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There are infinitely many limit points of the fractional parts of powers

ARTŪRAS DUBICKAS

Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania
E-mail: arturas.dubickas@maf.vu.lt

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Abstract. Suppose that $\alpha > 1$ is an algebraic number and $\xi > 0$ is a real number. We prove that the sequence of fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, has infinitely many limit points except when α is a PV-number and $\xi \in \mathbb{Q}(\alpha)$. For $\xi = 1$ and α being a rational non-integer number, this result was proved by Vijayaraghavan.

Keywords. Limit points; fractional parts; PV-numbers; Salem numbers.

1. Introduction

Let $\alpha > 1$ and $\xi > 0$ be real numbers. The problem of distribution of the fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, is a classical one. Some metrical results are well-known. Firstly, for fixed α, the fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, are uniformly distributed in [0, 1) for almost all ξ [17]. Secondly, for fixed ξ, the fractional parts $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, are uniformly distributed in [0, 1) for almost all α (see [11] and also [10] for a weaker result). However, for fixed pairs ξ, α, nearly nothing is known. Even the simple-looking Mahler's question [12] about the fractional parts $\{\xi(3/2)^n\}$, $n = 1, 2, 3, \ldots$, is far from being solved. (See, however, [9] and, for instance, see [1–3,7,8] for more recent work on this problem.)

One of the first results in this direction is due to Vijayaraghavan, who proved that the set of limit points of the sequence $\{(p/q)^n\}$, $n = 1, 2, 3, \ldots$, where $p > q > 1$ are integers satisfying $\gcd(p, q) = 1$, is infinite. In his note [15] (see also [16]) he gave two proofs of this fact: one due to himself and another due to A Weil. It was noticed later that the questions of distribution of $\{\xi\alpha^n\}$, $n = 1, 2, 3, \ldots$, for algebraic α are closely related to the size of conjugates of α. The algebraic integers $\alpha > 1$ whose conjugates other than α itself are all strictly inside the unit disc were named after Pisot and Vijayaraghavan and called *PV-numbers* (see [4] and [14]).

The aim of this paper is to prove the following generalization of the above mentioned result of Vijayaraghavan.

Theorem 1. *Let $\alpha > 1$ be an algebraic number and let $\xi > 0$ be a real number. Then the set $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, has only finitely many limit points if and only if α is a PV-number and $\xi \in \mathbb{Q}(\alpha)$.*

This theorem was already proved by Pisot in [13]. We give a different proof by developing the method of Vijayaraghavan [15]. In addition, we prove a stronger result for Salem numbers (see Lemma 3 below).

The 'if' part of the theorem is well-known. Indeed, let $\alpha = \alpha_1$ be a PV-number with conjugates, say, $\alpha_2, \dots, \alpha_d$. Assume that $\xi \in \mathbb{Q}(\alpha)$, that is, $\xi = (e_0 + e_1\alpha + \dots + e_{d-1}\alpha^{d-1})/L$ with $e_0, \dots, e_{d-1} \in \mathbb{Z}$ and $L \in \mathbb{N}$. By considering the trace of $L\xi\alpha^n$, namely, the sum over its conjugates, we have

$$\begin{aligned}\mathrm{Tr}(L\xi\alpha^n) &= e_0\mathrm{Tr}(\alpha^n) + \dots + e_{d-1}\mathrm{Tr}(\alpha^{n+d-1})\\ &= L[\xi\alpha^n] + L\{\xi\alpha^n\} + e_0\sum_{j=2}^{d}\alpha_j^n + \dots + e_{d-1}\sum_{j=2}^{d}\alpha_j^{n+d-1}.\end{aligned}$$

Since $\mathrm{Tr}(L\xi\alpha^n) - L[\xi\alpha^n]$ is an integer and, for each fixed k, the sum $\sum_{j=2}^{d}\alpha_j^{n+k}$ tends to zero as $n \to \infty$, we deduce that the set of limit points of $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, is a subset of $\{0, 1/L, \dots, (L-1)/L, 1\}$.

So in the proof below we only need to prove the 'only if' part, namely, that in all other cases the set of limit points of $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, is infinite.

We remark that the theorem does not apply to transcendental numbers $\alpha > 1$. It is not known, for instance, whether the sets $\{e^n\}$, $n \in \mathbb{N}$, and $\{\pi^n\}$, $n \in \mathbb{N}$, have one or more than one limit point.

