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On the series $\sum_{k=1}^{\infty} \binom{3k}{k}^{-1} k^{-n} x^k$

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Abstract. In this paper we investigate the series $\sum_{k=1}^{\infty} \binom{3k}{k}^{-1} k^{-n} x^k$. Obtaining some integral representations of them, we evaluated the sum of them explicitly for $n = 0, 1, 2$.

Keywords. Inverse binomial series; hypergeometric series; polylogarithms; integral representations.

1. Introduction

After Apéry [2] proved the irrationality of $\zeta(2)$ and $\zeta(3)$, where ζ is the Riemann-zeta function defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \operatorname{Re} s > 1,$$

by employing the series representations

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}},$$

many authors have considered the series involving inverse binomial coefficients and they obtained many interesting results. We have a similar series representation for $\zeta(4)$:

$$\zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}},$$

see [8]. Some other related interesting results involving binomial coefficients can be found in Chapter 9 of [4], [3], [5–9] and [11, 12].

Motivated by such results we shall consider here the following family of sums:

$$S(n, m; x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n \binom{3mk}{mk}}.$$

A good way to approach these series is to try and find their integral representations. In this way we can evaluate many of them explicitly.

In this paper, we will use, as usual, the following definitions and identities for the Euler's gamma function Γ, beta function β, polylogarithms $Li_n(z)$ and generalized hypergeometric series ${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q\colon x)$,

$$\beta(s,t) = \int_0^1 u^{s-1}(1-u)^{t-1}\mathrm{d}u = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad \text{for } s>0, t>0, \tag{1.1}$$

(see Theorem 7.69 of [4]),

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} = \frac{(-1)^{n-1}}{(n-1)!}\int_0^1 \frac{z\log^{n-1}\phi \mathrm{d}\phi}{1-z\phi} \quad \text{for } |z|\le 1, \tag{1.2}$$

$$Li_n(z^m) = m^{n-1}\sum_{k=1}^{m} Li_n(\omega^k z), \tag{1.3}$$

where m is a positive integer and $\omega = \mathrm{e}^{2\pi i/m}$. The mth primitive root of unity is called factorization formula for polylogarithm series and

$${}_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q\colon x) = \sum_{k=1}^{\infty} \frac{(a_1)_k(a_2)_k\dots(a_p)_k x^k}{(b_1)_k(b_2)_k\dots(b_q)_k},$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

For further properties of polylogarithms and hypergeometric series and related functions, see [10] and Chapter 2 of [1], respectively. Almost all results given here were obtained using identities (1.1) and (1.2) extensively.

2. Main results

The main results of this paper are the following two theorems.

Theorem 2.1. *For $|x|\le 27/4$ and $n=2,3,\dots$ we have*

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} = \frac{(-1)^{n-1}}{(n-2)!}\int_0^{\alpha(x)} u\log^{n-2}\left[\frac{1}{x}\frac{(1-\mathrm{e}^u)^3}{\mathrm{e}^{2u}}\right]\mathrm{d}u + \frac{4(-1)^{n-2}}{3(n-2)!}\int_0^{\beta(x)} v\log^{n-2}\left[\frac{[1+2\cos[(2v+2\pi)/3]]^3}{2x[1+\cos[(2v+2\pi)/3]]}\right]\mathrm{d}v, \tag{2.1}$$

where

$$\alpha(x) = \log\left[\frac{\phi^3+1}{[\phi+1]^3}\right], \quad \beta(x) = 3\arctan\left[\frac{\sqrt{3}}{2\phi-1}\right]$$

and

$$\phi(x) := \left[\frac{27-2x+3[81-12x]^{1/2}}{2x}\right]^{1/3}.$$

Proof. We start with identity (1.1).

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} &= \sum_{k=1}^{\infty}\frac{x^k(k!)(2k)!}{k^n(3k)!}\\ &= \sum_{k=1}^{\infty}\frac{x^k\Gamma(k+1)\Gamma(2k+1)}{k^n\Gamma(3k+1)}\\ &= \sum_{k=1}^{\infty}\frac{x^k\Gamma(k)\Gamma(2k+1)}{k^{n-1}\Gamma(3k+1)}\\ &= \sum_{k=1}^{\infty}\frac{x^k}{k^{n-1}}\beta(k,2k+1)\\ &= \sum_{k=1}^{\infty}\frac{x^k}{k^{n-1}}\int_0^1 t^{k-1}(1-t)^{2k}\mathrm{d}t.\end{aligned}$$

