Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 3, August 2005, pp. 347–368.

Wavelet characterization of Hörmander symbol class $S^m_{\rho,\delta}$ and applications

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MS received 12 March 2005

Abstract. In this paper, we characterize the symbol in Hörmander symbol class $S^m_{\rho,\delta}$ $(m \in R, \rho, \delta \geq 0)$ by its wavelet coefficients. Consequently, we analyse the kernel-distribution property for the symbol in the symbol class $S^m_{\rho,\delta}$ $(m \in R, \rho > 0, \delta \geq 0)$ which is more general than known results; for non-regular symbol operators, we establish sharp L^2-continuity which is better than Calderón and Vaillancourt's result, and establish L^p $(1 \leq p \leq \infty)$ continuity which is new and sharp. Our new idea is to analyse the symbol operators in phase space with relative wavelets, and to establish the kernel distribution property and the operator's continuity on the basis of the wavelets coefficients in phase space.

Keywords. Hörmander's symbol; wavelet; kernel distribution; operator's continuity.

1. Introduction

A symbol $\sigma(x, \xi) \in S'(R^n \times R^n)$ can define a symbol operator $\sigma(x, D)$: $S(R^n) \to S'(R^n)$ by the following formula:

$$\sigma(x, D) f(x) = \int e^{ix\xi} \sigma(x, \xi) \hat{f}(\xi)\, d\xi, \tag{1.1}$$

where $\hat{f}(\xi)$ is the Fourier transformation of function $f(x)$. When Hörmander studied pseudodifferential operators, he introduced Hörmander's symbol class $S^m_{\rho,\delta}$ $(m \in R, \rho, \delta \geq 0)$. One writes $\sigma(x, \xi) \in S^m_{\rho,\delta}$, if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}, \quad \forall \alpha, \beta \in N^n. \tag{1.2}$$

But we did not know what are the elements in Hörmander class $S^m_{\rho,\delta}$ $(m \in R, \rho, \delta \geq 0)$ before. Professor Meyer [12] proposed me to study such a kind of pseudodifferential operators with wavelets.

All of us know that wavelet theory has made a great success in the study of function spaces, and symbols were introduced as a representation of operators. In this sense, operators could be viewed as matrix under the usual wavelet bases for function spaces, and one hopes that the above class of operators could be characterized by the operators whose matrices under the respective wavelet basis are privileged on the diagonal. But this is not true except for the case where the operators themselves and their conjugate operator all belong to $\mathrm{Op}S^m_{1,1}$ (see [12]). In refs [6,17,18] one used the Beylkin–Coifman–Meyer–Rokhlin algorithm and its generalization to characterize the kernel-distribution of operators by their wavelet coefficients. In analysing Calderón–Zygmund operators, Yang treated their kernel-distributions as usual distribution in $2n$ dimensions. In analysing symbol operators in $\mathrm{Op}S^m_{1,\delta}$ $(0 \le \delta \le 1)$, he treated their kernel-distribution like distributions in $2n$ dimensions where different coordinates play different roles. Further, one developed pseudo-annular decomposition to study operator's continuity on the basis of wavelet characterization (see [4,11]). But there exists difficulties to find unconditional bases for general symbol operators in $\mathrm{Op}S^m_{\rho,\delta}$ by considering their kernel-distributions. Here, we treat directly the symbols as distributions in phase space and *our first aim* is to characterize all these symbol classes with wavelet coefficients.

Besov spaces $B^{m,\infty}_{\infty}$ is a little bigger than Hölder spaces C^m_b. But the latter has no unconditional basis, and wavelets cannot characterize it; the former has unconditional basis, and wavelets can characterize it. Hence we replace $S^m_{\rho,\delta}$ by $\tilde{S}^m_{\rho,\delta}$. One writes $\sigma(x,\xi) \in \tilde{S}^m_{\rho,\delta}$, if

$$\|\partial^{\beta}_{\xi}\sigma(x,\xi)\|_{B^{\alpha,\infty}_{\infty}} \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta\alpha}, \quad \forall \alpha \in N, \beta \in N^n. \tag{1.3}$$

We have the following theorem.

