and B will be unital C^*-algebras.

2. Preliminaries

Let I and J be countable index sets. In this section we briefly recall the definitions and basic properties of Hilbert C^*-modules and frames in Hilbert C^*-modules. For information about frames in Hilbert spaces we refer to [3,14,5,19]. Our reference for C^*-algebras is [17,18]. For a C^*-algebra A if $a \in A$ is positive we write $a \geq 0$ and A^+ denotes the set of positive elements of A.

DEFINITION 2.1

Let A be a unital C^*-algebra and let H be a left A-module, such that the linear structures of A and H are compatible. H is a *pre-Hilbert A-module* if H is equipped with an A-valued inner product $\langle ., . \rangle$: $H \times H \to A$, that is sesquilinear, positive definite and respects the module action. In other words,

(i) $\langle x, x \rangle \geq 0$ for all $x \in H$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;
(ii) $\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$ for all $a \in A$ and $x, y, z \in H$;
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in H$.

For $x \in H$, we define $\|x\| = \|\langle x, x \rangle\|^{1/2}$. If H is complete with $\|.\|$, it is called a *Hilbert A-module* or a *Hilbert C^*-module* over A. For every a in C^*-algebra A, we have $|a| = (a^*a)^{1/2}$ and the A-valued norm on H is defined by $|x| = \langle x, x \rangle^{1/2}$ for $x \in H$.

DEFINITION 2.2

Let H be a Hilbert A-module. A family $\{x_i\}_{i \in I}$ of elements of H is a *frame* for H, if there exist constants $0 < A \leq B < \infty$, such that for all $x \in H$,

$$A\langle x, x \rangle \leq \sum_{i \in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq B\langle x, x \rangle. \tag{1}$$

The numbers A and B are called lower and upper bound of the frame, respectively. If $A = B = \lambda$, the frame is λ-*tight*. If $A = B = 1$, it is called a *normalized tight frame* or a *Parseval* frame. If the sum in the middle of (1) is convergent in norm, the frame is called *standard*.

If $\{x_i\}_{i \in I}$ is a standard frame in a finitely or countably generated Hilbert A-module, it has a unique operator $S \in \mathrm{End}_A^*(H)$, where $\mathrm{End}_A^*(H)$ is the set of adjointable A-linear maps on H, such that for every $x \in H$,

$$x = \sum_{i \in I} \langle x, Sx_i \rangle x_i = \sum_{i \in I} \langle x, x_i \rangle Sx_i.$$

Moreover S is positive and invertible.

DEFINITION 2.3

Let H be a Hilbert A-module, and let $v \in H$. We say that v is a basic element if $e = \langle v, v \rangle$ is a minimal projection in A, i.e. $eAe = \mathbb{C}e$. A system $\{v_\lambda : \lambda \in \Lambda\}$ of basic elements of

H is called orthonormal if $\langle v_\lambda, v_\mu\rangle = 0$ for all $\lambda \neq \mu$. An *orthonormal basis* for H is an orthonormal system which generates a dense submodule of H.

3. Main results

Let A and B be C^*-algebras, E a Hilbert A-module and let F be a Hilbert B-module. We take $A \otimes B$ as the completion of $A \otimes_{\text{alg}} B$ with the spatial norm. Hence $A \otimes B$ is a C^*-algebra and for every $a \in A$, $b \in B$ we have $\|a \otimes b\| = \|a\| \cdot \|b\|$. The algebraic tensor product $E \otimes_{\text{alg}} F$ is a pre-Hilbert $A \otimes B$-module with module action

$$(a \otimes b)(x \otimes y) = ax \otimes by \qquad (a \in A,\ b \in B,\ x \in E,\ y \in F),$$

and $A \otimes B$-valued inner product

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle \otimes \langle y_1, y_2\rangle \qquad (x_1, x_2 \in E,\ y_1, y_2 \in F).$$

We also know that for $z = \sum_{i=1}^n x_i \otimes y_i$ in $E \otimes_{\text{alg}} F$ we have

$$\langle z, z\rangle = \sum_{i,j} \langle x_i, x_j\rangle \otimes \langle y_i, y_j\rangle \geq 0$$

and $\langle z, z\rangle = 0$ if and only if $z = 0$. Just as in the case of ordinary pre-Hilbert space, we can form the completion $E \otimes F$ of $E \otimes_{\text{alg}} F$, which is a Hilbert $A \otimes B$-module. It is called the *tensor product* of E and F (see [16]). We note that if $a \in A^+$ and $b \in B^+$, then $a \otimes b \in (A \otimes B)^+$. Plainly if a, b are hermitian elements of A and $a \geq b$, then for every positive element x of B, we have $a \otimes x \geq b \otimes x$.

