Proc. Indian Acad. Sci. (Math. Sci.) Vol. 116, No. 2, May 2006, pp. 137–146.

On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann hypothesis

BHASKAR BAGCHI

Indian Statistical Institute, R.V. College Post, Bangalore 560 059, India
E-mail: bbagchi@isibang.ac.in

MS received 21 May 2005; revised 29 April 2006

Abstract. There has been a surge of interest of late in an old result of Nyman and Beurling giving a Hilbert space formulation of the Riemann hypothesis. Many authors have contributed to this circle of ideas, culminating in a beautiful refinement due to Baez-Duarte. The purpose of this little survey is to dis-entangle the resulting web of complications, and reveal the essential simplicity of the main results.

Keywords. Riemann hypothesis; Hilbert space; total set; semigroups.

Let $\mathcal{H}$ denote the weighted l^2-space consisting of all sequences $a = \{a_n : n \in \mathbb{N}\}$ of complex numbers such that $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n(n+1)} < \infty$. For any two vectors $a, b \in \mathcal{H}$, their inner product is given by $\langle a, b\rangle = \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n(n+1)}$. Notice that all bounded sequences of complex numbers are vectors in this Hilbert space. For $l = 1, 2, 3, \ldots$ let $\gamma_l \in \mathcal{H}$ be the sequence

$$\gamma_l = \left\{ \left\{ \frac{n}{l} \right\} : n = 1, 2, 3, \ldots \right\}.$$

(Here and in what follows, $\{x\}$ is the fractional part of a real number x.) Also, let $\gamma \in \mathcal{H}$ denote the constant sequence

$$\gamma = \{1, 1, 1, \ldots\}.$$

Recall that a set A of vectors in a Hilbert space $\mathcal{H}$ is said to be *total* if the set of all finite linear combinations of elements of A is dense in $\mathcal{H}$, i.e., if no proper closed subspace of the Hilbert space contains the set A. In terms of these few notions and notations, the recent result of Baez-Duarte from [2] can be given the following dramatic formulation.

Theorem 1. *The following statements are equivalent*:

(i) *The Riemann hypothesis*,
(ii) γ *belongs to the closed linear span of* $\{\gamma_l : l = 1, 2, 3, \ldots\}$, *and*
(iii) *the set* $\{\gamma_l : l = 1, 2, 3, \ldots\}$ *is total in* $\mathcal{H}$.

We hasten to add that this is not the statement that the reader will see in Baez-Duarte's paper. For one thing, the implications (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) are not mentioned in this paper: perhaps the author thinks of them as 'well-known to experts'. (In such contexts, an expert is usually defined to be a person who has the relevant piece of information.) Moreover, the main result in [2] is not the implication (i) $\Longrightarrow$ (ii) itself, but a 'unitarily equivalent' version thereof. More precisely, the result actually proved in [2] is the implication (i) $\Longrightarrow$ (ii) of Theorem 7 below. In fact, we could not locate in the existing literature the statement (iii) of Theorem 1 (equivalently, of Theorem 7) as a reformulation of the Riemann hypothesis. This result may be new. It reveals the Riemann hypothesis as a version of the central theme of harmonic analysis: that more or less arbitrary sequences (subject to mild growth restrictions) can be arbitrarily well approximated by superpositions of a class of simple periodic sequences (in this instance, the sequences γ_l).

A second point worth noting is that the particular weight sequence $\left\{\frac{1}{n(n+1)}\right\}$ used above is not crucial for the validity of Theorem 1 (though this is the sequence which occurs naturally in its proof). Indeed, any weight sequence $\{w_n\colon n = 1, 2, 3, \ldots\}$ satisfying $\frac{c_1}{n^2} \leq w_n \leq \frac{c_2}{n^2}$ for all n (for constants $0 < c_1 \leq c_2$) would serve equally well. This is because the identity map is an invertible linear operator (hence carrying total sets to total sets) between any two of these weighted l^2-spaces.