In some cases the theorem cannot be strengthened. Suppose, for instance, that α is a rational integer $\alpha = b \geq 2$ (which is a PV-number) and $\xi = \sum_{k=0}^{\infty} b^{-k!}$ (which is a transcendental Liouville number, so $\xi \notin \mathbb{Q}(b) = \mathbb{Q}$). Then the set of limit points of the sequence $\{\xi b^n\}$, $n = 1, 2, 3, \dots$, is $\{0, b^{-1}, b^{-2}, b^{-3}, \dots\}$. Evidently, this set is countable.

2. Sketch of the proof and auxiliary results

From now on, let us assume that $\alpha = \alpha_1 > 1$ is a fixed algebraic number with conjugates $\alpha_2, \dots, \alpha_d$ and with minimal polynomial $a_d z^d + a_{d-1}z^{d-1} + \dots + a_0 \in \mathbb{Z}[z]$. Set $L(\alpha) = |a_0| + |a_1| + \dots + |a_d|$. Suppose that $\xi > 0$ is a real number satisfying $\xi \notin \mathbb{Q}(\alpha)$ in case α is a PV-number.

Recall that an algebraic integer $\alpha > 1$ is called a *Salem number* if its conjugates are all in the unit disc $|z| \leq 1$ with at least one conjugate lying on $|z| = 1$. The next lemma is part of Theorem 1 in [8]. (Here and below, $\|x\| := \min(\{x\}, 1 - \{x\})$.)

Lemma 2. *Let $\alpha > 1$ be a real algebraic number and let $\xi > 0$ be a real number. If $\|\xi\alpha^n\| < 1/L(\alpha)$ for every $n \in \mathbb{N}$ then α is a PV-number or a Salem number and $\xi \in \mathbb{Q}(\alpha)$.*

Suppose that the set S of limit points of $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, is finite, say, $S = \{\mu_1, \mu_2, \dots, \mu_q\}$. With this assumption, we will show in §3 that, for any $\varepsilon > 0$, there exist three positive integers m, r, L, where $m > r$, such that

$$\|L\xi(\alpha^m - \alpha^r)\alpha^n\| < 2\varepsilon$$

for every $n \in \mathbb{N}$. Taking $\varepsilon < 1/2L(\alpha)$, by Lemma 2, we conclude that α is a PV-number or a Salem number and $L\xi(\alpha^m - \alpha^r) \in \mathbb{Q}(\alpha)$, that is, $\xi \in \mathbb{Q}(\alpha)$.

However, the case when α is a PV-number and $\xi \in \mathbb{Q}(\alpha)$ is already treated in the 'if' part of the theorem. So the only case that remains to be settled is when α is a Salem number and $\xi \in \mathbb{Q}(\alpha)$. We will then prove even more than required.

Lemma 3. *Suppose that α is a Salem number and $\xi > 0$ belongs to $\mathbb{Q}(\alpha)$. Then there is an interval $I = I(\xi, \alpha) \subset [0, 1]$ of positive length such that each point $\zeta \in I$ is a limit point of the set $\{\xi\alpha^n\}$, $n \in \mathbb{N}$.*

We will prove Lemma 3 in §4. Finally, recall that the sequence $b_1, b_2, b_3, \dots$ is called *ultimately periodic* if there is a $t \in \mathbb{N}$ such that $b_{n+t} = b_n$ for every $n \geq n_0$. If $n_0 = 1$, then the sequence $b_1, b_2, b_3, \dots$ is called *purely periodic*. The next lemma was proved in [6]. It will be used in the proof of Lemma 3.

Lemma 4. *Let $d, L \in \mathbb{N}$ and $A_{d-1}, \dots, A_0 \in \mathbb{Z}$, where $A_0 \neq 0$ and $\gcd(A_0, L) = 1$. Then the sequence of integers $b_1, b_2, b_3, \dots$ satisfying the linear recurrence sequence*

$$b_{k+d} + A_{d-1}b_{k+d-1} + \cdots + A_1 b_{k+1} + A_0 b_k = 0,$$

where $k = 1, 2, 3, \dots$, is purely periodic modulo L.

3. Differences of fractional parts are close to an integer

Suppose that the set S of limit points of $\{\xi\alpha^n\}$, $n \in \mathbb{N}$, is finite, say, $S = \{\mu_1, \dots, \mu_q\}$. Let $\mathcal{D}(S)$ be the set of all differences $\mu_i - \mu_j$, where $\mu_i \geq \mu_j$ belong to S. It is possible that the set $S \cup \mathcal{D}(S)$ (which is a subset of $[0, 1]$) contains some rational numbers. For instance, $\mathcal{D}(S)$ always contains 0. Let L be the least common multiple of the denominators of all rational numbers that belong to $S \cup \mathcal{D}(S)$. (Of course, $L := 1$ if $(S \cup \mathcal{D}(S)) \cap \mathbb{Q} = \{0\}$ or $\{0, 1\}$.)

