

Matrix multiplication operators on Banach function spaces

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Abstract. In this paper, we study the matrix multiplication operators on Banach function spaces and discuss their applications in semigroups for solving the abstract Cauchy problem.

Keywords. Banach function spaces; closed operators; compact operators; Fredholm operators; matrix multiplication operators; semigroups.

1. Introduction

Let (Ω, Σ, μ) be a σ -finite complete measure space and \mathbf{C} be the field of complex numbers. By $L(\mu, \mathbf{C}^N)$, we denote the linear space of all equivalence classes of \mathbf{C}^N -valued Σ -measurable functions on Ω that are identified μ -a.e. and are considered as column vectors.

Let M_0 denote the linear space of all functions in $L(\mu, \mathbf{C}^N)$ that are finite a.e. With the topology of convergence in measure on the sets of finite measure, it is a metrizable space.

The \mathbf{C}^N -valued Banach function space X is defined as

$$X = \{f \in L(\mu, \mathbf{C}^N) : \|f\|_X < \infty\},$$

where $\|\cdot\|_X$ is a function norm on X such that for each $f, g, f_n \in L(\mu, \mathbf{C}^N)$, $n \in \mathbf{N}$, we have

- (i) $0 \leq \|f(x)\|_{\mathbf{C}^N} \leq \|g(x)\|_{\mathbf{C}^N}$ for μ -a.e. $x \in \Omega \Rightarrow \|f\|_X \leq \|g\|_X$,
- (ii) $0 \leq (f_n)_i \nearrow (f)_i$ μ -a.e. for each $i = 1, 2, \dots, N \Rightarrow \|f_n\|_X \nearrow \|f\|_X$, and
- (iii) $E \in \Sigma$ with $\mu(E) < \infty$ implies that $1_E \cdot z \in X$, for each $z \in \mathbf{C}^N$, and

$$\int_E \|f(x)\|_{\mathbf{C}^N} d\mu(x) \leq C_E \|f\|_X,$$

for some constant $0 < C_E < \infty$, depending on E but independent of f , where 1_E is the characteristic function of the set E .

DEFINITION 1

A function f in a Banach function space X is said to have *absolutely continuous norm* in X if $\|f 1_{E_n}\|_X \rightarrow 0$ for every sequence $\{E_n\}_{n=1}^\infty$ of μ -measurable sets in Ω satisfying $E_n \rightarrow \emptyset$, μ -a.e., where $E_n \rightarrow \emptyset$ means that $1_{E_n} \rightarrow 0$, μ -a.e.

Let X_a be the set of all functions in X having *absolutely continuous norm*. If $X_a = X$, then we say that X has *absolutely continuous norm*.

Let X_b be the closure of the set of all μ -simple functions in X . Then, we have

$$X_a \subseteq X_b \subseteq X.$$

Throughout this paper, we assume that $X = X_b$, that is, the simple functions are dense in X . In case X has *absolutely continuous norm*, we have $X_a = X_b = X$ and so its Banach space dual X^* and its associate space X' coincide, where X' is defined as

$$X' = \{g \in L(\mu, \mathbf{C}^N) : \|g\|_{X'} < \infty\}$$

and

$$\|g\|_{X'} = \sup \left\{ \left| \int_{\Omega} \langle f(x), g(x) \rangle d\mu(x) \right| : f \in X, \|f\|_X \leq 1 \right\},$$

where $L(\mu, \mathbf{C}^N)'$ denotes the corresponding space of equivalence classes of \mathbf{C}^N -valued functions considered as row vectors. Note that $\langle f(x), g(x) \rangle$ is the usual product of a row matrix formed by $g(x)$ into a column matrix formed by $f(x)$.

The monotone convergence theorem holds in every Banach function space X in the form of the weak Fatou property (see axiom (ii)). Also note that X_a is the largest subspace of X for which the suitable dominated convergence theorem holds (see Proposition 3.6, p. 16 of [2]). So this fact can be used to easily generalise those results on L^p -spaces to the general Banach function spaces having absolutely continuous norm in which the dominated convergence theorem is required. It is due to this fact that our results in §3 follow on similar lines as in L^p -spaces without any extra effort.