In what follows, we shall adopt the standard practice (in analytic number theory) of denoting a complex variable by $s = \sigma + it$. Thus σ and t are the real and imaginary parts of the complex number s. Recall that *Riemann's zeta function* is the analytic function defined on the half-plane $\{\sigma > 1\}$ by the absolutely convergent series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. The completed zeta function ζ^* is defined on this half-plane by $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, where Γ is Euler's gamma function. As Riemann discovered, ζ^* has a meromorphic continuation to the entire complex plane with only two (simple) poles: at $s = 0$ and at $s = 1$. Further, it satisfies the functional equation $\zeta^*(1-s) = \zeta^*(s)$ for all s. Since Γ has poles at the non-positive integers (and nowhere else), it follows that ζ has trivial zeros at the negative even integers. Further, since ζ is real-valued on the real line, its zeros occur in conjugate pairs. This trivial observation, along with the (highly non-trivial) functional equation, shows that the non-trivial zeros of the zeta function are symmetrically situated about the so-called *critical line* $\left\{\sigma = \frac{1}{2}\right\}$. *The Riemann hypothesis* (RH) conjectures that all these non-trivial zeros actually lie on the critical line. In view of the symmetry mentioned above, this amounts to the conjecture that ζ has no zeros on the half-plane

$$\Omega = \left\{s = \sigma + it\colon \sigma > \frac{1}{2},\ -\infty < t < \infty\right\}.$$

In other words, the Riemann hypothesis is the statement that $\frac{1}{\zeta}$ is analytic on the half-plane Ω. This is the formulation of RH that we use in this article. Throughout this article, Ω stands for the half-plane $\left\{\sigma > \frac{1}{2}\right\}$.

Baez-Duarte's theorem refines an earlier result of the same type (Theorem 5 below) proved by Nyman and Beurling (cf. [6] and [1]). Our intention in this article is to point out that the entire gamut of these results is best seen inside the *Hardy space* $H^2(\Omega)$. Recall that this is the Hilbert space of all analytic functions F on Ω such that

$$\|F\|^2 := \sup_{\sigma > \frac{1}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\sigma + it)|^2 \,\mathrm{d}t < \infty.$$

It is known that any $F \in H^2(\Omega)$ has, almost everywhere on the critical line, a non-tangential boundary value F^* such that

$$\|F\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|F^*\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t.$$

Thus $H^2(\Omega)$ may be identified (via the isometric embedding $F \mapsto F^*$) with a closed subspace of the L^2-space of the critical line with respect to the Lebesgue measure scaled by the factor $\frac{1}{2\pi}$. (This scaling is to ensure that the Mellin transform $\mathcal{F}$, defined while proving Theorem 5 below, is an isometry.)

For $0 \leq \lambda \leq 1$, let $F_\lambda \in H^2(\Omega)$ be defined by

$$F_\lambda(s) = (\lambda^s - \lambda)\frac{\zeta(s)}{s}, \quad s \in \Omega.$$

Notice that the zero of the first factor at $s = 1$ cancels the pole of the second factor, so that F_λ, thus defined, is analytic on Ω. Also, in view of the well-known elementary estimate (see [7])

$$\zeta(s) = O(|s|^{\frac{1}{6}} \log |s|), \quad s \in \bar{\Omega}, \quad s \longrightarrow \infty,$$

the factor $\frac{1}{s}$ ensures that $F_\lambda \in H^2(\Omega)$ for $0 \leq \lambda \leq 1$. (Note that, in order to arrive at this conclusion, any exponent $< \frac{1}{2}$ in the above zeta estimate would have sufficed. But the exponent $\frac{1}{6}$ happens to be the simplest non-trivial estimate which occurs in the theory of the Riemann zeta function.) Indeed, under Riemann hypothesis we have the stronger estimate (Lindelof hypothesis)

$$\zeta(s) = O(|s|^{\epsilon}) \quad \text{as } |s| \longrightarrow \infty, \text{uniformly for } s \in \bar{\Omega}, \tag{1}$$

for each $\epsilon > 0$. (More precisely, under RH, this estimate holds uniformly on the complement of any given neighbourhood of 1 in $\bar{\Omega}$.)

Finally, for $l = 1, 2, 3, \ldots$, let $G_l \in H^2(\Omega)$ be defined by $G_l = F_{\frac{1}{l}}$. Thus,

$$G_l(s) = (l^{-s} - l^{-1})\frac{\zeta(s)}{s}, \quad s \in \Omega.$$

Also, let $E \in H^2(\Omega)$ be defined by

$$E(s) = \frac{1}{s}, \quad s \in \Omega.$$

In terms of these notations, the most naural formulation of the Nyman–Beurling–Baez-Duarte theorem is the following:

Theorem 2. *The following statements are equivalent*:

(i) *The Riemann hypothesis*,
(ii) *E belongs to the closed linear span of the set* $\{G_l: l = 1, 2, 3, \ldots\}$, *and*
(iii) *E belongs to the closed linear span of the set* $\{F_\lambda: 0 \leq \lambda \leq 1\}$.