Consider the set S_L of limit points of $\{L\xi\alpha^n\}$, $n \in \mathbb{N}$.

Lemma 5. *S_L is a subset of* $\{0, \{L\mu_1\}, \dots, \{L\mu_q\}, 1\}$.

Proof. Note that

$$L\{\xi\alpha^n\} - \{L\xi\alpha^n\} = [L\xi\alpha^n] - L[\xi\alpha^n]$$

is a non-negative integer. Therefore each element of S_L is of the form $L\mu_i - n_i$ with integer $n_i \geq 0$. Evidently, $S_L - \{0, 1\}$ is a subset of the interval $(0, 1)$. Consequently, $n_i = [L\mu_i]$ for each μ_i satisfying $L\mu_i \notin \mathbb{Z}$. This proves the lemma. □

Lemma 6. *The set* $(S_L \cup \mathcal{D}(S_L)) \cap \mathbb{Q}$ *is either* $\{0, 1\}$ *or* $\{0\}$.

Proof. Of course, for any rational μ_i, by the definition of L, we have $\{L\mu_i\} = 0$. By Lemma 5, we deduce that $S_L \subset \{0, \dots, \{L\mu\}, \dots, 1\}$, where μ runs over every irrational element of S, so that $S_L \cap \mathbb{Q} \subset \{0, 1\}$. The difference $\{L\mu_i\} - \{L\mu_j\} = L(\mu_i - \mu_j) - [L\mu_i] + [L\mu_j]$, where $\mu_i, \mu_j \in S$, is either irrational or, by the definition of L, an integer. Hence, $\mathcal{D}(S_L)$ contains at most two rational elements, namely, 0 and 1. This proves the lemma. □

Write

$$x_n = [L\xi\alpha^n] \quad \text{and} \quad y_n = \{L\xi\alpha^n\}.$$

Then, as $a_0\alpha^n + a_1\alpha^{n+1} + \cdots + a_d\alpha^{n+d} = 0$, we set

$$s_n := a_0 y_n + a_1 y_{n+1} + \cdots + a_d y_{n+d} = -a_0 x_n - a_1 x_{n+1} - \cdots - a_d x_{n+d}.$$

So s_n belongs to a finite set of integers for each $n \in \mathbb{N}$. (We remark that a key result which was proved in [7] is that the sequence $s_1, s_2, s_3, \dots$ is not ultimately periodic, unless α is a PV-number or a Salem number and $\xi \in \mathbb{Q}(\alpha)$. Lemma 2 given in §2 is an easy consequence of this result.)

Suppose that S_L contains g irrational elements. We denote $S_L^* = S_L - \{0, 1\}$. By Lemma 6, the set S_L contains at most two rational elements 0 and 1. Hence S_L contains at most $g + 2$ elements. By Lemma 6 again, the numbers $\eta - \eta'$, where $\eta, \eta' \in S_L$, $\eta > \eta'$, are all irrational except (possibly) when $(\eta, \eta') = (1, 0)$.

Set

$$\tau = \min \|a_d(\eta - \eta')\|,$$

where the minimum is taken over every pair $\eta, \eta' \in S_L^* \cup \{0, 1\}$, where $\eta > \eta'$, except for the pair $(\eta, \eta') = (1, 0)$. Since all these differences are irrational, we have $0 < \tau < 1/2$.

Recall that $s_n = a_0 y_n + \cdots + a_d y_{n+d}$ is an integer, where $y_n = \{L\xi\alpha^n\}$. Fix ε in the interval $0 < \varepsilon < \tau/2L(\alpha) < 1/4L(\alpha)$. Then the intervals $(\eta - \varepsilon, \eta + \varepsilon)$, where $\eta \in S_L^* \cup \{0, 1\}$, are disjoint. Furthermore, there is an integer N so large that y_n lies in an ε-neighbourhood of $\eta = \eta_n \in S_L$ for each $n \geq N$. We will write η_n for the element of S_L closest to y_n.