Theorem 1. *Given $m \in R$, $\rho, \delta \ge 0$, there exists an index set $\Lambda_{\rho,\delta}$, a group of wavelet basis $\{\Phi_\lambda(x,\xi)\}_{\lambda\in\Lambda_{\rho,\delta}}$ where $\Phi_\lambda(x,\xi) \in S(R^n \times R^n)$ and a group of number array spaces $N^m_{\rho,\delta}$ such that*

(i) *If $\sigma(x,\xi) \in \tilde{S}^m_{\rho,\delta}$, then there exists a unique sequence $\{a_\lambda\}_{\lambda\in\Lambda_{\rho,\delta}} \in N^m_{\rho,\delta}$ such that*

$$\sigma(x,\xi) = \sum_{\lambda\in\Lambda_{\rho,\delta}} a_\lambda \Phi_\lambda(x,\xi).$$

(ii) *Conversely, if $\{a_\lambda\}_{\lambda\in\Lambda_{\rho,\delta}} \in N^m_{\rho,\delta}$, then there exists a unique symbol $\sigma(x,\xi)$ such that the following formula is true in the sense of symbol*

$$\sigma(x,\xi) = \sum_{\lambda\in\Lambda_{\rho,\delta}} a_\lambda \Phi_\lambda(x,\xi) \in \tilde{S}^m_{\rho,\delta}.$$

Remark 1. Given $m \in R$, $\rho, \delta \ge 0$, by (1.2) and (1.3), it is easy to see that $S^m_{\rho,\delta} \subset \tilde{S}^m_{\rho,\delta}$ where their elements are almost the same, more precisely, $S^m_{\rho,\delta} \subsetneq \tilde{S}^m_{\rho,\delta} \subsetneq S^m_{\rho,\delta+\tau}$, $\forall\tau > 0$. Further, by the proof in Theorem 5 below, we know $S^m_{0,0} = \tilde{S}^m_{0,0}$.

Note that, if there exist a set S and a group of functions $\{\Phi_\lambda(x,\xi)\}_{\lambda\in S}$ satisfying that $\Phi_\lambda(x,\xi) \in S(R^n \times R^n)$ and $\{\Phi_\lambda(x,\xi)\}_{\lambda\in S}$ is an orthonormal basis in $L^2(R^n \times R^n)$, then for each distribution $\sigma(x,\xi) \in S'(R^n \times R^n)$ and for each $\lambda \in S$, we can define a unique number $a_\lambda = \langle\sigma(x,\xi), \Phi_\lambda(x,\xi)\rangle$. That is to say, there is an one-to-one relationship between the symbols in $S'(R^n \times R^n)$ and the number sequences $\{a_\lambda\}_{\lambda\in S}$. Thus $\{a_\lambda\}_{\lambda\in S}$ becomes a new representation for symbol — a wavelet representation. The difficulties to analyse operators with wavelets are to find the appropriate wavelet basis. The proof of Theorem 1 will be given in two sections: in Theorem 5 of §3, we find unconditional bases for $\tilde{S}^m_{0,\delta}$ $(\delta \ge 0)$; in §5, we characterize $\tilde{S}^m_{\rho,\delta}$ $(\rho > 0)$ with wavelet coefficients.

The second generation of Calderón–Zygmund operators studied kernel-distribution $k(x,z)$ where

$$k(x,z) = (2\pi)^{-n}\int \sigma(x,\xi)\,\mathrm{e}^{iz\xi}\,\mathrm{d}\xi$$

and

$$Tf(x) = \sigma(x, D)f(x) = \int k(x, z)f(x - z)\,\mathrm{d}z.$$

Meyer [12] and Stein [15] have established some relations for symbol and kernel-distribution for some special symbol class $S^m_{1,\delta}$. The *second aim* of this paper is to get a more general result by using Theorem 1 or more precisely, by using Theorem 6 in §4.