Inverting the order of summation and integration, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} &= \int_0^1\sum_{k=1}^{\infty}\frac{[xt(1-t)^2]^k}{k^{n-1}}\frac{\mathrm{d}t}{t}\\ &= \int_0^1\frac{Li_{n-1}[xt(1-t)^2]}{t}\mathrm{d}t \qquad (2.2)\\ &= \frac{(-1)^{n-2}}{(n-2)!}\int_0^1\left[\int_0^1\frac{xt(1-t)^2\log^{n-2}z}{1-xt(1-t)^2z}\mathrm{d}z\right]\frac{\mathrm{d}t}{t}, \qquad (2.3)\end{aligned}$$

where in the last step we employ identity (1.2). Inverting the order of integration here and leaving the justification of it at the end of the proof, we obtain

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} = \frac{(-1)^{n-1}}{(n-2)!}\int_0^1\frac{\log^{n-1}z}{z}\left[\int_0^1\frac{t^2-2t+1}{t^3-2t^2+t-[xz]^{-1}}\mathrm{d}t\right]\mathrm{d}z.$$

Making the change of variable $t = u + 2/3$ here, we find after some manipulations that

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} = \frac{(-1)^{n-2}}{3(n-2)!}\int_0^1\frac{\log^{n-2}z}{z}\left[\int_{-2/3}^{1/3}\frac{2u-2/3}{u^3-u/3+2/27-[xz]^{-1}}\mathrm{d}u\right]\mathrm{d}z.$$

Now making the change of variable

$$z = \frac{27}{x}\frac{y^3}{(y^3+1)^2}$$

in the first integral, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} &= \frac{(-1)^{n-1}}{(n-2)!}\int_0^{\phi(x)}\log^{n-2}\left[\frac{27}{x}\frac{y^3}{(y^3+1)^2}\right]\\ &\quad\times\left[\int_{-2/3}^{1/3}\frac{2u-2/3}{u^3-u/3-(1+y^6)/(27y^3)}\mathrm{d}u\right]\frac{1}{y}\frac{y^3-1}{y^3+1}\mathrm{d}y.\end{aligned}$$

Here

$$\phi(x)=\left[\frac{27-2x+3[81-12x]^{1/2}}{2x}\right]^{1/3}. \tag{2.4}$$

If we make the change of variable $t=3y/(y^2+1)$ in this integral, we find that

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}}=\frac{3(-1)^{n-1}}{(n-2)!}\int_0^{\lambda(x)}\log^{n-2}\left[\frac{27}{x}\frac{t^3}{(2t+3)(t-3)^2}\right]$$
$$\times\left[\int_{-2/3}^{1/3}\frac{2u-2/3}{u^3-u/3-(3-t^2)/3t^3}\,\mathrm{d}u\right]\frac{t+3}{t(t-3)(2t+3)}\mathrm{d}t, \tag{2.5}$$

where

$$\lambda(x)=\frac{3\phi(x)}{\phi(x)^2+1}$$

with $\phi(x)$ defined by (2.4). First, we compute the inner integral. By Cardano's method, the roots of the cubic equation $u^3-u/3-(3-t^2)/3t^3=0$ are

$$\alpha=1/t,\quad \beta=[-3-i\sqrt{27-12t^2}]/6t\quad\text{and}\quad\gamma=-3+i\sqrt{27-12t^2}]/6t.$$

Thus, we can factorize the integrand in the inner integral as

$$\frac{2u-2/3}{u^3-u/3-(3-t^2)/3t^3}=\frac{2t}{t+3}\frac{1}{u-1/t}-\frac{t}{t+3}\frac{2u+1/t}{u^2+u/t+1/t^2-1/3}$$
$$-\frac{2t+3}{t+3}\frac{1}{u^2+u/t+1/t^2-1/3}.$$

Integrating both sides of this equation from $-2/3$ to 1/3 and then simplifying it, we find that

$$\int_{-2/3}^{1/3}\frac{2u-2/3}{u^3-u/3-(3-t^2)/3t^3}\,\mathrm{d}u$$
$$=\frac{3t}{t+3}\log\left[\frac{3-t}{3+2t}\right]-\frac{2t+3}{t+3}\frac{6t}{[27-12t^2]^{1/2}}\arctan\left[\frac{3t}{5t-6}\sqrt{\frac{9-6t}{3+2t}}\right].$$