The plan of the proof is to verify (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i). As we shall see in a little while, except for the first implication ((i) $\Longrightarrow$ (ii)), all these implications are fairly straight forward. In order to prove (i) $\Longrightarrow$ (ii), we need recall that on the half-plane $\{\sigma > 1\}$, $\frac{1}{\zeta}$ is represented by an absolutely convergent Dirichlet series

$$\sum_{l=1}^{\infty} \mu(l) l^{-s} = \frac{1}{\zeta(s)}. \tag{2}$$

Here $\mu(\cdot)$ is the Mobius function. (To determine its formula, we may formally multiply this Dirichlet series by that of $\zeta(s)$ and equate coefficients to get the recurrence relation $\sum_{l|n}\mu(l) = \delta_{1n}$. Solving this, one can show that $\mu(\cdot)$ takes values in $\{0, +1, -1\}$ and hence the Dirichlet series for $\frac{1}{\zeta}$ is absolutely convergent on $\{\sigma > 1\}$. Indeed, $\mu(l) = 0$ if l has a repeated prime factor, $\mu(l) = +1$ if l has an even number of distinct prime factors, and $\mu(l) = -1$ if l has an odd number of distinct prime factors. But, for our limited purposes, all this is unnecessary.) What we need is an old theorem of Littlewood (see [7]) to the effect that for the validity of the Riemann hypothesis, it is necessary (and sufficient) that the Dirichlet series displayed above converges uniformly on compact subsets of Ω. Actually, we need the following quantitative version of this theorem of Littlewood.

Lemma 3. *If the Riemann hypothesis holds then for each* $\epsilon > 0$ *and each* $\delta > 0$, *we have* $\sum_{l=1}^{L}\mu(l)l^{-s} = O((|t|+1)^{\delta})$ *uniformly for* $L = 1, 2, 3, \ldots$ *and uniformly for* $s = \sigma + it$ *in the half-plane* $\left\{\sigma > \frac{1}{2} + \epsilon\right\}$. (*Thus the implied constant depends only on* ϵ *and* δ.)

Since Lemma 3 is more or less well-known, we omit its proof. It may be proved by a minor variation in the original proof of Littlewood's theorem quoted above. (Note that, with the aid of a little 'normal family' argument, Littlewood's theorem itself is an easy consequence of this lemma.)

Proof of Theorem 2. (i) $\Rightarrow$ (ii). Assume RH. For positive integers L and any small real number $\epsilon > 0$, let $H_{L,\epsilon} \in H^2(\Omega)$ be defined by

$$H_{L,\epsilon} = \sum_{l=1}^{L}\frac{\mu(l)}{l^{\epsilon}}G_l.$$

Thus each $H_{L,\epsilon}$ is in the linear span of $\{G_l : l \geq 1\}$. Note that

$$H_{L,\epsilon}(s) = \frac{\zeta(s)}{s}\left(\sum_{l=1}^{L}\frac{\mu(l)}{l^{s+\epsilon}} - \sum_{l=1}^{L}\frac{\mu(l)}{l^{1+\epsilon}}\right), \quad s \in \bar{\Omega}.$$

Therefore, by the theorem of Littlewood quoted above, for any fixed $\epsilon > 0$,

$$H_{L,\epsilon}(s) \longrightarrow H_{\epsilon}(s) \quad \text{for } s \text{ in the critical line, as } L \longrightarrow \infty.$$

Here,

$$H_{\epsilon}(s) := \frac{\zeta(s)}{s}\left(\frac{1}{\zeta(s+\epsilon)} - \frac{1}{\zeta(1+\epsilon)}\right).$$

Also, by the estimate (1) and Lemma 3, $H_{L,\epsilon}$ is bounded by an absolutely square integrable function on the critical line. Therefore, by Lebesgue's dominated convergence theorem, we have, for each fixed $\epsilon > 0$,

$$H_{L,\epsilon} \longrightarrow H_{\epsilon} \quad \text{in the norm of } H^2(\Omega) \text{ as } L \longrightarrow \infty.$$