Lemma 3.1. *Let $\{u_i\}_{i\in I}$ be a frame for E with frame bounds A and B, and let $\{v_j\}_{j\in J}$ be a frame for F with frame bounds C and D. Then $\{u_i \otimes v_j\}_{i\in I, j\in J}$ is a frame for $E \otimes F$ with frame bounds AC and BD. In particular, if $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ are tight or Parseval frames, then so is $\{u_i \otimes v_j\}_{i\in I, j\in J}$.*

Proof. Let $x \in E$ and $y \in F$. Then we have

$$A\langle x, x\rangle \leq \sum_{i\in I} \langle x, u_i\rangle\langle u_i, x\rangle \leq B\langle x, x\rangle, \tag{2}$$

$$C\langle y, y\rangle \leq \sum_{j\in J} \langle y, v_j\rangle\langle v_j, y\rangle \leq D\langle y, y\rangle. \tag{3}$$

Therefore

$$\begin{aligned} A\langle x, x\rangle \otimes \langle y, y\rangle &\leq \sum_i \langle x, u_i\rangle\langle u_i, x\rangle \otimes \langle y, y\rangle \\ &\leq B\langle x, x\rangle \otimes \langle y, y\rangle. \end{aligned}$$

Now by (3), we have

$$\begin{aligned} AC\langle x, x\rangle \otimes \langle y, y\rangle &\leq \sum_i \sum_j \langle x, u_i\rangle\langle u_i, x\rangle \otimes \langle y, v_j\rangle\langle v_j, y\rangle \\ &\leq B\langle x, x\rangle \otimes \sum_j \langle y, v_j\rangle\langle v_j, y\rangle \\ &\leq BD\langle x, x\rangle \otimes \langle y, y\rangle. \end{aligned}$$

Consequently we have

$$AC\langle x\otimes y, x\otimes y\rangle \leq \sum_i \sum_j \langle x\otimes y, u_i\otimes v_j\rangle\langle u_i\otimes v_j, x\otimes y\rangle$$

$$\leq BD\langle x\otimes y, x\otimes y\rangle.$$

From these inequalities it follows that for all $z=\sum_{k=1}^n x_k\otimes y_k$ in $E\otimes_{\mathrm{alg}} F$,

$$AC\langle z,z\rangle \leq \sum_{i,j}\langle z, u_i\otimes v_j\rangle\langle u_i\otimes v_j, z\rangle \leq BD\langle z,z\rangle. \tag{4}$$

Hence relation (4) holds for all z in $E\otimes F$. □

From Theorem 1 of [2] and the above lemma we have the following result.

Theorem 3.2. *Let E be a Hilbert A-module and F be a Hilbert B-module. Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be orthonormal bases in E and F, respectively. Then $\{u_i\otimes v_j\}_{i\in I, j\in J}$ is an orthonormal basis for $E\otimes F$.*

Proof. It is clear that each $u_i\otimes v_j$ is a basic element of $E\otimes F$ and $\{u_i\otimes v_j\}_{i\in I,j\in J}$ is an orthonormal system in $E\otimes F$. Now for each $x\in E$ and each $y\in F$, we have $x=\sum_{i\in I}\langle x,u_i\rangle u_i$ and $y=\sum_{j\in J}\langle y,v_j\rangle v_j$. Hence

$$x\otimes y=\sum_{i\in I}\sum_{j\in J}\langle x\otimes y, u_i\otimes v_j\rangle u_i\otimes v_j.$$

Similar to the above lemma we can show that for each z in $E\otimes F$, we have $z=\sum_{i\in I}\sum_{j\in J}\langle z, u_i\otimes v_j\rangle u_i\otimes v_j$. But Bakic and Guljas in Theorem 1 of [2] showed that if W is a Hilbert C^*-module over a C^*-algebra A, and $(v_\lambda)_{\lambda\in\Lambda}$ is an orthonormal system in W, then $(v_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis for W if and only if for every $w\in W$, $w=\sum\langle w, v_\lambda\rangle v_\lambda$. Now by using this fact we have the result. □

Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be standard frames for E and F, respectively. So $\{u_i\otimes v_j\}_{i\in I,j\in J}$ is a standard frame for $E\otimes F$.

Let S, S' and S'' be the frame operators of $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ and $\{u_i\otimes v_j\}_{i\in I,j\in J}$, respectively. So S is A-linear and S' is B-linear. Hence for every $x\in E$ and $y\in F$, we have $x=\sum_i\langle x, Su_i\rangle u_i$, $y=\sum_j\langle y, S'v_j\rangle v_j$. Therefore

$$x\otimes y=\sum_i\sum_j\langle x, Su_i\rangle u_i\otimes\langle y, S'v_j\rangle v_j$$

$$=\sum_i\sum_j(\langle x, Su_i\rangle\otimes\langle y, S'v_j\rangle)(u_i\otimes v_j)$$

$$=\sum_i\sum_j(\langle x\otimes y, Su_i\otimes S'v_j\rangle u_i\otimes v_j.$$

Now by the uniqueness of frame operator we have $S''(u_i\otimes v_j)=Su_i\otimes S'v_j$. Hence $S''=S\otimes S'$, which is a bounded $A\otimes B$-linear, self-adjoint, positive and invertible operator on $E\otimes F$. We note that $\|S''\|=\|S\otimes S'\|\leq\|S\|.\|S'\|$. Now we summarize the above results as follows:

Theorem 3.3. *Let $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ be standard frames in the Hilbert C^*-modules E and F, respectively. If S, S' and S'' are the frame operators of $\{u_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ and $\{u_i \otimes v_j\}_{i\in I, j\in J}$, respectively, then $S'' = S \otimes S'$.*

For the frame operator we prove the following result.

Lemma 3.4. *If $\{x_i\}_{i\in I}$ is a frame in Hilbert A-module X with frame operator S and $Q \in \mathrm{End}_A^*(X)$ is invertible, then $\{Qx_i\}_{i\in I}$ is a frame in X with frame operator $Q^{*-1}SQ^{-1}$.*

Proof. Let $\{x_i\}_{i\in I}$ be a frame of X with frame operator S. Then there exist constants A, $B > 0$ such that for every $x \in X$,

$$A\langle x, x\rangle \leq \sum_i |\langle x, x_i\rangle|^2 \leq B\langle x, x\rangle, \tag{5}$$

and $S^{-1}x = \sum_i \langle x, x_i\rangle x_i$. Since Q is invertible and $Q \in \mathrm{End}_A^*(X)$, then Q is a bounded A-linear map with invertible adjoint Q^*. So for every $x \in X$, we have

$$\|Q^{*-1}\|^{-1} \cdot |x| \leq |Q^*x| \leq \|Q^*\| \cdot |x|. \tag{6}$$

Since Q is A-linear, $QS^{-1}x = \sum_i \langle x, x_i\rangle Qx_i$. So $QS^{-1}Q^*(Q^{*-1}x) = \sum_i \langle Q^{*-1}x, Qx_i\rangle Qx_i$, because

$$\langle x, x_i\rangle = \langle Q^*Q^{*-1}x, x_i\rangle = \langle Q^{*-1}x, Qx_i\rangle.$$

Consequently, for every $x \in X$,

$$QS^{-1}Q^*(x) = \sum_i \langle x, Qx_i\rangle Qx_i. \tag{7}$$

Now by using (5) and (6) we have

$$\begin{aligned} A\|Q^{*-1}\|^{-2}\langle x, x\rangle &\leq A\langle Q^*x, Q^*x\rangle \\ &\leq \sum_i |\langle Q^*x, x_i\rangle|^2 \leq B\langle Q^*x, Q^*x\rangle \leq B\|Q^*\|^2\langle x, x\rangle. \end{aligned}$$

On the other hand, $\langle Q^*x, x_i\rangle = \langle x, Qx_i\rangle$, so $\{Qx_i\}_{i\in I}$ is a frame for X and by (7), $Q^{*-1}SQ^{-1} = (QS^{-1}Q^*)^{-1}$ is the frame operator of $\{Qx_i\}_{i\in I}$. □

Theorem 3.5. *If $Q \in \mathrm{End}_A^*(E)$ is an invertible A-linear map and $\{T_i\}_{i\in J}$ is a frame in $E \otimes F$ with frame operator S, then $\{(Q^* \otimes I)(T_i)\}_{i\in J}$ is a frame of $E \otimes F$ with frame operator $(Q \otimes I)^{-1}S(Q^* \otimes I)^{-1}$.*

Proof. Since $Q \in \mathrm{End}_A^*(E)$, $Q \otimes I \in \mathrm{End}_A^*(E \otimes F)$ with inverse $Q^{-1} \otimes I$. It is obvious that $Q \otimes I$ is $A \otimes B$-linear, adjointable, with adjoint $Q^* \otimes I$. An easy calculation shows that for every elementary tensor $x \otimes y$,

$$\begin{aligned} \|(Q \otimes I)(x \otimes y)\|^2 &= \|Q(x) \otimes y\|^2 = \|Q(x)\|^2 \cdot \|y\|^2 \\ &\leq \|Q\|^2 \cdot \|x\|^2 \cdot \|y\|^2 = \|Q\|^2 \cdot \|x \otimes y\|^2. \end{aligned}$$

So $Q \otimes I$ is bounded, and therefore it can be extended to $E \otimes F$. Similarly for $Q^{*-1} \otimes I$. Hence $Q \otimes I$ is $A \otimes B$-linear, adjointable with adjoint $Q^* \otimes I$, and as we mentioned in the proof of Lemma 3.4, Q^* is invertible and bounded. Hence for every $T \in E \otimes F$, we have

$$\|Q^{*-1}\|^{-1} \cdot |T| \leq |(Q^* \otimes I)T| \leq \|Q\| \cdot |T|. \tag{8}$$

Hence $Q \otimes I \in \mathrm{End}^*_{A\otimes B}(E \otimes F)$. Now by the above lemma we have the result. □

Now we generalize some of the results in [15] to frame of subspaces. First we recall the definition of frame of subspaces (for basic definitions and properties, see [4]).