Theorem 2. *Given $m \in R$, $\rho > 0$, $\delta \geq 0$. If $\sigma(x, \xi) \in S^m_{\rho,\delta}$, then $\forall \alpha, \beta \in N^n$, we have*

(i) *If $|z| \geq \frac{1}{2}$, then, $\forall \alpha, \beta \in N^n$, we have*

$$|\partial_x^\alpha \partial_z^\beta k(x, z)| \leq C_{\alpha,\beta,N}(1 + |z|)^{-N}, \forall N > 0.$$

(ii) *If $|z| \leq \frac{1}{2}$, then, $\forall \alpha \in N$, $\beta \in N^n$, we have*

$$\|\partial_z^\beta k(x, z)\|_{B^{\alpha,\infty}_\infty} \leq C_{\alpha,\beta,N}|z|^{-N}, \forall N \geq 0$$

and

$$n + m + \delta\alpha + \max(1, \rho)|\beta| < N\rho.$$

The proof of Theorem 2 will be given in §5.

The reason why we pay attention to the wavelet structure of operators is to analyse precisely operator's continuity. For example, $T1$ theorem and compensated compactness are well-known (see [2,4,9,11,16]). There are some problems which are hard to solve without wavelets. In [5,7,19], one uses wavelets and relative pseudo-annular decomposition to study the $T1$ theorem and the compensated theory and gets some good results. The *third aim* of this paper is to study the L^2-continuity and the L^p-continuity of non-regular symbol operators. On the basis of symbol's wavelet coefficients in phase space, we can apply a precise Huygens' principal (or a precise micro-analysis method) to study operator's continuity (see also [10]).

In [15], Stein studied the L^2-continuity of operators defined by the symbol in $S^0_{0,0} = C^\infty_b(R^{2n})$. In [3] and [8], one studied pseudodifferential operators in phase space. In [1], Calderón and Vaillancourt studied L^2-continuity of symbol operators where symbol $\sigma(x, \xi)$ belong to the Hölder space $C^{2n+1}_b(R^{2n})$ and in some sense, which is the special Besov space $B^{2n+1,\infty}_\infty(R^{2n}) = B^{2n+1,\infty}_\infty$. Here we reduce an index n for the order of smoothness and establish L^2-continuity also; in fact, for $s \leq n < s'$, we know that $B^{s',\infty}_\infty \subset B^{n,1}_\infty \subset B^{s,\infty}_\infty$. Further, we can construct a special operator to show that our result is sharp. That is our Theorem 3.

Theorem 3.

(i) *If $\sigma(x, \xi) \in B^{n,1}_\infty$, then we have*

$$\sigma(x, D) \text{ defines an operator which is continuous from } L^2(R^n) \text{ to } L^2(R^n). \tag{1.4}$$

(ii) *Conversely, for* $0 < s < n$, *there exists a symbol* $\sigma(x,\xi) \in B_{\infty}^{s,\infty}$ *but*

$$\sigma(x, D) \text{ is not continuous from } L^2(R^n) \text{ to } L^2(R^n). \tag{1.5}$$

In addition, if we strengthen a little the above assumption, we can consider L^p-continuity. Let $Q = \{x = (x_1, \ldots, x_n), 0 \leq x_i \leq 1, 1 \leq i \leq n\}$ be a unit cube in R^n. For $j \geq 0$ and $k \in Z^n$, denote $2^{-j}k + 2^{-j}Q = \{x\colon 2^j x - k \in Q\}$. Let I_n be the set which is composed by n elements in R^n which are the unit vectors in the direction of the axes. For arbitrary distribution $f(x)$ and for $e \in I_n$, $h \in R$, $m \in N$, let

$$\tau_{he} f(x) = f(x + he) - f(x) \quad \text{and} \quad \tau_{he}^m = (\tau_{he})^m.$$