Since $H_{L,\epsilon}$ is in the linear span of $\{G_l : l = 1, 2, 3, \ldots\}$, it follows that, for each $\epsilon > 0$, H_{ϵ} is in the closed linear span of $\{G_l : l = 1, 2, 3, \ldots\}$. Now note that, since ζ has a pole at $s = 1$,

$$H_{\epsilon}(s) \longrightarrow \frac{1}{s} = E(s) \text{ for } s \text{ in the critical line, as } \epsilon \searrow 0.$$

Therefore, in order to show that E is in the closed linear span of $\{G_l : l = 1, 2, 3, \dots\}$ and thus complete this part of the proof, it suffices to show that H_ϵ, $0 < \epsilon < \frac{1}{2}$, are uniformly bounded in modulus on the critical line by an absolutely square integrable function. Then, another application of Lebesgue's dominated convergence theorem would yield

$$H_\epsilon \longrightarrow E \text{ in the norm of } H^2(\Omega) \text{ as } \epsilon \searrow 0.$$

Consider the entire function $\xi(s) := s(1-s)\zeta^*(s) = s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$. It has the Hadamard factorisation

$$\xi(s) = \xi(0)\prod_\rho \left(1 - \frac{s}{\rho}\right),$$

where the product is over all the non-trivial zeros ρ of the Riemann zeta function. This product converges provided the zeros ρ and $1-\rho$ are grouped together. In consequence, with a similar bracketing, we have

$$|\xi(s)| = |\xi(0)| \prod_\rho \left|1 - \frac{s}{\rho}\right|.$$

Now, under RH, each ρ has real part $= \frac{1}{2}$. Therefore, for s in the closed half-plane $\bar{\Omega}$, we have $\left|1 - \frac{s}{\rho}\right| \le \left|1 - \frac{s+\epsilon}{\rho}\right|$. Multiplying this trivial inequality over all ρ, we get

$$|\xi(s)| \le |\xi(s+\epsilon)|, \quad s \in \bar{\Omega},\ \epsilon > 0.$$

(Aside: conversely, the above inequality clearly implies RH. Thus, this simple looking inequality is a reformulation of RH.) In other words, we have, for $s \in \bar{\Omega}$,

$$\left|\frac{\zeta(s)}{\zeta(s+\epsilon)}\right| \le \pi^{-\epsilon/2}\left|\frac{(s+\epsilon)(1-\epsilon-s)}{s(1-s)}\right|\left|\frac{\Gamma((s+\epsilon)/2)}{\Gamma(s/2)}\right| \le c\left|\frac{\Gamma((s+\epsilon)/2)}{\Gamma(s/2)}\right|$$

for some absolute constant $c > 0$. But, by Sterling's formula (see [5] for instance), the gamma ratio on the extreme right is bounded by constant times $|s|^{\epsilon/2}$, uniformly for $s \in \bar{\Omega}$. Therefore we get

$$\left|\frac{\zeta(s)}{\zeta(s+\epsilon)}\right| \le c|s|^{\epsilon/2}, \quad s \in \bar{\Omega},$$

for some other absolute constant $c > 0$. In conjunction with the estimate (1), this implies

$$|H_\epsilon(s)| \le c|s|^{-3/4}, \quad s \in \bar{\Omega},$$

for $0 < \epsilon < \frac{1}{2}$. Since $s \longmapsto c|s|^{-3/4}$ is square integrable on the critical line, we are done. This proves the implication (i) $\Rightarrow$ (ii).

Since $\{G_l : l = 1, 2, 3, \dots\} \subseteq \{F_\lambda : 0 \le \lambda \le 1\}$, the implication (ii) $\Rightarrow$ (iii) is trivial. To prove (iii) $\Rightarrow$ (i), suppose RH is false. Then there is a zeta-zero $\rho \in \Omega$. Since $\zeta(\rho) = 0$, it follows that $F_\lambda(\rho) = 0$ for all $\lambda \in (0, 1]$. Thus the set $\{F_\lambda : \lambda \in (0, 1]\}$ (and hence also its closed linear span) is contained in the proper closed subspace $\{F \in H^2(\Omega) : F(\rho) = 0\}$ of $H^2(\Omega)$. (It is a closed subspace since evaluation at any fixed $\rho \in \Omega$ is a continuous linear functional: $H^2(\Omega)$ is a functional Hilbert space.) Since E belongs to the closed linear span of this set, it follows that $0 = E(\rho) = \frac{1}{\rho}$. Hence $0 = 1$: the ultimate contradiction! This proves (iii) $\Longrightarrow$ (i). ■

Remark 4. Since $\mu(l) = 0$ unless l is square-free, the functions $H_{L,\epsilon}$ introduced in the course of the above proof are in the linear span of the set $\{G_l\colon l \text{ square-free}\}$. Thus, the proof actually shows that RH implies (and hence is equivalent to) that E belongs to the closed linear span of the thinner set $\{G_l\colon l \text{ square-free}\}$ in $H^2(\Omega)$.