DEFINITION 3.6

Let H be a separable Hilbert space and let $\{v_i\}_{i\in I}$ be a sequence of weights, i.e., $v_i > o$ for all $i \in I$. A sequence $\{W_i\}_{i\in I}$ of closed subspaces of H is a *frame of subspaces* with respect to $\{v_i\}_{i\in I}$ if there exist real numbers $A, B > 0$ such that for every $x \in H$,

$$A\|x\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 \leq B\|x\|^2,$$

where for each $i \in I$, π_{W_i} is the orthogonal projection of H onto W_i. Similar to frames, A and B are called the frame bounds. 1f $A = B = \lambda$, the frame of subspaces is λ-tight and it is a Parseval frame of subspaces if $A = B = 1$.

Let H and K be Hilbert spaces and let W, Z be closed subspaces of H and K, respectively. Then $\pi_W \otimes \pi_Z\colon H \otimes_{\mathrm{alg}} K \to W \otimes Z$ is a bounded linear map, and it can be extended to a bounded linear map from $H \otimes K$ into $W \otimes Z$. We also denote it by $\pi_W \otimes \pi_Z$ and clearly it is surjective. Hence $\pi_W \otimes \pi_Z$ is the orthogonal projection of $H \otimes K$ onto $W \otimes Z$.

Theorem 3.7. *Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{u_i\}_{i\in I}$ for H, with frame bounds A, B, and let $\{Z_j\}_{j\in J}$ be a frame of subspaces with respect to $\{v_j\}_{j\in J}$ for K with frame bounds A', B'. Then $\{W_i \otimes Z_j\}_{i\in I, j\in J}$ is a frame of subspaces with respect to $\{u_i v_j\}_{i\in I, j\in J}$ for $H \otimes K$ with frame bounds AA' and BB'. It is tight or Parseval if $\{W_i\}_i$ and $\{Z_j\}$ are tight or Parseval.*

Proof. Let $x \otimes y$ be an elementary tensor. Then $A\|x\|^2 \leq \sum_{i\in I} u_i^2 \|\pi_{W_i}(x)\|^2 \leq B\|x\|^2$ and $A'\|y\|^2 \leq \sum_{j\in J} v_j^2 \|\pi_{Z_j}(y)\|^2 \leq B'\|y\|^2$.

A simple calculation shows that

$$AA'\|x \otimes y\|^2 \leq \sum_i \sum_j u_i^2 v_j^2 \|\pi_{W_i}(x)\|^2 \cdot \|\pi_{Z_j}(y\|^2$$

$$\leq BB'\|x \otimes y\|^2.$$

Hence

$$AA'\|x \otimes y\|^2 \leq \sum_{i,j} u_i^2 v_j^2 \|\pi_{W_i}(x) \otimes \pi_{Z_j}(y)\|^2 \leq BB'\|x \otimes y\|^2.$$

Therefore

$$AA'\|x \otimes y\|^2 \leq \sum_{i,j} u_i^2 v_j^2 \|\pi_{W_i} \otimes \pi_{Z_j}(x \otimes y)\|^2 \leq BB'\|x \otimes y\|^2. \tag{9}$$

Consequently, for every $z = \sum_{l=1}^n x_l \otimes y_l$ in $H \otimes_{\mathrm{alg}} K$ and every z in $H \otimes K$, the relation (9) holds. Hence we have the result. □

Now we try to generalize a known result of frames (Proposition 3.1 of [15]) to frames of subspaces.

DEFINITION 3.8

Let $\{W_i\}_{i\in I}$ be a frame of subspaces for H with respect to $\{v_i\}_{i\in I}$. Then the frame operator $S_{W,v}$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is defined by

$$S_{W,v}(x) = \sum_{i\in I} v_i^2 \pi_{W_i}(x), \quad x \in H$$

COROLLARY 3.9

With the hypothesis in Theorem 3.7, *if $S_{W,u}$ and $S_{Z,v}$ are frame operators for $\{W_i\}_{i\in I}$, $\{u_i\}$ and $\{Z_j\}$, $\{v_j\}$, respectively, then $S_{W,u} \otimes S_{Z,v}$ is the frame operator for $\{W_i \otimes Z_j\}_{i\in I, j\in J}$ and $\{u_i v_j\}_{i\in I, j\in J}$.*