For $j \geq 1$, $X = (x, \xi) \in R^{2n}$, e $\in I_{2n}$, denote

$$\sigma_{j,e}(X) = \tau_{2^{-j}e}^n \sigma(X).$$

Denote

$$\omega(0) = \sup_{k \in Z^n} \int_{k+Q} \int_{R^n} |\sigma(x, \xi)| \, \mathrm{d}x \, \mathrm{d}\xi,$$

and for $j \geq 1$, denote

$$\omega(j) = \sup_{k \in Z^n, e \in I_{2n}} \int_{2^{-j}k + 2^{-j}Q} \int_{R^n} |\sigma_{j,e}(x, \xi)| \, \mathrm{d}x \, \mathrm{d}\xi.$$

We say that $\sigma(x, \xi) \in B^s$, if

$$\sum_j 2^{(n+s)j} \omega(j) < \infty.$$

By (2.7) and (7.1) below, we know that $B^n \subsetneq B_{\infty}^{n,1}$. Now we establish L^p-continuity.

Theorem 4.

(i) *If* $\sigma(x, \xi)$ *satisfies the condition*

$$\sigma(x, \xi) \in B^n, \tag{1.6}$$

then for $1 \leq p \leq \infty$, *we have*

$$\sigma(x, D) \text{ is continuous from } L^p(R^n) \text{ to } L^p(R^n). \tag{1.7}$$

(ii) *Conversely, for* $0 < s < n$, *there exists* $\sigma(x, \xi)$ *satisfies the condition*

$$\sigma(x, \xi) \in B^s, \tag{1.8}$$

but for $1 \leq p \leq \infty$, *we have*

$$\sigma(x, D) \text{ is not continuous from } L^p(R^n) \text{ to } L^p(R^n). \tag{1.9}$$

The difficulty to study operator's continuity is to find an appropriate operator's decomposition such that the relative operators have some pseudo-orthogonality. Our new idea is to establish the operator's continuity in Theorems 3 and 4 on the basis of wavelet characterization in phase space, and the proof will be given in the last two sections of this paper.

Remark 2. Meyer wrote in his famous book [12] that, for a long time, the study of operators stayed in two isolated classes—Calderón–Zygmund operators and symbol operators. On one hand, one has found wavelet characterization for Calderón–Zygmund operators and established such operator's continuity and also commutator operator's continuity (see [5,7,13,17,19]). On the other hand, one has given a wavelet representation for symbol operators and developed relative methods to study operator's continuity in this paper and in other papers (see [18]). That is to say, we can study both Calderón–Zygmund operators and symbol operators under their wavelet representation.

2. Preliminaries

At the begin of this section, we introduce some notations for wavelets and prove some wavelet properties.

In this paper, we use a wavelet basis which is a tensor product of the wavelets in dimension 1. When we characterize $S^m_{\rho,\delta}$ in §§3 and 4 and when we analyse kernel-distribution in §5, we always use Meyer's wavelets; but in other cases, when we prove operator's continuity, we need Meyer's wavelets; when we construct special operators to prove that our results are sharp, we need sufficiently regular Daubechies' wavelets. In dimension 1, denote the father wavelet by $\Phi^0(x)$ and the mother wavelet by $\Phi^1(x)$. In high dimension, for $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$, denote

$$\Phi^{\epsilon}(x) = \prod_{i=1}^{n} \Phi^{\epsilon_i}(x_i) \quad \text{and} \quad \Phi^0(x) = \Phi^{(0,\dots,0)}(x). \tag{2.1}$$

For $j \in Z, k \in Z^n$, denote

$$f_{j,k}(x) = 2^{nj/2} f(2^j x - k). \tag{2.2}$$

Let $\{V_j\}_{j\in Z}$ be an orthogonal multi-resolution analysis in $L^2(R^n)$ and $V_{j+1} = V_j \oplus W_j$. Then $\{\Phi^0_{j,k}(x)\}_{k\in Z^n}$ is an orthonormal wavelet basis in V_j and $\{\Phi^{\epsilon}_{j,k}(x)\}_{\epsilon\in\{0,1\}^n\backslash\{0\},k\in Z^n}$ is an orthonormal wavelet basis in W_j and $L^2(R^n) = V_0 \bigoplus_{j\geq 0} W_j$. Let P_j be the projector operator from $L^2(R^n)$ to V_j and let Q_j be the projector operator from $L^2(R^n)$ to W_j. It is easy to see that $P_0 + \sum_{j\geq 0} Q_j$ is the unit operator I. Let