Now let $L^2((0,1])$ be the Hilbert space of complex-valued absolutely square integrable functions (modulo almost everywhere equality) on the interval $(0, 1]$. For $0 \leq \lambda \leq 1$, let $f_\lambda \in L^2((0,1])$ be defined by

$$f_\lambda(x) = \left\{\frac{\lambda}{x}\right\} - \lambda\left\{\frac{1}{x}\right\}, \quad x \in (0,1].$$

(Recall that $\{\cdot\}$ stands for the fractional part.) Let $\mathbf{1} \in L^2((0,1])$ denote the constant function $= 1$ on $(0, 1]$. Thus,

$$\mathbf{1}(x) = 1,\ x \in (0,1].$$

In terms of these notations, the original theorem of Nyman and Beurling may be stated as follows:

Theorem 5. *The following statements are equivalent*:

(i) *The Riemann hypothesis*,
(ii) $\mathbf{1}$ *is in the closed linear span in* $L^2((0,1])$ *of the set* $\{f_\lambda\colon 0 \leq \lambda \leq 1\}$,
(iii) *the set* $\{f_\lambda\colon 0 \leq \lambda \leq 1\}$ *is total in* $L^2((0,1])$.

Proof. One defines the *Fourier–Mellin transform* $\mathcal{F}\colon L^2((0,1]) \longrightarrow H^2(\Omega)$ by

$$\mathcal{F}(f)(s) = \int_0^\infty x^{s-1} f(x)\mathrm{d}x, \quad s \in \Omega,\ f \in L^2((0,1]). \tag{3}$$

It is well-known that $\mathcal{F}$, thus defined, is an isometry. For completeness, we sketch a proof. Since $s \longmapsto (x \longmapsto x^{s-1})$ is an $L^2((0,1])$-valued analytic function on Ω, it follows that $\mathcal{F}(f)$ is analytic on Ω for each $f \in L^2((0,1])$. For $\lambda \in [0,1]$, let $\Psi_\lambda \in L^2((0,1])$ denote the indicator function of the interval $(0, \lambda)$. Using the well-known identity

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{iux}}{1+x^2}\mathrm{d}x = \mathrm{e}^{-|u|}, \quad u \in \mathbb{R},$$

one sees that $\|\mathcal{F}(\Psi_\lambda)\|^2 = \|\Psi_\lambda\|^2 < \infty$ – hence $\mathcal{F}(\Psi_\lambda) \in H^2(\Omega)$ – and, more generally, $\|\mathcal{F}(\Psi_\lambda) - \mathcal{F}(\Psi_\mu)\|^2 = \|\Psi_\lambda - \Psi_\mu\|^2$ for $\lambda, \mu \in [0,1]$. Since $\{\Psi_\lambda\colon \lambda \in [0,1]\}$ is a total subset of $L^2((0,1])$, this implies that $\mathcal{F}$ maps $L^2((0,1])$ isometrically into $H^2(\Omega)$.

We begin with a computation of the Melin transform of f_λ.

Claim.

$$\mathcal{F}(f_\lambda) = -F_\lambda, \quad 0 \leq \lambda \leq 1. \tag{4}$$

To verify this claim, begin with $s = \sigma + it,\ \sigma > 1$. Then,

$$\begin{aligned}\int_0^1 \left\{\frac{\lambda}{x}\right\} x^{s-1}\mathrm{d}x &= \lambda\int_0^1 x^{s-2}\mathrm{d}x - \int_0^1 \left\lfloor\frac{\lambda}{x}\right\rfloor x^{s-1}\mathrm{d}x\\ &= \frac{\lambda}{s-1} - \int_0^1 \left\lfloor\frac{\lambda}{x}\right\rfloor x^{s-1}\mathrm{d}x.\end{aligned}$$

But,

$$\int_0^1 \left\lfloor \frac{\lambda}{x} \right\rfloor x^{s-1}\mathrm{d}x = \sum_{n=1}^{\infty} n \int_{\lambda/(n+1)}^{\lambda/n} x^{s-1}\mathrm{d}x$$

$$= \frac{\lambda^s}{s} \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}).$$