Replacing this in (2.5), we obtain after some simplification

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}}=S_1+S_2, \tag{2.6}$$

where

$$S_1=\frac{9(-1)^{n-1}}{(n-2)!}\int_0^{\lambda(x)}\log^{n-2}\left[\frac{27}{x}\frac{t^3}{(2t+3)(t-3)^2}\right]\log\left[\frac{3-t}{2t+3}\right]\frac{\mathrm{d}t}{(t-3)(2t+3)} \tag{2.7}$$

and

$$S_2 = \frac{18(-1)^{n-2}}{(n-2)!} \int_0^{\lambda(x)} \log^{n-2}\left[\frac{27}{x}\frac{t^3}{(2t+3)(t-3)^2}\right] \times \arctan\left[\frac{3t}{5t-6}\sqrt{\frac{9-6t}{3+2t}}\right] \frac{\mathrm{d}t}{(t-3)[27-12t^2]^{1/2}}. \tag{2.8}$$

Now we simplify these two integrals. If we make in (2.7) the change of variable

$$u = \log\left[\frac{3-t}{2t+3}\right],$$

we find that

$$S_1 = \frac{(-1)^{n-1}}{(n-2)!} \int_0^{\alpha(x)} u \log^{n-2}\left[\frac{1}{x}\frac{(1-\mathrm{e}^u)^3}{\mathrm{e}^{2u}}\right] \mathrm{d}u, \tag{2.9}$$

where

$$\alpha(x) = \log\left[\frac{3-\lambda(x)}{3+2\lambda(x)}\right].$$

In (2.8), making the change of variable

$$y = \sqrt{\frac{9-6t}{3+2t}},$$

we arrive at the following:

$$S_2 = \frac{4(-1)^{n-2}}{(n-2)!} \int_{\sqrt{3}}^{\lambda_1(x)} \log^{n-2}\left[\frac{(3-y^2)^3}{4x(y^2+1)^2}\right] \frac{(3\arctan y - \pi)}{y^2+1}\, \mathrm{d}y \tag{2.10}$$

since

$$\arctan\left[\frac{y^3-3y}{3y^2-1}\right] = 3\arctan y - \pi, \quad \text{for } y > 0$$

where

$$\lambda_1(x) = \sqrt{\frac{9-6\lambda(x)}{3+2\lambda(x)}}.$$

We need to induce one more change of variable to bring (2.10) in a simple form. Setting $v = 3\arctan y - \pi$ here, we get

$$S_2 = \frac{4(-1)^{n-2}}{3(n-2)!} \int_0^{\beta(x)} v \log^{n-2}\left[\frac{[1+2\cos[(2v+2\pi)/3]]^3}{2x[1+\cos[(2v+2\pi)/3]]}\right] \mathrm{d}v, \tag{2.11}$$

where

$$\beta(x) = 3\arctan\sqrt{\frac{9-6\lambda(x)}{3+2\lambda(x)}} - \pi = 3\arctan\frac{\sqrt{3}|\phi(x)-1|}{\phi(x)+1} - \pi$$
$$= 3\arctan\left[\frac{\sqrt{3}}{1-2\phi(x)}\right].$$

Substituting the values of S_1 and S_2 from (2.9) and (2.11) in (2.6), we get

$$\sum_{k=1}^{\infty}\frac{x^k}{k^n\binom{3k}{k}} = \frac{(-1)^{n-1}}{(n-2)!}\int_0^{\alpha(x)} u\log^{n-2}\left[\frac{1}{x}\frac{(1-\mathrm{e}^u)^3}{\mathrm{e}^{2u}}\right]\mathrm{d}u$$
$$+\frac{4(-1)^{n-2}}{3(n-2)!}\int_0^{\beta(x)} v\log^{n-2}\left[\frac{[1+2\cos[(2v+2\pi)/3]]^3}{2x[1+\cos[(2v+2\pi)/3]]}\right]\mathrm{d}v,$$

where

$$\alpha(x) = \log\left[\frac{\phi(x)^3+1}{(\phi(x)+1)^3}\right] \quad \text{and} \quad \beta(x) = 3\arctan\left[\frac{\sqrt{3}}{1-2\phi(x)}\right].$$