Proof. Let $x \otimes y$ be an elementary tensor. Therefore

$$\begin{aligned} S_{W,u} \otimes S_{Z,v}(x \otimes y) &= S_{W,u}(x) \otimes S_{Z,v}(y) \\ &= \sum_i u_i^2 \pi_{W_i}(x) \otimes \sum_j v_j^2 \pi_{Z_j}(y) \\ &= \sum_i \sum_j u_i^2 v_j^2 (\pi_{W_i} \otimes \pi_{Z_j})(x \otimes y). \end{aligned}$$

Now the uniqueness of frame operator implies that $S_{W,u} \otimes S_{Z,v}$ is the desired frame operator. $\square$

Remark 3.10. Let H and K be Hilbert spaces. A map $T\colon H \longrightarrow K$ is *antilinear* (or conjugate linear) if $T(\lambda x + y) = \bar{\lambda} T(x) + T(y)$ for all $\lambda \in \mathbb{C}$ and $x, y \in H$. By the techniques in [8], $H \otimes K$ is the set of anti-linear maps $T\colon K \to H$ with the norm $\|.\|$ defined by

$$\|T\| = \sup\{\|Ty\|\colon y \in K,\ \|y\| \le 1\}.$$

So $W_i \otimes Z_j$ is the set of anti-linear maps $T\colon Z_j \to W_i$ and therefore $\pi_{W_i} \otimes \pi_{Z_j}$ is the map which assigns to every $T \in H \otimes K$, the restriction of $\pi_{W_i} \circ T$ to Z_j, i.e. $\pi_{W_i} \circ T|Z_j$.

4. Resolution of the identity

In this section we present the notion of ℓ^2-resolution of the identity with lower resolution bound in tensor product of Hilbert spaces (for more information see [4,9]).

DEFINITION 4.1

Let I be a countable index set and let H be a Hilbert space. Let $\{v_i\}_{i\in I}$ be a family of weights, i.e., for all i, $v_i > 0$. Then a family of bounded operators $\{T_i\}_{i\in I}$ on H is called a ℓ^2-*resolution of the identity with lower resolution bound* with respect to $\{v_i\}_{i\in I}$ on H if there are positive real numbers C and D such that for all $f \in H$,

(i) $C\|f\|^2 \le \sum_{i\in I} v_i^{-2}\|T_i(f)\|^2 \le D\|f\|^2$,
(ii) $f = \sum_{i\in I} T_i(f)$ (and the series converges unconditionally for every $f \in H$).

The optimal values of C and D are called the *bounds* of the resolution of the identity.

PROPOSITION 4.2

Let $\{T_i\}_{i\in I}$ be a ℓ^2-resolution of the identity with lower resolution bound with respect to $\{v_i\}_{i\in I}$ on H, and let $\{S_j\}_{j\in J}$ be a ℓ^2-resolution of the identity with lower resolution bound with respect to $\{u_j\}_{j\in J}$ on K. Then $\{T_i \otimes S_j\}_{i\in I, j\in J}$ is a ℓ^2-resolution of the identity with lower resolution bound with respect to $\{v_i u_j\}_{i\in I, j\in J}$ on $H \otimes K$.

Proof. Let $f \in H$, $g \in K$. Then $f = \sum_{i\in I} T_i(f)$, $g = \sum_{j\in J} S_j(g)$, and consequently

$$\begin{aligned}\sum_{i,j}(T_i \otimes S_j)(f\otimes g) &= \sum_{i,j} T_i(f)\otimes S_j(g)\\ &= \sum_i T_i(f)\otimes \sum_j S_j(g) = f\otimes g.\end{aligned}$$

Since both the series $f = \sum_{i\in I} T_i(f)$ and $g = \sum_{j\in J} S_j(g)$ are unconditionally convergent, the above series is unconditionally convergent. So for every $h \in H \otimes_{\mathrm{alg}} K$ and consequently for every $h \in H \otimes K$ the above relation holds. Let C, D and C', D' be the bounds of the resolutions $\{T_i\}$ and $\{S_j\}$, respectively. Then for every $f \in H$, $g \in K$ we have

$$\begin{aligned}CC'\|f\otimes g\|^2 = CC'\|f\|^2\cdot\|g\|^2 &\le C'\sum_i v_i^{-2}\|T_i(f)\|^2\cdot\|g\|^2\\ &\le \sum_i v_i^{-2}\|T_i(f)\|^2\cdot\sum_j u_j^{-2}\|S_j g\|^2\\ &= \sum_{i,j} v_i^{-2}u_j^{-2}\|(T_i\otimes S_j)(f\otimes g)\|^2\\ &\le DD'\|f\otimes g\|^2. \qquad (10)\end{aligned}$$

Now by using the fact that

$$\|(T\otimes S)\left(\sum_{i=1}^n f_i\otimes g_i\right)\|^2 = \|T\left(\sum_{i=1}^n f_i\right)\|^2\cdot\|S\left(\sum_{i=1}^n g_i\right)\|^2,$$

and $\|\sum_{i=1}^n f_i\otimes g_i\|^2 = \|\sum_{i=1}^n f_i\|^2\cdot\|\sum_{i=1}^n g_i\|^2$, we conclude that for every $h = \sum_{i=1}^n f_i\otimes g_i$ and consequently for every $h \in H\otimes K$ the relation (10) holds. □

From the above proposition and Proposition 3.26 of [4] we have the following result.