Now, the partial sum $\sum_{n=1}^{N} n(n^{-s} - (n+1)^{-s})$ telescopes to $-N(N+1)^{-s} + \sum_{n=1}^{N} n^{-s}$. Since $\sigma > 1$, letting $N \longrightarrow \infty$, we get $\sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}) = \zeta(s)$. Thus,

$$\int_0^1 \left\{ \frac{\lambda}{x} \right\} x^{s-1}\mathrm{d}x = \frac{\lambda}{s-1} - \lambda^s \frac{\zeta(s)}{s}.$$

In particular, taking $\lambda = 1$ here, one gets

$$\int_0^1 \left\{ \frac{1}{x} \right\} x^{s-1}\mathrm{d}x = \frac{1}{s-1} - \frac{\zeta(s)}{s}.$$

Multiplying the second equation by λ and subtracting the result from the first, we arrive at

$$\int_0^1 f_\lambda(x) x^{s-1}\mathrm{d}x = -(\lambda^s - \lambda)\frac{\zeta(s)}{s} = -F_\lambda(s)$$

for s in the half-plane $\{\sigma > 1\}$. Since both sides of this equation are analytic in the bigger half-plane Ω, this equation continues to hold for $s \in \Omega$. This proves the Claim.

(i) $\Longrightarrow$ (ii). Assume RH. Then, by Theorem 2, $E = \mathcal{F}(\mathbf{1})$ belongs to the closed linear span of $\{F_\lambda = -\mathcal{F}(f_\lambda)\colon 0 \le \lambda \le 1\}$. Since $\mathcal{F}$ is an isometry, this shows that $\mathbf{1}$ belongs to the closed linear span of the set $\{f_\lambda\colon 0 \le \lambda \le 1\}$. Thus (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (iii). Let $\mathbf{1}$ be in the closed linear span in $L^2((0,1])$ of $\{f_\lambda\colon 0 \le \lambda \le 1\}$. Applying $\mathcal{F}$, it follows that E is in the closed linear span (say $\mathcal{N}$) of $\{F_\lambda\colon 0 \le \lambda \le 1\}$. For $\mu \in (0,1]$, let $\Theta_\mu \in H^\infty(\Omega)$ (the Banach algebra of bounded analytic functions on Ω) be defined by

$$\Theta_\mu(s) = \mu^{s-\frac{1}{2}}, \quad s \in \Omega.$$

We have $|\Theta_\mu(s)| = 1$ for s in the critical line. That is, Θ_μ is an inner function. In consequence, the linear operators $M_\mu\colon H^2(\Omega) \longrightarrow H^2(\Omega)$ defined by

$$M_\mu(F) = \Theta_\mu F \text{ (point-wise product)}, \; F \in H^2(\Omega),$$

are isometries. (Since $\Theta_\lambda\Theta_\mu = \Theta_{\lambda\mu}$, it follows that $M_\lambda M_\mu = M_{\lambda\mu}$ for $\lambda, \mu \in (0,1]$. Thus $\{M_\mu\colon \mu \in (0,1]\}$ is a semi-group of isometries on $H^2(\Omega)$ modelled after the multiplicative semi-group $(0,1]$.) Trivially, for $0 \le \lambda \le 1$ and $0 < \mu \le 1$, we have

$$M_\mu(F_\lambda) = \Theta_\mu F_\lambda = \mu^{-1/2}(F_{\lambda\mu} - \lambda F_\mu).$$

This shows that the closed subspace $\mathcal{N}$ spanned by the F_λ's is invariant under the semi-group $\{M_\mu\colon \mu \in (0,1]\}$:

$$M_\mu(\mathcal{N}) \subseteq \mathcal{N}, \quad \mu \in (0,1].$$

Since $E \in \mathcal{N}$, it follows that $M_\mu(E) \in \mathcal{N}$ for $\mu \in (0, 1]$. But we have the trivial computation

$$\mathcal{F}(\Psi_\lambda) = \lambda^{1/2} M_\lambda(E), \quad 0 < \lambda \le 1.$$