For details on Banach function spaces, we refer to [2, 13, 14].

For a measurable function $u: \Omega \rightarrow M_N(\mathbf{C})$, the set of all $N \times N$ matrices over \mathbf{C} , the multiplication transformation $M_u: L(\mu, \mathbf{C}^N) \mapsto L(\mu, \mathbf{C}^N)$ is defined as

$$M_u(f) = u \cdot f, \quad \text{for all } f \in L(\mu, \mathbf{C}^N).$$

Using the arguments given in the introduction on p. 517 of [23] and p. 163, Theorem 1.2 of [6] we can easily prove the next result.

Theorem 1.1. *The multiplication operator M_u is a bounded operator on a Banach function space X if and only if $u \in L^\infty(\mu, M_N(\mathbf{C}))$, the space of all $M_N(\mathbf{C})$ -valued essentially-bounded measurable functions. Moreover, we have*

$$\|M_u\|_{X \mapsto X} = \|u\|_\infty := \inf_{q \in [u]} \sup_{x \in \Omega} \|q(x)\|, \quad (1.1)$$

$$\|M_u\|_{X \mapsto X} = \inf\{M \geq 0: \mu(\{x \in \Omega: \|u(x)\| > M\}) = 0\}, \quad (1.2)$$

where $\|u(x)\|$ is the operator norm of $u(x)$ in $\mathbf{B}(\mathbf{C}^N)$ induced by $\|\cdot\|_{\mathbf{C}^N}$ and $[u]$ is the equivalence class of all measurable functions that are μ -a.e. equal to u .

Note that for $N = 1$, $L^\infty(\mu, M_N(\mathbf{C}))$ is the space of essentially bounded \mathbf{C} -valued measurable functions denoted as $L^\infty(\mu)$.

For details on matrix analysis, see [10].

DEFINITION 2

For $u \in L(\mu, M_N(\mathbf{C}))$, an operator $(M_u, D(M_u))$ defined on $X(\mathbf{C}^N)$ by $(M_u f(x)) = u(x) \cdot f(x)$ for $x \in \Omega$ and for all $f \in D(M_u) = \{f \in X: u \cdot f \in X\}$ is called a matrix multiplication operator.

We consider an abstract Banach space-valued linear initial value problem of the form

$$\dot{v}(t) = Av(t), \quad t \geq 0,$$

$$v(0) = x,$$

where the independent variable t represents time, $v(\cdot)$ is a function with values in a \mathbf{C}^N -valued Banach function space $X = X(\mathbf{C}^N)$, where \mathbf{C}^N denotes the N -dimensional complex space. Since all the norms on \mathbf{C}^N are equivalent, we choose $\|\cdot\|_{\mathbf{C}^N} = \|\cdot\|_{\text{sup}}$. Note that this norm induces the matrix norm $\|(a_{ij})_{N \times N}\| = \max_{i=1, \dots, N} \sum_{j=1}^N |a_{ij}|$ on $M_N(\mathbf{C})$, that is, the norm of a matrix is given by the maximal absolute sum of its rows.

Also, $A: D(A) \subseteq X \mapsto X$ is a linear operator and $x \in X$ is the initial value. This problem is called an abstract Cauchy problem (ACP) associated with a matrix multiplication operator $(M_u, D(M_u))$ and the initial value x .

A function $u: \mathbf{R}_+ \mapsto X$ forms a solution of ACP, if u is a continuously differentiable function with respect to X and $u(t) \in D(A)$, for each $t \geq 0$ (see [17]).

If the operator A is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$, then for each $x \in D(A)$, the function

$$u: t \mapsto u(t) = T(t)x$$

is the unique solution of ACP. See [17] and [19] for details on ACP and semigroups. Note that every well-posed Cauchy problem is solved by a strongly continuous semigroup and each such semigroup is the solution semigroup of a well-posed Cauchy problem (see [16]).

In this paper, we extend the results of [6, 9, 12, 24] to general Banach function spaces. Note that the separable Banach function spaces form a subclass of the absolutely continuous ones. The examples of Banach function spaces having absolutely continuous norm are L^p -spaces, Orlicz spaces with Δ_2 -conditions [20], Lorentz spaces [2], separable Orlicz–Lorentz spaces [11], etc. In the second section, we study the compactness, closedness, invertibility and Fredholmness properties of a multiplication operator M_u on a Banach function space X . In the third section, we discuss the abstract Cauchy problem associated with the operator $(M_u, D(M_u))$ on X .