Consider the vectors $Z_h := (\eta_h, \eta_{h+1}, \dots, \eta_{h+d})$ for $h = N, N + 1, \dots$. There are at most $(g+2)^{d+1}$ different vectors in S_L^{d+1}. So there are two integers, say, m and r satisfying $m > r \geq N$, such that $Z_m = Z_r$. Subtracting s_{r+n} from s_{m+n} yields

$$\begin{aligned} s_{m+n} - s_{r+n} = a_0(y_{m+n} - y_{r+n}) + \cdots + a_{d-1}(y_{m+d-1+n} - y_{r+d-1+n}) \\ + a_d(y_{m+d+n} - y_{r+d+n}) \end{aligned}$$

for $n = 0, 1, \dots$. Writing $y_h = \eta_h + (y_h - \eta_h)$ and using $|y_h - \eta_h| < \varepsilon$, we deduce that

$$\begin{aligned} &\|a_0(\eta_{m+n} - \eta_{r+n}) + \cdots + a_{d-1}(\eta_{m+d-1+n} - \eta_{r+d-1+n}) \\ &\quad + a_d(\eta_{m+d+n} - \eta_{r+d+n})\| < 2\varepsilon L(\alpha) < \tau. \end{aligned}$$

We next claim that the difference $\eta_{m+n} - \eta_{r+n}$ belongs to the set $\{0, 1, -1\}$ for each $n \geq 0$. Since $Z_m = Z_r$, we have $\eta_{m+n} = \eta_{r+n}$ for every $n = 0, 1, \dots, d$. For the contradiction, assume that l is the smallest positive integer for which $\eta_{m+d+l} - \eta_{r+d+l} \notin \{0, 1, -1\}$. In particular, this implies that $\eta_{m+j+l} - \eta_{r+j+l} \in \mathbb{Z}$ for $j = 0, 1, \dots, d - 1$. Hence

$$\begin{aligned} &\|a_d(\eta_{m+d+l} - \eta_{r+d+l})\| \\ &\quad = \|a_0(\eta_{m+l} - \eta_{r+l}) + \cdots + a_d(\eta_{m+d+l} - \eta_{r+d+l})\| < \tau. \end{aligned}$$

By the choice of τ, this is impossible, unless $\eta_{m+d+l} = \eta_{r+d+l}$ or $\{\eta_{m+d+l}, \eta_{r+d+l}\} = \{0, 1\}$. However, in both cases, we have $\eta_{m+d+l} - \eta_{r+d+l} \in \{0, 1, -1\}$, a contradiction.

Note that, since $\eta_h \in [0, 1]$, we have $\eta_{m+n} - \eta_{r+n} \in \{0, 1, -1\}$ if and only if either $\eta_{m+n} = \eta_{r+n}$ or $\{\eta_{m+n}, \eta_{r+n}\} = \{0, 1\}$. Obviously, $\eta_{m+n} = \eta_{r+n}$ implies that the difference between the fractional parts $y_{m+n} = \{L\xi\alpha^{m+n}\}$ and $y_{r+n} = \{L\xi\alpha^{r+n}\}$ is smaller than 2ε. The alternative case, namely, $\{\eta_{m+n}, \eta_{r+n}\} = \{0, 1\}$ occurs when one of the numbers $\{L\xi\alpha^{m+n}\}$, $\{L\xi\alpha^{r+n}\}$ is smaller than ε and another is greater than $1 - \varepsilon$. So, in both cases, we have

$$\|L\xi\alpha^{m+n} - L\xi\alpha^{r+n}\| < 2\varepsilon$$

for every $n \geq 0$. Thus, we established the existence of three positive integers m, r, L, where $m > r$, such that $\|L\xi(\alpha^m - \alpha^r)\alpha^n\| < 2\varepsilon$ for every $n \in \mathbb{N}$ (as required in §2).

4. Salem numbers

In this section, we will prove Lemma 3 and thus complete the proof of the theorem. Suppose that α is a Salem number. Let us write the conjugates of α in the form α^{-1}, $\mathrm{e}^{\phi_1\sqrt{-1}}, \mathrm{e}^{-\phi_1\sqrt{-1}}, \ldots, \mathrm{e}^{\phi_m\sqrt{-1}}, \mathrm{e}^{-\phi_m\sqrt{-1}}$, where $d = 2m + 2$ and where the arguments $\phi_1, \ldots, \phi_m$ belong to the interval $(0, \pi)$. As above, we set $\xi = (e_0 + e_1\alpha + \cdots + e_{d-1}\alpha^{d-1})/L$ with $e_0, \ldots, e_{d-1} \in \mathbb{Z}$ and $L \in \mathbb{N}$. Now, by considering the trace of $L\xi\alpha^n$, we have

$$\begin{aligned}\mathrm{Tr}(L\xi\alpha^n) &= e_0\mathrm{Tr}(\alpha^n) + \cdots + e_{d-1}\mathrm{Tr}(\alpha^{n+d-1})\\ &= L[\xi\alpha^n] + L\{\xi\alpha^n\} + (e_0 + \cdots + e_{d-1}\alpha^{-d+1})\alpha^{-n}\\ &\quad + 2\sum_{j=1}^{m}(e_0\cos(n\phi_j) + e_1\cos((n+1)\phi_j)\\ &\quad + \cdots + e_{d-1}\cos((n+d-1)\phi_j)).\end{aligned}$$