Thus, $\{\mathcal{F}(\Psi_\lambda): 0 \le \lambda \le 1\}$ is contained in the closed linear span $\mathcal{N}$ of $\{\mathcal{F}(f_\lambda): 0 \le \lambda \le 1\}$. Since $\mathcal{F}$ is an isometry, it follows that $\{\Psi_\lambda: 0 \le \lambda \le 1\}$ is contained in the closed linear span in $L^2((0, 1])$ of the set $\{f_\lambda: 0 \le \lambda \le 1\}$. Since the first set is clearly total in $L^2((0, 1])$, it follows that so is the second. Thus (ii) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i). Clearly (iii) implies that the closed linear span of $\{f_\lambda: 0 \le \lambda \le 1\}$ contains $\mathbf{1}$ and hence, applying $\mathcal{F}$, the closed linear span of $\{F_\lambda: 0 \le \lambda \le 1\}$ contains E. Therefore, by Theorem 2, Riemann hypothesis follows. Thus (iii) $\Longrightarrow$ (i). ■

Remark 6. It is instructive to compare the proof of Theorem 5 with Beurling's original proof as given in [4]. Our proof makes it clear that the heart of the matter is very simple: Riemann hypothesis amounts to the existence of approximate inverses to the zeta function in a suitable function space (viz. the weighted Hardy space of analytic functions on Ω with the weight function $|E(s)|^2$). The simplification in its proof is achieved by Baez-Duarte's perfectly natural and yet vastly illuminating observation that, under RH, these approximate inverses are provided by the partial sums of the Dirichlet series for $\frac{1}{\zeta}$. In contrast, Beurling's original proof is a clever and ill-motivated application of Phragmen–Lindelof type arguments. (We have not seen Nyman's original proof.) To be fair, we should however point out that such arguments are now hidden under the carpet: they occur in the proofs (not presented here) of the conditional estimate (1) and Lemma 3.

Let $\mathcal{M}$ be the closed subspace of $L^2((0, 1])$ consisting of the functions which are almost everywhere constant on each of the sub-intervals $\left(\frac{1}{n+1}, \frac{1}{n}\right]$, $n = 1, 2, 3, \ldots$. Since each element of $\mathcal{M}$ is almost everywhere equal to a unique function which is everywhere constant on these sub-intervals, we may (and do) think of $\mathcal{M}$ as the space of all such (genuine) piece-wise constant functions. As a closed subspace of a Hilbert space, $\mathcal{M}$ is a Hilbert space in its own right.

For $l = 1, 2, 3, \ldots$, let $g_l \in L^2((0, 1])$ be defined by

$$g_l(x) = \left\{\frac{1}{lx}\right\} - \frac{1}{l}\left\{\frac{1}{x}\right\}, \quad x \in (0, 1].$$

Thus, $g_l = f_{1/l}$, $l = 1, 2, 3, \ldots$.

Notice that we have $g_l(x) = \frac{1}{l}\left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{lx} \right\rfloor$. Also, for $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, $n = 1, 2, 3, \ldots$, $\frac{1}{lx} \in \left[\frac{n}{l}, \frac{n+1}{l}\right)$, and no integer can be in the interior of the latter interval, so that $\left\lfloor \frac{1}{lx} \right\rfloor = \left\lfloor \frac{n}{l} \right\rfloor$; also, $\left\lfloor \frac{1}{x} \right\rfloor = n$ for $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Thus we get

$$g_l(x) = g_l\left(\frac{1}{n}\right) = \left\{\frac{n}{l}\right\}, \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]. \tag{5}$$

In consequence,

$$g_l \in \mathcal{M}, \quad l = 1, 2, 3, \ldots.$$

The refinement due to Baez-Duarte of the Beurling–Nyman theorem may now be stated as follows. (However, as already stated, the implication (i) $\Longrightarrow$ (ii) of this theorem is its only part which explicitly occurs in [2].)

Theorem 7. *The following are equivalent*:

(i) *The Riemann hypothesis*,
(ii) $\mathbf{1}$ *belongs to the closed linear span of* $\{g_l : l = 1, 2, 3, \dots\}$, *and*
(iii) $\{g_l : l = 1, 2, 3, \dots\}$ *is a total set in* $\mathcal{M}$.

Proof. Putting $\lambda = \frac{1}{l}$ in the formula (4), we get

$$\mathcal{F}(g_l) = -G_l, \quad l = 1, 2, 3, \dots.$$

Since, under RH, $E = \mathcal{F}(\mathbf{1})$ is in the closed linear span of $\{G_l = -\mathcal{F}(g_l) : l = 1, 2, 3, \dots\}$ and $\mathcal{F}$ is an isometry, it follows that $\mathbf{1}$ is in the closed linear span of $\{g_l : l = 1, 2, 3, \dots\}$. Thus (i) $\Longrightarrow$ (ii).