2. Properties of multiplication operators

For a measurable function $u: \Omega \mapsto M_N(\mathbf{C})$, we call the set

$$u_{\text{ess}}(\Omega) = \{\lambda \in \mathbf{C}: \mu(\{s \in \Omega: \|u(s) - \lambda\| < \epsilon\}) \neq 0, \quad \forall \epsilon > 0\},$$

its essential range and define the associated multiplication operator M_u on the space X by $M_u(f) = u \cdot f$ for each f in the domain

$$D(M_u) = \{f \in X: u \cdot f \in X\}.$$

In case M_u is a bounded linear operator on X , we have $D(M_u) = X$.

PROPOSITION 2.1

Let $(M_u, D(M_u))$ be the multiplication operator on a Banach function space X induced by some measurable function $u: \Omega \mapsto M_N(\mathbf{C})$. Then the following statements hold:

- (i) The operator $(M_u, D(M_u))$ is closed and densely defined, in case X has absolutely continuous norm.
- (ii) The operator M_u has a bounded inverse if and only if $0 \notin u_{\text{ess}}(\Omega)$. In this case, $M_{u^{-1}} = M_r$, for some measurable $r: \Omega \mapsto M_N(\mathbf{C})$ defined by

$$r(s) = \begin{cases} (u(s))^{-1} & \text{if } u(s) \neq 0, \\ 0 & \text{if } u(s) = 0, \end{cases}$$

for each $s \in \Omega$.

- (iii) The spectrum of M_u is the essential range of u , i.e., $\sigma(M_u) = u_{\text{ess}}(\Omega)$.

The proof follows on similar lines as in L^p -spaces (see [6] and [7]).

Theorem 2.2. For $N = 1$, the set of all bounded multiplication operators on a Banach function space X of \mathbf{C} -valued functions forms a maximal abelian subalgebra of $\mathbf{B}(X)$, the space of all bounded linear operators on X .

Proof. Let $\mathbf{m} = \{M_u: u \in L^\infty(\mu)\}$. Clearly, \mathbf{m} is an abelian subalgebra of $\mathbf{B}(X)$. We prove that \mathbf{m} is maximal, i.e., if A commutes with \mathbf{m} , then $A \in \mathbf{m}$.

Let $e: \Omega \rightarrow \mathbf{C}$ be the unity function. Let $v = Ae$ and $E \in \Sigma$. Then

$$A1_E = AM_{1_E}e = M_{1_E}Ae = 1_Ev = v1_E = M_v1_E.$$

We claim $v \in L^\infty(\mu)$. Suppose that the set

$$F_n = \{x \in \Omega: |v(x)| > n\}$$

has a positive measure for each $n \geq 1$. By the finite subset property of the underlying measure space, we assume $\mu(F_n) < \infty$. Then, $1_{F_n} \in X$ and we have

$$\begin{aligned} n1_{F_n}(x) &\leq 1_{F_n}|v(x)| \\ \Rightarrow n\|1_{F_n}\|_X &\leq \|M_v1_{F_n}\|_X \\ \Rightarrow n\|1_{F_n}\|_X &\leq \|A1_{F_n}\|_X \end{aligned}$$

for each $n \in \mathbf{N}$, which contradicts the boundedness of A . Therefore $v \in L^\infty(\mu)$. Since the set of μ -simple functions is dense in X , we have $A = M_v$. This proves that $A \in \mathbf{m}$ and so \mathbf{m} is a maximal abelian subalgebra of $\mathbf{B}(X)$.

Theorem 2.3. Let $X = X(\mathbf{C})$ be a Banach function space of \mathbf{C} -valued measurable functions on Ω and $M_u \in \mathbf{B}(X)$ for some $u \in L^\infty(\mu)$. Then M_u has closed range if and only if there exists some $\delta > 0$ such that $|u(x)| \geq \delta$, for μ -almost all $x \in \text{support}(u) = S$.