COROLLARY 4.3

With the hypothesis in Corollary 3.9, if $T_i = \pi_{W_i} S_{W,v_i}$ and $S_j = \pi_{Z_j} S_{Z,u_j}$, then $\{v_i^2 u_j^2 T_i \otimes S_j\}_{i\in I, j\in J}$ is a ℓ^2-resolution of the identity with lower resolution bound with respect to $\{v_i u_j\}_{i\in I, j\in J}$ on $H\otimes K$ and for all $z \in H\otimes K$,

$$\frac{C}{D^2}\cdot\frac{C'}{D'^2}\|z\|^2 \leq \sum_{i\in I}\sum_{j\in J} v_i^2 u_j^2 \|(T_i\otimes S_j)(z)\|^2 \leq \frac{D}{C^2}\cdot\frac{D'}{C'^2}\|z\|^2.$$

5. Frame representation

Let H be a separable Hilbert space, and let G be a discrete countable abelian group. Let $\pi\colon G\to B(H)$ be a unitary representation of G on H. If there is a vector $v\in H$ such that $\{\pi(g)v|g\in G\}$ is a frame for H, then the representation π is called a *frame representation*. Let $\hat{G}$ denote the dual group of G, i.e., the group of characters on G and let λ be the normalized Haar measure on $\hat{G}$. Let $\pi : G \longrightarrow B(H)$ be a frame representation with frame vector v. As we have in [1,13,17] there is a spectral measure E on $\hat{G}$ such that

$$\pi(g) = \int_{\hat{G}} g(\xi)\mathrm{d}E(\xi).$$

Since π is a frame representation, by using the results in §2 of [1] and the properties of spectral measure there is a unitary operator $U\colon H \longrightarrow L^2(F,\lambda|F)$, where F is a measurable subset of $\hat{G}$ with $\lambda(F)>0$ and $\lambda|F$ is the restriction of Haar measure λ to F such that U interwines the spectral measure on H and the canonical spectral measure on $\hat{G}$. The operator U is called the *decomposition operator*. Moreover π is unitarily equivalent to the representation $\sigma\colon G\longrightarrow B(L^2(F,\lambda|F))$ defined by $\sigma(g)=M_g$, where M_g is the multiplication operator with symbol g. In fact, $U^*M_gU=\pi(g)$.

We also note that if θ_v is the analysis operator of H for frame vector v, then $\theta_v\pi(g) = L_g\theta_v$, where $L_g\colon \ell^2(G)\longrightarrow \ell^2(G)$ is defined by $(L_gx)(h)=x(g^{-1}h)$ for all $h\in G$. In fact, if J is the range of θ_v, then the representation π of G is unitarily equivalent to $\rho = L_g|J$ (see Lemma 3 of [1]). For more details see [1] or [13].

Let H and K be separable Hilbert spaces and let $\pi\colon G_1\to B(H)$ and $\sigma\colon G_2\to B(K)$ be frame representations on H and K with frame vectors $v\in H$ and $w\in K$, respectively. Since G_1 and G_2 are discrete countable abelian groups, their direct sum $G=G_1\oplus G_2$ is a discrete countable abelian group. Hence we can consider the representation $\pi\otimes\sigma\colon G\to B(H\otimes K)$ defined by

$$(\pi\otimes\sigma)(g,h) = \pi g\otimes\sigma h, \quad (g,h)\in G.$$

Since $\{\pi(g)v\colon g\in G_1\}$ is a frame for H and $\{\sigma(h)w\colon h\in G_2\}$ is a frame for K, by Lemma 3.1 and the definition of $\pi\otimes\sigma$,

$$\{\pi\otimes\sigma(g,h)(v\otimes w) : (g,h)\in G\} = \{(\pi g)v\otimes(\sigma h)w : (g,h)\in G\}$$

is a frame for $H\otimes K$. So $\pi\otimes\sigma$ is a frame representation of $H\otimes K$ with frame vector $v\otimes w$. Moreover, if θ_v and θ_w are the analysis operators of H and K for frame vectors v and w, respectively, then $\theta_v\otimes\theta_w$ is the analysis operator of $H\otimes K$ for frame vector $v\otimes w$. Hence we have proved the following result.