To complete the proof of Theorem 2.1 we need to justify the inversion made in (2.3). In the inner integral in (2.3), we induce the change of variable $z = 1/u$ to get

$$\int_0^1\left[\int_0^1\frac{xt(1-t)^2\log^{n-2}z}{1-xt(1-t)^2z}\,\mathrm{d}z\right]\mathrm{d}t = \int_0^1\left[\int_1^{\infty}\frac{x(1-t)^2\log^{n-2}u}{u^2-xt(1-t)^2u}\,\mathrm{d}u\right]\mathrm{d}t.$$

Since for every $0 \le t \le 1$, $-27/4 \le x \le 27/4$ and $u \ge 1$,

$$\frac{(1-t)^2\log^{n-2}u}{u^2-xt(1-t)^2u} \le \frac{\log^{n-2}u}{u^2-u}$$

and the improper integral

$$\int_1^{\infty}\frac{\log^{n-2}u}{u^2-u}\,\mathrm{d}u$$

is convergent, and

$$\int_1^{\infty}\frac{x(1-t)^2\log^{n-2}u}{u^2-xt(1-t)^2u}\,\mathrm{d}u$$

is uniformly convergent. This justifies the inversion of the order of the integrals in (2.3) and hence the proof of Theorem 2.1 is complete. □

The next theorem gives a generalization of Theorem 2.1.

Theorem 2.2. *For $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$ and $|x| \le (27/4)^m$, we have*

$$S(n, m\colon x) = m^{n-1}\sum_{j=1}^{m} S(n, 1\colon \omega^j x^{1/m}), \tag{2.12}$$

where $\omega = \mathrm{e}^{2\pi i/m}$ is a primitive root of unity.

Proof.

$$S(n, m\colon x) = m \sum_{k=1}^{\infty} \frac{x^k \Gamma(mk)\Gamma(2mk+1)}{k^{n-1}\Gamma(3mk+1)}$$

$$= m \sum_{k=1}^{\infty} \frac{x^k}{k^{n-1}} \beta(mk, 2mk+1)$$

$$= m \sum_{k=1}^{\infty} \frac{x^k}{k^{n-1}} \int_0^1 t^{mk-1}(1-t)^{2mk}\mathrm{d}t.$$

Inverting the order of summation and integration, we find that

$$S(n, m\colon x) = m \int_0^1 \sum_{k=1}^{\infty} \frac{[(t(1-t)^2 x^{1/m})^m]^k}{k^{n-1}} \frac{\mathrm{d}t}{t}$$

$$= m \int_0^1 \frac{Li_{n-1}[(t(1-t)^2 x^{1/m})^m]}{t}\mathrm{d}t$$

$$= m^{n-1} \sum_{j=1}^{m} \int_0^1 \frac{Li_{n-1}(\omega^j t(1-t)^2 x^{1/m})}{t}\mathrm{d}t$$

$$= m^{n-1} \sum_{j=1}^{m} 1 \int_0^1 \sum_{k=1}^{\infty} \frac{[\omega^j t(1-t)^2 x^{1/m}]^k}{k^{n-1}} \frac{\mathrm{d}t}{t}.$$

Inverting the order of summation and integration, we get

$$S(n, m\colon x) = m^{n-1} \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{[\omega^{jk} x^{1/m}]^k}{k^{n-1}} \int_0^1 t^{k-1}(1-t)^{2k}\mathrm{d}t$$

$$= m^{n-1} \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{[\omega^j x^{1/m}]^k}{k^n \binom{3k}{k}}$$

$$= m^{n-1} \sum_{j=1}^{m} S(n, 1\colon \omega^j x^{1/m}),$$

completing the proof of Theorem 2.2. □

COROLLARY 2.3

For $m = 1, 2, 3, \ldots$ *and* $|x| \le (27/4)^m$ *we have*

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2 \binom{3mk}{mk}} = m \sum_{k=1}^{m} \left\{ 6 \arctan^2 \left[\frac{\sqrt{3}}{2\phi(\omega^k x^{1/m}) - 1} \right] - \frac{1}{2} \log^2 \left[\frac{1 + [\phi(\omega^k x^{1/m})]^3}{[1 + \phi(\omega^k x^{1/m})]^3} \right] \right\}, \tag{2.13}$$

where $\omega = \mathrm{e}^{2\pi i/m}$ *is a primitive root of unity.*

Proof. Setting $n = 2$ in (2.12) we get the desired result. □

3. Applications

Putting some particuler values for n and x in Theorems 2.1 and 2.2, we can make many explicit evaluations.