Proof. Suppose $|u(x)| \geq \delta$ for μ -almost all $x \in S$. Then using the same technique as in converse part of Theorem 1.2 in p. 163 of [6] with $N = 1$, we have

$$|(M_u|_S f)(x)| \geq \delta|f(x)|,$$

for μ -a.e. $x \in \Omega$ and each $f \in X$. As the norm $\|\cdot\|_X$ is increasing on X , so we have

$$\|M_u|_S f\|_X \geq \delta \|f\|_X,$$

for each $f \in X$. This implies that M_u is invertible. Thus, we conclude that M_u has closed range.

Conversely, suppose M_u has closed range. Then there exists some $\delta > 0$ such that

$$\|M_u h\|_X \geq \delta \|h\|_X, \text{ for each } h \in X. \quad (2.1)$$

Let $E = \{x \in \Omega: |u(x)| < \delta/2\}$ be such that $\mu(E) > 0$. So, there exists a measurable set $F \subseteq E$ such that $1_F \in X$. Further,

$$|(M_u 1_F)(x)| = |u(x) 1_F(x)| < \delta |1_F(x)|,$$

implies that

$$\|M_u 1_F\|_X \leq \delta \|1_F\|_X,$$

which contradicts (2.1) and so $\mu(E) = 0$. Therefore $|u(x)| \geq \delta/2$, for μ -almost all $x \in S$.

Theorem 2.4. *Let $X = X(\mathbf{C}^N)$ be a Banach function space of \mathbf{C}^N -valued measurable functions on Ω and $M_u \in \mathbf{B}(X)$. Then M_u is a compact operator if and only if $X(N, \epsilon, \mathbf{C}^N)$ is finite-dimensional, for each $\epsilon > 0$, where*

$$N = N(u, \epsilon) = \{x \in \Omega: \|u(x)\| \geq \epsilon\}$$

and

$$X(N, \epsilon, \mathbf{C}^N) = \{f \in X: f(x) = \mathbf{0} \text{ for } x \notin N\}.$$

Proof. Suppose M_u is a compact operator. Then, its restriction to the invariant subspace $X(N, \epsilon, \mathbf{C}^N)$ is compact. Also, $M_u|_{X(N, \epsilon, \mathbf{C}^N)}$ has closed range in $X(N, \epsilon, \mathbf{C}^N)$. But $M_u|_{X(N, \epsilon, \mathbf{C}^N)}$ is invertible. Therefore, $X(N, \epsilon, \mathbf{C}^N)$ is finite-dimensional for each $\epsilon > 0$.

Conversely, suppose $X(N, \epsilon, \mathbf{C}^N)$ is finite-dimensional for each $\epsilon > 0$. In particular, $X(N, 1/n, \mathbf{C}^N)$ is finite-dimensional for each $n \geq 1$. Take $w \in L^\infty(\mu, M_N(\mathbf{C}))$ and define $w_n: \Omega \rightarrow M_N(\mathbf{C})$, by

$$w_{n,ij}(x) = \begin{cases} w_{ij}(x) & \text{if } x \in A_n, \\ 0 & \text{if } x \notin A_n, \end{cases}$$

for each $i, j = 1, 2, \dots, N$, where $A_n = \{x \in \Omega: \|w(x)\| \geq 1/n\}$.

Then for each $f \in X$, we have

$$\begin{aligned} \|((M_{w_n} - M_w)f)(x)\|_{\mathbf{C}^N} &\leq \|(w_n(x) - w(x))f(x)1_{A_n}(x)\|_{\mathbf{C}^N} \\ &\quad + \|(w_n(x) - w(x))f(x)1_{\Omega \setminus A_n}(x)\|_{\mathbf{C}^N} \\ &= \|w(x)f(x)1_{\Omega \setminus A_n}(x)\|_{\mathbf{C}^N} \leq \frac{1}{n} \|f(x)\|_{\mathbf{C}^N}, \end{aligned}$$

for μ -a.e., $x \in \Omega$. Thus, for each $f \in X$, we have

$$\|(M_{w_n} - M_w)f\|_X \leq 1/n \|f\|_X.$$

This implies that $M_{w_n} \rightarrow M_w$ as $n \rightarrow \infty$. But each M_{w_n} is of finite rank and thus M_w is a compact operator.