Setting

$$U(z) := e_0 + e_1\cos z + \cdots + e_{d-1}\cos((d-1)z)$$

and

$$V(z) := e_1\sin z + \cdots + e_{d-1}\sin((d-1)z),$$

we can write

$$\begin{aligned}&\sum_{j=1}^{m}(e_0\cos(n\phi_j) + e_1\cos((n+1)\phi_j) + \cdots + e_{d-1}\cos((n+d-1)\phi_j))\\ &\quad = \sum_{j=1}^{m}(U(\phi_j)\cos(n\phi_j) - V(\phi_j)\sin(n\phi_j)).\end{aligned}$$

It follows that the sum

$$\begin{aligned}&L\{\xi\alpha^n\} + 2\sum_{j=1}^{m}(U(\phi_j)\cos(n\phi_j) - V(\phi_j)\sin(n\phi_j))\\ &\quad = \mathrm{Tr}(L\xi\alpha^n) - L[\xi\alpha^n] - (e_0 + \cdots + e_{d-1}\alpha^{-d+1})\alpha^{-n}\end{aligned}$$

is close to an integer for each sufficiently large n. Moreover, the sequence of integers $b_n := \mathrm{Tr}(L\xi\alpha^n)$, $n = 1, 2, 3, \ldots$, satisfies the linear recurrence sequence

$$a_d b_{n+d} + a_{d-1}b_{n+d-1} + \cdots + a_0 b_n = 0$$

for $n = 1, 2, 3, \ldots$. The fact that α is a Salem number implies that $a_d = a_0 = 1$, so we can apply Lemma 4. It follows that the sequence $b_n = \mathrm{Tr}(L\xi\alpha^n)$, $n = 1, 2, \ldots$, is purely periodic modulo L. Let q be the length of its period, so that the numbers $b_q, b_{2q}, b_{3q}, \ldots$ are all equal modulo L. Then, there is an integer ℓ in the range $0 \le \ell \le L - 1$ such that

$$L\{\xi\alpha^{qn}\} + 2R_n \to \ell$$

as $n \to \infty$. Here, $R_n := \sum_{j=1}^{m}(U(\phi_j)\cos(qn\phi_j) - V(\phi_j)\sin(qn\phi_j))$.

Note that

$$U(\phi)\cos(qn\phi) - V(\phi)\sin(qn\phi)$$
$$= \sqrt{U(\phi)^2 + V(\phi)^2}\cos(qn\phi - \arctan(V(\phi)/U(\phi)))$$

for each real number ϕ. The numbers $\phi_1, \dots, \phi_m$ and π are linearly independent over $\mathbb{Q}$ (see, for example, p. 32 of [14]). Hence, by Kronecker's theorem, for arbitrary m numbers $\theta_1, \dots, \theta_m \in [-1, 1]$ there is an $n \in \mathbb{N}$ such that the value of $U(\phi_j)\cos(qn\phi_j) - V(\phi_j)\sin(qn\phi_j)$ lies close to $\theta_j\sqrt{U(\phi_j)^2 + V(\phi_j)^2}$ for every $j = 1, \dots, m$.

It follows that the sequence R_n, $n = 1, 2, \dots$, is dense in the interval $[-H, H]$, where $H = \sum_{j=1}^{m}\sqrt{U(\phi_j)^2 + V(\phi_j)^2}$. Clearly, $H > 0$, because $H = 0$ yields $e_0 + e_1\alpha' + \dots + e_{d-1}\alpha'^{d-1} = U(\phi_1) + \sqrt{-1}V(\phi_1) = 0$, where $\alpha' = \mathrm{e}^{\phi_1\sqrt{-1}}$ is conjugate to α. This is impossible, because the degree of α' over $\mathbb{Q}$ equals d.

Now, since $\{\xi\alpha^{qn}\} + 2R_n/L \to \ell/L$ as $n \to \infty$ and since the values of $2R_n$, $n \in \mathbb{N}$, are dense in $[-2H, 0]$, we see that there exists an interval $[\ell/L, \ell/L + \delta]$, where δ is a positive number, such that every $\zeta \in [\ell/L, \ell/L + \delta]$ is a limit point of the set $\{\xi\alpha^{qn}\}$, $n \in \mathbb{N}$. This completes the proof of Lemma 3. □

See also [5,18,19] for other recent results concerning integer and fractional parts of Salem numbers.

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