Now, for positive integers m, define the linear operators $T_m : \mathcal{M} \longrightarrow \mathcal{M}$ by

$$(T_m f)(x) = \begin{cases} m^{1/2} f(mx), & \text{if } x \in \left(0, \frac{1}{m}\right], \\ 0, & \text{if } x \in \left(\frac{1}{m}, 1\right]. \end{cases}$$

Clearly each T_m is an isometry. (We have $T_m T_n = T_{mn}$ – thus $\{T_m : m = 1, 2, 3, \dots\}$ is a semigroup of isometries modelled after the multiplicative semi-group of positive integers.) Also, it is easy to see that

$$T_m(g_l) = m^{1/2} \left(g_{lm} - \frac{g_m}{l} \right)$$

for any two positive integers l, m. Thus the closed linear span $\mathcal{K}$ of the vectors g_l, $l = 1, 2, 3, \dots$ is invariant under this semi-group. Further, letting $\Phi_n \in \mathcal{M}$ denote the indicator function of the interval $\left(0, \frac{1}{n}\right]$, one has

$$T_m(\Phi_n) = m^{1/2} \Phi_{mn}.$$

Thus, if $\mathcal{K}$ contains $\mathbf{1} = \Phi_1$ then it contains Φ_n for all n. Since $\{\Phi_n : n = 1, 2, 3, \dots\}$ is clearly a total subset of $\mathcal{M}$, it then follows that $\mathcal{K} = \mathcal{M}$, so that $\{g_l : l = 1, 2, 3, \dots\}$ is a total subset of $\mathcal{M}$. Thus (ii) $\Longrightarrow$ (iii).

Lastly, if $\{g_l : l = 1, 2, 3, \dots\}$ is a total subset of $\mathcal{M}$ then, in particular its closed linear span contains $\mathbf{1}$, and hence the closed linear span of $\{G_l = -\mathcal{F}(g_l)\}$ contains $E = \mathcal{F}(\mathbf{1})$, so that RH follows by Theorem 2. Thus (iii) $\Longrightarrow$ (i). ■

Proof of Theorem 1. Let $U : \mathcal{M} \longrightarrow \mathcal{H}$ be the unitary defined by

$$U(f) = \left\{ f\left(\frac{1}{n}\right) : n = 1, 2, 3, \dots \right\}, \quad f \in \mathcal{M}.$$

Since $U(\mathbf{1}) = \gamma$ and (in view of equation (5)) $U(g_l) = \gamma_l$, this theorem is a straightforward reformulation of theorem 7. ■

Remark 8. In view of Remark 4, Riemann hypothesis actually implies (and hence is equivalent to) the statement that γ belongs to the closed linear span in $\mathcal{H}$ of the much thinner set $\{\gamma_l : l \text{ square-free}\}$.

So where does the undoubtedly elegant reformulation of RH in Theorem 1 leave us? One possible approach is as follows. For positive integers L, let $D(L)$ denote the distance of the vector $\gamma \in \mathcal{H}$ from the $(L-1)$-dimensional subspace of $\mathcal{H}$ spanned by $\gamma_1, \gamma_2, \ldots, \gamma_L$. In view of Theorem 1, RH is equivalent to the statement $D(L) \longrightarrow 0$ as $L \longrightarrow \infty$. So one might try to estimate $D(L)$. Indeed, as a discrete analogue of a conjecture of Baez-Duarte *et al* [3], one might expect that $D^2(L)$ is asymptotically equal to $\frac{A}{\log L}$ for $A = 2 + C - \log(4\pi)$, where C is Euler's constant. (But, of course, this is far stronger than RH itself.) A standard formula gives $D^2(L)$ as a ratio of two Gram determinants, i.e., determinants with the inner products $\langle \gamma_l, \gamma_m \rangle$ as entries. It is easy to write down these inner products as finite sums involving the logarithmic derivative of the gamma function. But such formulae are hardly suitable for calculation/estimation of determinants. In any case, it will be a sad day for Mathematics when (and if) the Riemann hypothesis is proved by a brute-force calculation! Surely a dramatically new and deep idea is called for. But then, as a wise man once said, it is fool-hardy to predict – specially the future!

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