The next result gives the necessary and sufficient conditions for a multiplication operator M_u on a Banach function space $X = X(\mathbf{C})$ to be a Fredholm operator thereby generalising the results in [12] for Orlicz spaces and [24] for L^p -spaces. Here, we take $N = 1$.

Theorem 2.5. *Suppose (Ω, Σ, μ) is a non-atomic measure space and $M_u \in \mathbf{B}(X)$, where $X = X(\mathbf{C})$ is a Banach function space having absolutely continuous norm. Then, the following are equivalent.*

- (i) M_u is invertible.
- (ii) M_u is Fredholm.
- (iii) The range of M_u , that is, $R(M_u)$ is closed and $\text{codim}(R(M_u)) < \infty$.
- (iv) $|u(x)| \geq \delta$ for μ -a.e. $x \in \Omega$ for some $\delta > 0$.

Proof. We prove (iii) \Rightarrow (iv) only, as the other implications are obvious by using the previous results. Suppose $R(M_u)$ is closed and $\text{codim}(R(M_u)) < \infty$.

We claim M_u is onto. Suppose the contrary. Then there exists $f_\circ \in X \setminus R(M_u)$. Since $R(M_u)$ is closed, there exists $g_\circ \in X^*$, the dual (associate) space of X such that

$$\int_{\Omega} \langle f_\circ, g_\circ \rangle d\mu = 1 \quad (2.2)$$

and

$$\int_{\Omega} \langle (M_u f), g_\circ \rangle d\mu = 0, \text{ for each } f \in X. \quad (2.3)$$

Now (2.2) yields that the set

$$E_\delta = \{x \in \Omega: \text{Re} \langle f_\circ(x), g_\circ(x) \rangle \geq \delta\}$$

has positive μ -measure for some $\delta > 0$. As μ is non-atomic, we can choose a sequence $\{E_n\}$ of subsets of E_δ with $0 < \mu(E_n) < \infty$ and $E_m \cap E_n = \emptyset$ for $m \neq n$.

Put $g_n = 1_{E_n} g_\circ$. Clearly, $g_n \in X^*$, as $\|g_n\|_{X^*} \leq \|g_\circ\|_{X^*}$ and $g_n \neq 0$, as

$$\text{Re} \int_{\Omega} \langle f_\circ, g_n \rangle d\mu = \text{Re} \int_{E_n} \langle f_\circ, g_\circ \rangle d\mu \geq \delta \mu(E_n) > 0,$$

for each n . Also, for each $f \in X$, $1_{E_n} f \in X$ and so (2.3) implies that

$$\begin{aligned} \int_{\Omega} \langle f, (M_u^* g_n) \rangle d\mu &= \int_{\Omega} \langle M_u f, g_n \rangle d\mu = \int_{\Omega} \langle M_u f, 1_{E_n} g_\circ \rangle d\mu \\ &= \int_{\Omega} \langle M_u 1_{E_n} f, g_\circ \rangle d\mu = 0, \end{aligned}$$

where M_u^* is the conjugate operator of M_u , which implies that $M_u^* g_n = 0$ a.e. and so $g_n \in \ker(M_u^*)$. Since all the sets in $\{E_n\}$ are disjoint, the sequence $\{g_n\}$ forms a linearly-independent subset of $\ker(M_u^*)$. This contradicts the fact that

$$\dim \ker(M_u^*) = \text{codim } R(M_u) < \infty.$$

So M_u is onto.

Let $\ker(u) = \{x \in \Omega: u(x) = 0\}$. Then $\mu(\ker(u)) = 0$. Since $\mu(\ker(u)) > 0$, there is an $F \subseteq \ker(u)$ with $0 < \mu(F) < \infty$. Thus, $1_F \in X \setminus R(M_u)$ which contradicts the surjectiveness of M_u . For $n \geq 1$, put

$$G_n = \left\{ x \in \Omega: \frac{\|u\|_\infty}{(n+1)^2} < |u(x)| \leq \frac{\|u\|_\infty}{n^2} \right\}$$

and

$$T = \{n \in \mathbb{N}: \mu(G_n) > 0\}.$$

Clearly $\Omega = \bigcup_{n=1}^\infty G_n$ and $\mu(G_n) < \infty$, for each $n \geq 1$. Take