$$\Lambda_n = \{\lambda = (\epsilon, j, k), \epsilon \in \{0, 1\}^n, j \geq 0, k \in Z^n; \\ \text{and if } j > 0, \text{ then } \epsilon \neq 0\}. \tag{2.3}$$

Then $\{\Phi^{\epsilon}_{j,k}(x)\}_{(\epsilon,j,k)\in\Lambda_n}$ is an orthonormal wavelet basis in $L^2(R^n)$. $\forall\epsilon \in \{0, 1\}^n$, there exists $\{g^{\epsilon}_k\}_{k\in Z^n}$ such that

$$\Phi^{\epsilon}(x) = \sum_k g^{\epsilon}_k \Phi^0(2x - k). \tag{2.4}$$

$\forall\epsilon = (\epsilon_1, \ldots, \epsilon_n) \neq 0$, let τ_ϵ denote the smallest number i such that $\epsilon_i \neq 0$ and let $e_\epsilon = e_{\tau_\epsilon}$ denote the vector where the τ_ϵ coordinate is 1 and the rest are 0. For any sequence $\{a_k\}_{k\in Z_n}$, let $\tau^0 = S^0$ be the unit operator satisfying $\tau^0 a_k = S^0 a_k = a_k$; for $e_i \in I_n$ where its ith element is 1 and the rest are 0; and for $s \in N$, let

$$\tau_{\pm e_i} a_k = a_{k\pm e_i} - a_k \quad \text{and} \quad S_{e_i} a_k = -\sum_{l=-\infty}^{-1+k_i} a_{(k_1,\ldots,k_{i-1},l,k_{i+1},\ldots,k_n)}, \tag{2.5}$$

and let $\tau^s_{\pm e_i} = (\tau_{\pm e_i})^s$ and $S^s_{e_i} = (S_{e_i})^s$. Further, for $\alpha \in N^n$, let

$$\tau^\alpha_\pm = \prod_{i=1}^n \tau^{\alpha_i}_{\pm e_i} \quad \text{and} \quad S^\alpha = \prod_{i=1}^n S^{\alpha_i}_{e_i}. \tag{2.6}$$

For $e_i \in I_n$ such that the ith element of e_i is 1 and for $s \in N$, let $S^0_{e_i} f(x) = f(x)$ and $S_{e_i} f(x) = -\sum_{l=-\infty}^{-1} f(x - le_i)$, and let $S^s_{e_i} = (S_{e_i})^s$; further, for $\alpha \in N^n$, let $S^\alpha = \prod_{i=1}^n S^{\alpha_i}_{e_i}$.

Lemma 1.

(i) *For* $\epsilon \in \{0,1\}^n\backslash 0$ *and* $s \in N$, $\tilde\Phi^{\epsilon,s}(x) = \sum_k (S^s_{e_\epsilon} g^\epsilon_k)\Phi^0(2x-k)$ *satisfies* $\Phi^\epsilon(x) = \tau^s_{-\frac{1}{2}e_\epsilon}\tilde\Phi^{\epsilon,s}(x)$; *and further, if* $\Phi^\epsilon(x)$ *are Meyer's wavelets, then* $\tilde\Phi^{\epsilon,s}(x) \in S(R^n)$; *if* $\Phi^\epsilon(x)$ *are Daubechies' wavelets and* s *is less than the index of divergence moment of wavelets, then* $\tilde\Phi^{\epsilon,s}(x)$ *have compact support.*

(ii) *For Meyer's wavelet,* $\forall\beta \in N^n$, $S^\beta(\partial^\beta\Phi^0)(x) \in S(R^n)$.

Proof.