Theorem 5.1. *Let $\pi\colon G_1\to B(H)$ and $\sigma\colon G_2\to B(K)$ be frame representations with frame vectors v and w, respectively. Then $\pi\otimes\sigma\colon G_1\oplus G_2\to B(H\otimes K)$ is a frame representation with frame vector $v\otimes w$. If θ_v and θ_w are the analysis operators for frame vectors v and w, respectively, then $\theta_v\otimes\theta_w$ is the analysis operator for $v\otimes w$.* □

For the decomposition operators we have the following result.

Theorem 5.2. *With the hypothesis in Theorem* 5.1, *suppose that* $U\colon H \to L^2(E, \lambda|E)$ *and* $V\colon K \to L^2(F, \lambda|F)$ *are the decomposition operators of* π *and* σ, *respectively, then* $U \otimes V\colon H \otimes K \to L^2(E \oplus F, \lambda \times \mu|E \otimes F)$ *is the decomposition operator of* $\pi \otimes \sigma$.

Proof. It is clear that $(G_1 \oplus G_2)^\wedge = \hat{G}_1 \oplus \hat{G}_2$. If $U\colon H \longrightarrow L^2(E, \lambda|E)$ and $V\colon K \longrightarrow L^2(F, \mu|F)$, where $\hat{G}_1 \supseteq E$, $\hat{G}_2 \supseteq F$, then $\hat{G}_1 \oplus \hat{G}_2 \supseteq E \oplus F$ and $U \otimes V : H \otimes K \longrightarrow L^2(E \oplus F, \lambda \times \mu|E \oplus F)$, where $\lambda \times \mu$ is the product measure of λ and μ. We note that for every $x \in H$, $y \in K$, the function $(U \otimes V)(x \otimes y) = Ux \otimes Vy$ defined on $E \oplus F$ by $(Ux \otimes Vy)(\zeta, \eta) = (Ux)(\zeta).(Vy)(\eta)$ and since $L^2(E, \lambda|E) \otimes L^2(F, \lambda|F)$ is isomorphic to $L^2(E \oplus F, \lambda \times \mu|E \oplus F)$ we can take $Ux \otimes Vy \in L^2(E \oplus F, \lambda \times \mu|E \oplus F)$. Since G_1 and G_2 form an orthonormal basis of $L^2(\hat{G}_1, \lambda)$ and $L^2(\hat{G}_2, \mu)$, respectively (Corollary 4.26 of [8]), a simple calculation shows that

$$\begin{aligned}\|Uv \otimes Vw\|^2 &= \|\chi_{E\oplus F} \cdot Uv \otimes Vw\|^2 \\ &= \int_{\hat{G}_1} |\chi_E(\zeta)Uv(\zeta)|^2 \mathrm{d}\lambda \cdot \int_{\hat{G}_2} |\chi_F(\eta)Vw(\eta)|^2 \mathrm{d}\mu \\ &= \|\chi_E Uv\|^2 \cdot \|\chi_F Vw\|^2 = \|Uv\|^2 \cdot \|Vw\|^2 < \infty. \qquad \square\end{aligned}$$

COROLLARY 5.3

Let $\{\pi(g)v\}_{g\in G_1}$ *and* $\{\sigma(h)w\}_{h\in G_2}$ *be frames for* H *and* K *with frame bounds* A_1, B_1 *and* A_2, B_2, *respectively. Then* $\{(\pi\otimes\sigma)(g,h)(v\otimes w)\}_{g\in G_1, h\in G_2}$ *is a frame with frame bounds* A_1A_2 *and* B_1B_2.

Proof. First we note that for all $x \in H$,

$$\sum_{g\in G_1} |\langle x, \pi(g)v\rangle|^2 = \sum_{g\in G} \int_{\hat{G}_1} |Ux(\zeta)Uv(\zeta)|^2 \mathrm{d}\lambda = \|(Ux)(Uv)\|^2$$

and

$$A_1\|x\|^2 \le \sum_{g\in G_1} |\langle x, \pi(g)v\rangle|^2 \le B_1\|x\|^2, \quad \text{for all } x \in H.$$

Similarly

$$A_2\|y\|^2 \le \sum_{h\in G_2} |\langle y, \sigma(h)w\rangle|^2 \le B_2\|y\|^2, \quad \text{for all } y \in K.$$

Hence for every elementary tensor $x \otimes y$ we have $\|x \otimes y\| = \|x\|.\|y\|$ and

$$\begin{aligned}&\sum_{g\in G_1}\sum_{h\in G_2} |\langle x\otimes y, \pi(g)\otimes\sigma(h)(v\otimes w)\rangle|^2 \\ &\quad= \int_{\hat{G}_1}\int_{\hat{G}_2} |v(x)|^2 \cdot |Uv|^2 \cdot |w(x)|^2 \cdot |Vw|^2 d(\lambda\times\mu) \\ &\quad= \|(Ux)(Uv)\|^2 \cdot \|(Vy)(Vw)\|^2.\end{aligned}$$

So we have the result. $\square$

We can also state similar results for Bessel vectors.