Let $|x| \leq 27/4$. If we set $n = 2$ in (2.1) we find by the help of Gauss multiplication formula for Euler's gamma function:

$$\begin{aligned} S(2,1\colon x) &= \sum_{k=1}^{\infty} \frac{x^k}{k^2\binom{3k}{k}} = \frac{x}{3}\,{}_4F_3\left(1,1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3},2;\frac{4x}{27}\right) \\ &= 6\arctan^2\left[\frac{\sqrt{3}}{2\phi-1}\right] - \frac{1}{2}\log^2\left[\frac{\phi^3+1}{(\phi+1)^3}\right]. \end{aligned} \tag{3.1}$$

Differentiating (3.1) with respect to x and then multiplying by x we get for $|x| < 27/4$:

$$\begin{aligned} S(1,1\colon x) &= \sum_{k=1}^{\infty} \frac{x^k}{k\binom{3k}{k}} = \frac{x}{3}\,{}_3F_2\left(1,1,\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{4x}{27}\right) \\ &= \frac{1}{\sqrt{27-4x}}\left\{\arctan\left[\frac{\sqrt{3}}{2\phi-1}\right]\frac{18\phi}{1-\phi+\phi^2}\right. \\ &\qquad \left. -\log\left[\frac{\phi^3+1}{(\phi+1)^3}\right]\frac{3\sqrt{3}\phi(1-\phi)}{1+\phi^3}\right\}. \end{aligned} \tag{3.2}$$

Differentiating both sides of (3.2) with respect to x and then multiplying by x we get for $|x| < 27/4$:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{x^k}{\binom{3k}{k}} &= {}_3F_2\left(1,\frac{3}{2},2;\frac{4}{3},\frac{5}{3}:\frac{4x}{27}\right) \\ &= \left[\frac{36\phi x}{(27-4x)^{3/2}(1-\phi+\phi^2)} - \frac{18\sqrt{3}(1-\phi^2)\phi}{(1-\phi+\phi^2)^2(27-4x)}\right] \\ &\quad \times \arctan\left[\frac{\sqrt{3}}{2\phi-1}\right] \\ &\quad + \left[\frac{9\phi(1-2\phi-2\phi^3+\phi^4)}{(1+\phi^3)^2(27-4x)} - \frac{6\sqrt{3}(1-\phi)\phi x}{(27-4x)^{3/2}(1+\phi^3)}\right] \\ &\quad \times \log\left[\frac{1+\phi^3}{(1+\phi)^3}\right] + \frac{108\phi^3}{(27-4x)(1+\phi^3)^2}, \end{aligned} \tag{3.3}$$

where

$$\phi = \phi(x) = \left[\frac{27-2x+3[81-12x]^{1/2}}{2x}\right]^{1/3},$$

as defined by (2.4).

Putting $x = 27/4$ and $n = 2$ in (2.1) yields

$$\sum_{k=1}^{\infty}\frac{(27/4)^k}{k^2\binom{3k}{k}} = \frac{2\pi^2}{3} - 2\log^2 2. \tag{3.4}$$

Let $n = 2$ and $x = 6$ in (2.1). Then

$$\sum_{k=1}^{\infty}\frac{6^k}{k^2\binom{3k}{k}} = 6\arctan^2\left[\frac{\sqrt{3}}{2^{4/3}-1}\right] - \frac{1}{2}\log^2(2^{1/3}-1). \tag{3.5}$$

Set $x = 1/2$ in (3.1) to get

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{3k}{k}2^k} = \frac{1}{24}\pi^2 - \frac{1}{2}\log^2 2. \tag{3.6}$$

Set $x = 1$ in (3.1) to get

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{3k}{k}} = 6\arctan^2\left[\frac{\sqrt{3}}{1-[100+12\sqrt{69}]^{1/3}}\right] - \frac{1}{2}\log^2\left[\frac{12(9+\sqrt{69})}{[2+(100+12\sqrt{69})^{1/3}]^3}\right]. \tag{3.7}$$

Set $x = -1/4$ in (3.1) to get

$$\sum_{k=1}^{\infty}\frac{(-1/4)^k}{k^2\binom{3k}{k}} = 6\,\mathrm{arc\,cot}^2(2\sqrt{3}+\sqrt{7}) - \frac{1}{2}\log^2 2. \tag{3.8}$$

Put $x = 1/2$ in (3.2) to get

$$\sum_{k=1}^{\infty}\frac{1}{k\binom{3k}{k}2^k} = \frac{1}{10}\pi - \frac{1}{5}\log 2. \tag{3.9}$$