$$f = \sum_{n \in T} \frac{u \cdot 1_{G_n}}{\|1_{G_n}\|_X}.$$

Then

$$\begin{aligned} |f(x)| &= \left| \sum_{n \in T} \frac{u(x) 1_{G_n}(x)}{\|1_{G_n}\|_X} \right| \\ &\leq \sum_{n \in T} \frac{|u(x) 1_{G_n}(x)|}{\|1_{G_n}\|_X} \\ &\leq \sum_{n \in T} \frac{\|u\|_\infty}{n^2 \|1_{G_n} \cdot z\|_X}, \end{aligned}$$

for each $x \in \Omega$, this proves that

$$\|f\|_X \leq \sum_{n \in T} \frac{\|u\|_\infty}{n^2} \leq \|u\|_\infty \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

Therefore $f \in X$ and so there exists some $g \in X$ such that $M_u g = f$ with

$$g = \frac{f}{u} = \sum_{n \in T} \frac{1_{G_n}}{\|1_{G_n}\|_X}.$$

Since the sets G_n are disjoint, using the absolute continuity of X (Theorem 4.1, p. 20 of [2]), we have

$$\begin{aligned} \|g\|_X &= \sup_{h \in X^*, \|h\|_{X^*} \leq 1} \left| \int_\Omega \prec \sum_{n \in T} \frac{1_{G_n}}{\|1_{G_n}\|_X}, h \succ d\mu \right| \\ &= \left(\sum_{n \in T} \frac{1}{\|1_{G_n}\|_X} \right) \sup_{h \in X^*, \|h\|_{X^*} \leq 1} \left| \int_\Omega \prec 1_{G_n}, h \succ d\mu \right| \\ &= \sum_{n \in T} 1. \end{aligned}$$

Then, $\|g\|_X < \infty$ if and only if T is finite. So there is some $m > 0$ such that for $n \geq m$, $\mu(G_n) = 0$ and $\mu(\ker(u)) = 0$, which further implies that

$$\begin{aligned} \mu\{x \in \Omega: \|u(x)\| \leq \|u\|_\infty/m^2\} &= \mu((\bigcup_{n=m}^\infty G_n) \cup \ker(u)) \\ &\leq \mu((\bigcup_{n=1}^\infty G_n) \cup \ker(u)) = 0, \end{aligned}$$

that is, $\|u(x)\| \geq \|u\|_\infty/m^2 = \delta$, a.e. on X . This proves (iv).

3. Semigroups of multiplication operators

In this section, we study some applications of the multiplication operators on X in semigroup theory. Using Proposition 2.1(i), we see that M_u is a closed and densely defined multiplication operator defined on an absolutely continuous Banach function space X .

The essential spectrum of a multiplication operator is defined as

$$\sigma_{\text{ess}}(M_u) = \bigcup_{x \in \Omega} \sigma_{\text{ess}}(u(x)) = \bigcap_{p \in [u]} \overline{\bigcup_{x \in \Omega} \sigma(p(x))},$$

where $\sigma(p(x))$ denotes the spectrum of the matrix $p(x)$ and $[u]$ is the equivalence class of all measurable functions that are μ -a.e. equal to u .

For an open ϵ -disk U_ϵ with center 0, we have

$$\begin{aligned} \sigma_{\text{ess}}(M_u) &= \bigcup_{x \in X} \sigma_{\text{ess}}(u(x)) \\ &= \{z \in \mathbf{C} : \mu(\{x \in \Omega : \sigma(u(x)) \cap z + U_\epsilon\}) > 0, \quad \forall \epsilon > 0\}. \end{aligned}$$

Using Propositions 4.11 and 4.12, p. 32 of [7], it is easy to prove the result.

PROPOSITION 3.1

Suppose $(M_u, D(M_u))$ is a matrix multiplication operator on the Banach function space X with non-void resolvent set $\rho(M_u)$. Then its spectrum is given by

$$\sigma(M_u) = \overline{\bigcup_{x \in \Omega} \sigma_{\text{ess}}(u(x))}.$$

Remark 1. If $\Omega = \mathbf{R}^m$ with the Lebesgue measure μ and u is a continuous function with non-void resolvent, then we have

$$\sigma(M_u) = \overline{\bigcup_{x \in \Omega} \sigma(u(x))}.$$

The stability of the solutions of the abstract Cauchy problem are determined by the spectral bound of the corresponding operator, [6].