DEFINITION 5.4

Let $\pi\colon G \longrightarrow B(H)$ be a frame representation with frame vector v. We say $v' \in H$ is a Bessel vector for the frame representation if there exists $C_2 > 0$ such that for all $x \in H$,

$$\sum_{g\in G} |\langle x, \pi(g)v'\rangle|^2 \leq C_2\|x\|^2.$$

Lemma 5.5. *Suppose π and σ are frame representations on H and K with frame vectors v and w, respectively. If v' and w' are Bessel vectors for π and σ, respectively, then $v' \otimes w'$ is a Bessel vector for $\pi \otimes \sigma$.*

Proof. By Theorem 5.1, $\pi \otimes \sigma$ is a frame representation with frame vector $v \otimes w$, and since v' and w' are Bessel vectors for π and σ, respectively, there are constants C_2 and C_2' such that

$$\sum_{g\in \hat{G}_1} |\langle x, \pi(g)v'\rangle|^2 \leq C_2\|x\|^2, \quad x \in H,$$

$$\sum_{h\in \hat{G}_2} |\langle y, \sigma(h)w'\rangle|^2 \leq C_2'\|y\|^2, \quad y \in K.$$

Hence for every elementary tensor $x \otimes y$ we have

$$\sum_{g\in \hat{G}_1}\sum_{h\in \hat{G}_2} |\langle x\otimes y, \pi\otimes\sigma(g,h)(v'\otimes w')\rangle|^2 \leq C_2C_2'\|x\otimes y\|^2.$$

As we have in \$ 4, the above relation holds for every $z = \sum_{i=1}^n x_i \otimes y_i$ and so for every $z \in H \otimes K$. Therefore $v' \otimes w'$ is a Bessel vector for $\pi \otimes \sigma$. □

Acknowledgement

The authors express their gratitude to the referee for carefully reading and several valuable pointers which improved the manuscript.

References

[1] Aldroubi A, Larson D, Tang W and Weber E, Geometric aspects of frame representation of Abelian groups, *Trans. Am. Math. Soc.* **356(120)** (2004) 4767–4786

[2] Bakic D and Guljas B, Hilbert C^*-modules over C^*-algebras of compact operators, *Acta Sci. Math. (Szeged)* **68(1–2)** (2002) 249–269

[3] Casazza P G, The art of frame theory, *Taiwanese J. Math.* **4** (2000) 129–201

[4] Casazza P G and Kutyniok G, Frames of subspaces, *Contemp. Math.* vol. 345, *Am. Math. Soc.* (RI: Providence) (2004) pp. 87–113

[5] Christensen O, An introduction to frames and Riesz bases (Boston: Birkhauser) (2003)

[6] Daubechies I, Grassman A and Meyer Y, Painless nonorthogonal expansions, *J. Math. Phys.* **27** (1986) 1271–1283

[7] Duffin R J and Schaeffer A C, A class of nonharmonic Fourier series, *Am. Math. Soc.* **72** (1952) 341–366

[8] Folland G B, A course in abstract harmonic analysis (Boca Raton Florida: CRC Press) (1995)

[9] Fornasier M, Decomposition of Hilbert spaces, in: local construction of global frames (ed.) B Bojanov, Proc. Int. Conf. on Constructive Function Theory, Varna, 2002, DARBA (Sofia) (2003)

[10] Frank M and Larson D R, A module frame concept for Hilbert C^*-modules, in: Functional and harmonic analysis of wavelets, San Antonio, Tx, Jan. 1999. *Contemp Math.* vol. 247, *Am. Math. Soc.* (RI: Providence) (1999) pp. 207–233

[11] Frank M and Larson D R, Frames in Hilbert C^*-modules and C^*-algebras, *J. Op. Theory* **48** (2002) 273–314

[12] Gabor D, Theory of Communications, *J. Inst. Electr. Eng. (London)* **93(3)** (1946) 429–457

[13] Han D and Larson D R, Frames, bases and group representations, *Mem. Am. Math. Soc.* **147** (2000) 697

[14] Heil C E and Walnut D F, Continuous and discrete wavelet transforms, *SIAM Review* **31** (1989) 628–666

[15] Khosravi A and Asgari M S, Frames and bases in tensor product of Hilbert spaces, *Int. J. Math.* **4(6)** (2003) 527–538

[16] Lance E C, Hilbert C^*-modules – a toolkit for operator algebraists, *London Math. Soc.* Lecture Note Ser., (Cambridge, England: Cambridge Univ. Press) (1995) vol. 210

[17] Murphy G J, C^*-algebra and operator theory (San Diego, California: Academic Press) (1990)

[18] Wegge-Olsen N E, K-Theory and C^*-algebras, a friendly approach (Oxford, England: Oxford Univ. Press) (1993)

[19] Young R, An introduction to nonharmonic Fourier series (New York: Academic Press) (1980)