Set $x = 6$ in (3.2) to get

$$\sum_{k=1}^{\infty}\frac{6^k}{k\binom{3k}{k}} = \sqrt{3}2^{4/3}(1+2^{1/3})\arctan\left[\frac{\sqrt{3}}{2^{4/3}-1}\right] - 2^{1/3}(1-2^{1/3})\log(2^{1/3}-1). \tag{3.10}$$

Let $x = 1/2$ and $n = 0$ in (3.3) to get

$$\sum_{k=1}^{\infty}\frac{1}{\binom{3k}{k}2^k} = \frac{2}{25} - \frac{6}{125}\log 2 + \frac{11}{250}\pi. \tag{3.11}$$

Here, observe that $\phi(1/2) = 2+\sqrt{3}$. Let $x = -1/4$ in (3.3) to get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{\binom{3k}{k}4^k} = -\frac{1}{28} - \frac{3}{32}\log 2 + \frac{39}{112\sqrt{7}}\mathrm{arc\,cot}(2\sqrt{3}+\sqrt{7}). \tag{3.12}$$

Note that $\phi(-1/4) = -(5+\sqrt{21})/2$. Set $x = 6$ in (3.3) to get

$$\sum_{k=1}^{\infty} \frac{6^k}{\binom{3k}{k}} = 2\left(240 + 96.2^{1/3} + 75.2^{2/3}\right)^{1/2} \arctan\left(\frac{\sqrt{3}}{2^{4/3}-1}\right) + 2^{1/3}(4.2^{1/3}-5)\log(2^{1/3}-1) + 8. \tag{3.13}$$

Setting $x = 1$ in (3.3) we obtain

$$\sum_{k=1}^{\infty} \frac{1}{\binom{3k}{k}} = \left[\frac{36\sqrt{23}\tau}{529(1-\tau+\tau^2)} - \frac{18\sqrt{3}(1-\tau^2)\tau}{23(1-\tau+\tau^2)^2}\right] \arctan\left[\frac{\sqrt{3}}{2\tau-1}\right] + \left[\frac{9\tau(1-2\tau-2\tau^3+\tau^4)}{23(1+\tau^3)^2} - \frac{6\sqrt{69}(1-\tau)\tau}{529(1+\tau^3)}\right] \times \log\left[\frac{1+\tau^3}{(1+\tau)^3}\right] + \frac{108\tau^3}{23(1+\tau^3)^2}, \tag{3.14}$$

where

$$\tau = \left(\frac{25+3\sqrt{69}}{2}\right)^{1/3}.$$

Substituting $m = 2$ and $m = 3$ in (2.13) we get

$$\sum_{k=1}^{\infty} \frac{x^{3k}}{k^2\binom{6k}{2k}} = 12\arctan^2\left[\frac{\sqrt{3}}{2\phi(-x)-1}\right] + 12\arctan^2\left[\frac{\sqrt{3}}{2\phi(x)-1}\right] - \log^2\left[\frac{\phi(-x)^3+1}{[\phi(-x)+1]^3}\right] - \log^2\left[\frac{\phi(x)^3+1}{[\phi(x)+1]^3}\right]$$

and

$$\sum_{k=1}^{\infty} \frac{x^{3k}}{k^2\binom{9k}{3k}} = 18\arctan^2\left[\frac{\sqrt{3}}{2\phi(x)-1}\right] + 18\arctan^2\left[\frac{\sqrt{3}}{2\phi(ax)-1}\right] + 18\arctan^2\left[\frac{\sqrt{3}}{2\phi(a^2x)-1}\right] - \frac{3}{2}\log^2\left[\frac{\phi(x)^3+1}{[\phi(x)+1]^3}\right] - \frac{3}{2}\log^2\left[\frac{\phi(ax)^3+1}{[\phi(ax)+1]^3}\right] - \frac{3}{2}\log^2\left[\frac{\phi(a^2x)^3+1}{[\phi(a^2x)+1]^3}\right],$$

where $a = (i\sqrt{3}-1)/2$.

Of these results, eqs (3.6), (3.9), (3.11) and (3.12) have been evaluated by Borwein and Girgensohn [7] experimentally by the method called *integer relation algorithm* which does not constitute a mathematical proof. So their results are just conjectural. Our results verify Borwein and Girgensohn's experimental evaluations. All the results we obtained seem to be new.

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