COROLLARY 3.2

A matrix multiplication operator M_u is bounded if and only if its spectrum $\sigma(M_u)$ is bounded.

There are a number of functional analytic approaches to the well-posedness of solutions of abstract Cauchy problems (see, for example, [3, 7–9]). The proofs of the following results are on the similar lines as in [9] for L^p -spaces, so we only state the results.

Theorem 3.3. *A matrix multiplication operator M_u generates a semigroup on X having absolutely continuous norm if and only if*

$$\overline{\lim_{t \rightarrow 0}} \|e^{tu}\|_\infty < \infty. \quad (3.1)$$

In case M_u generates the semigroup $\{T(t)\}_{t \geq 0}$, then we have $T(t) = M_{e^{tu}}$. On the other hand, if $\{M_{e^{tu}}\}_{t \geq 0}$ defines a semigroup on X , then its generator is given by M_u .

Theorem 3.4. *If the matrix multiplication operator $(M_u, D(M_u))$ generates the semigroup $\{T(t)\}_{t \geq 0}$, then*

$$\sigma(T(t)) = \overline{e^{t\sigma(M_u)}}.$$

DEFINITION 3 [9, 13]

Let $(A, D(A))$ be a linear operator on a Banach space F such that there exists some constants $m \in \mathbf{N}$, $M > 0$, $w \in \mathbf{R}$, the real line and a strongly continuous family $(S(t))_{t \geq 0}$ in $\mathbf{B}(F)$, the space of all the bounded linear operators on F with

$$\|S(t)\| \leq M e^{wt}, \text{ for each } t \geq 0$$

such that its resolvent operator $R(\lambda, A) = (\lambda - A)^{-1}$ exists and is given by

$$R(\lambda, A)y = \lambda^m \int_0^\infty e^{-\lambda t} S(t)y \, dt,$$

for $y \in F$ and $\lambda > w$. Then $(A, D(A))$ is the generator of an m -times integrated semigroup $(S(t))_{t \geq 0}$.

The generator of an m -times integrated semigroups yields the well-posedness of the abstract Cauchy problem, since there exists some constants M and w as above such that for all initial values $u_0 \in D(A^{m+1})$, we get a unique solution $u(\cdot)$ with the property

$$\|u(t)\| \leq M e^{wt} \|x\|_{A^m},$$

for each $t \geq 0$, where $\|\cdot\|_{A^m}$ denotes the m th-graph norm of $(A, D(A))$ (see [17]).

Theorem 3.5. *Let $(M_u, D(M_u))$ be a matrix multiplication operator on a Banach function space $X = X(\mathbf{C}^N)$. Then the following are equivalent.*

- (i) *The operator $(M_u, D(M_u))$ is the generator of an integrated semigroup.*
- (ii) *There exists a constant $w \in \mathbf{R}$ such that*

$$\sigma(M_u) \subseteq \{z \in \mathbf{C} : \operatorname{Re} z \leq w\}.$$

- (iii) *The resolvent set of M_u is non-empty such that*

$$\operatorname{ess\,sup}_{x \in \Omega} s(u(x)) \leq w,$$

for some $w \in \mathbf{R}$, where $s(u(x))$ denotes the spectral bound of the matrix $u(x) \in M_N(\mathbf{C})$.

Remark 2. As in Corollary 4.9 of [17], we see that the multiplication operator $(M_u, D(M_u))$ satisfying one of the conditions in the statement of the above-stated theorem is a generator of a $(2N + 1)$ -times integrated semigroup.

For a multiplication operator $(M_u, D(M_u))$ on $X = X(\mathbf{C}^N)$, we see that it is a generator of a strongly continuous semigroup if and only if there is some $c > 0$ such that

$$\|e^{tu}\|_\infty < c, \text{ for } t \in [0, 1].$$

Remark 3. Theorem 1 on p. 16 of [9] is also true for Banach function spaces $X = X(\mathbf{C}^N)$. Further, if X has absolutely continuous norm, then p. 163, Theorem 2 of [9], p. 164, Proposition 1 of [9] and p. 165, Proposition 2 of [9] can be proved for such Banach function spaces.

Remark 4. The matrix multiplication operator $(M_u, D(M_u))$ on X also generates some subclasses of the strongly continuous semigroups on X such as analytic, differentiable, norm-continuous semigroups, etc. See [7, 8, 19] for details on these types of semigroups.

Also, we see that the analyticity of semigroups on X depends on the spectrum $\sigma(M_u)$ of the matrix multiplication operator $(M_u, D(M_u))$ on X .

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References

- [1] Abramovich Y A, Aliprantis C D and Burkinshaw O, Multiplication and compact-friendly operators, *Positivity* **1** (1997) 171–180
- [2] Bennett C and Sharpley R, Interpolation of operators, *Pure Appl. Math.* (London: Academic Press) (1988) vol. 129
- [3] Butzer P L and Berens H, Semigroups of operators and approximation, *Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen Band* (New York Inc.: Springer-Verlag) (1967) vol. 145
- [4] Conway J B, A first course in functional analysis, *Graduate Texts in Math.* (New York: Springer-Verlag) (1974)
- [5] Douglas R G, Banach algebra techniques in operator theory (New York Inc.: Springer-Verlag) (1972)
- [6] Engel K J, Operator matrices and systems of evolution equations (preprint)
- [7] Engel K J and Nagel R, One-parameter semigroups for linear evolution equations, *Graduate Texts in Math.* (New York: Springer-Verlag) (2000)
- [8] Goldstein J A, Semigroups of linear operators and applications (New York: Oxford University Press) (1985)
- [9] Holderieth A, Matrix multiplication operators generating one parameter semigroups, *Semigroup Forum* **142** (1991) 155–166
- [10] Horn R A and Johnson C R, Matrix analysis, 3rd edition (Cambridge University Press) (1990)
- [11] Hudzik H, Kamińska A and Mastyló M, On the dual of Orlicz–Lorentz spaces, *Proc. Am. Math. Soc.* **130(6)** (2002) 1645–1654
- [12] Komal B S and Gupta S, Multiplication operators between Orlicz spaces, *Integral Equations and Operator Theory* **41** (2001) 324–330
- [13] Lindenstrauss J and Tzafriri L, Classical Banach spaces II, Function spaces (Berlin-New York: Springer-Verlag) (1979)
- [14] Maligranda L, Orlicz spaces and interpolation, *Seminars in Math.* (Brazil: Univ. Estadual de Campinas, Campinas, SP) (1989) vol. 5
- [15] Nagel R, One-parameter semigroups for positive operators, *Lecture Notes Math.* (Berlin: Springer-Verlag) (1986) vol. 1184
- [16] Nagel R, Semigroup methods for nonautonomous Cauchy problems in: Evolution equations (eds) G Ferreyra, G Goldstein and F Nuebrander, *Lect. Notes Pure Appl. Math.* (1995) vol. 168, pp. 301–316
- [17] Nuebrander F, Integrated semigroups and their applications to the abstract Cauchy problem, *Pacific J. Math.* **135** (1988) 111–155
- [18] Partington J R, Linear operators and linear systems (Cambridge University Press) (2004)
- [19] Pazy A, Semigroups of linear operators and applications to partial differential equations (Berlin-Heidelberg-New York-Tokyo: Springer-Verlag) (1986)
- [20] Rao M M and Ren Z D, Theory of Orlicz spaces (New York: Marcel Dekker) (1991)

- [21] Rao M M and Ren Z D, Applications of Orlicz spaces (New York: Marcel Dekker) (2002)
- [22] Singh R K and Manhas J S, Composition operators on function spaces, North-Holland Math. Studies (Amsterdam: North-Holland) (1993) vol. 179
- [23] Sirotkin G G, Compact-friendly multiplication operators on Banach function spaces, *J. Funct. Anal.* **192** (2002) 517–523
- [24] Takagi H and Yokouchi K, Fredholm composition operators, *Integral Equations and Operator Theory* **16** (1993) 267–276