(i) For $\epsilon \in \{0,1\}^n\backslash 0$ and $s \in N$, by the scale equation $\Phi^\epsilon(x) = \sum_k g^\epsilon_k \Phi^0(2x-k)$ and by the construction of $\tilde\Phi^{\epsilon,s}(x)$, we have $\Phi^\epsilon(x) = \tau^s_{-\frac{1}{2}e_\epsilon}\tilde\Phi^{\epsilon,s}(x)$. Further, by divergence moment properties of wavelets, we have:

  (1) If $\Phi^\epsilon(x)$ are Meyer's wavelets, then $|S^s_{e_\epsilon} g^\epsilon_k| \leq C_{s,N}(1+|k|)^{-N}, \forall N > 0$ and hence $\tilde\Phi^{\epsilon,s}(x) \in S(R^n)$.

  (2) If $\Phi^\epsilon(x)$ are Daubechies' wavelets and s is less than the index of divergence moment of wavelets, then there exists C_s such that, for $|k| \geq C_s$, $S^s_{e_\epsilon} g^\epsilon_k = 0$, and hence $\tilde\Phi^{\epsilon,s}(x)$ have compact support.

(ii) For Meyer's wavelet, $\forall\beta \in N^n$, $\sum_k k^\beta \Phi^0(x-k)$ are polynomials $P_\beta(x)$ where the degree of x_i is β_i; hence we have $S^\beta(\partial^\beta\Phi^0)(x) \in S(R^n)$. Or we can prove (ii) by the fact that the Fourier transformation of $S^\beta(\partial^\beta\Phi^0)(x)$ is equal to $f_\beta(\xi) = C_\beta \xi^\beta \prod_{j=1}^n (1-\mathrm{e}^{i\xi_j})^{-\beta_j}\hat\Phi^0(\xi)$; since $\mathrm{supp}\hat\Phi^0(\xi) \subset [-\frac{4\pi}{3}, \frac{4\pi}{3}]^n$, so $f_\beta(\xi) \in S(R^n)$.

Besov spaces $B^{s,q}_p$, which were introduced systematically by Peetre [14] can be characterized with their wavelet coefficients (see [12] and [19]). For $f(x) = \sum_{\lambda=(\epsilon,j,k)\in\Lambda_n} a_\lambda \Phi_\lambda(x)$, we have the following.

Lemma 2.

$$f(x) \in B^{s,q}_p(R^n) \Longleftrightarrow \left(\sum_{j\geq 0} 2^{jq(s+\frac{n}{2}-\frac{n}{p})}\left(\sum_{\epsilon,k}|a_\lambda|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty. \tag{2.7}$$

Secondly, we introduce a useful and simple inequality which would be used often in this paper.

Lemma 3. *$\forall \alpha \geq 1, m \in R, x, y \in R^n$, we have*

$$(1+|x|)^m \leq (1+|y|)^m(1+\alpha|x-y|)^{|m|}. \tag{2.8}$$

Proof. It is evident for $m \geq 0$. If $m < 0$, then we have

$$(1+|x|)^m \leq (1+|y|)^m(1+|x-y|)^{|m|} \leq (1+|y|)^m(1+\alpha|x-y|)^{|m|}.$$

At the end of this section, a variation of the result in [15] (which discusses the operator's continuity) will be introduced. For $j \geq 0$ and $m = (k,l) \in Z^{2n}$, let $T^*_{j,m}$ be the relative conjugate operators of operators $T_{j,m}$. Then we have the following lemma.

Lemma 4. *Suppose that $T_{j,m}$ satisfies the following three conditions*:

$$\|T_{j,m}\|_{L^2\to L^2} \leq C, \tag{2.9}$$

$$\|T_{j,k_1,l_1}T^*_{j,k_2,l_2}\|_{L^2\to L^2} \leq C(1+4^{-j}|k_1-k_2|)^{-2N_0}(1+|l_1-l_2|)^{-2N_0}, \tag{2.10}$$

$$\|T^*_{j,k_1,l_1}T_{j,k_2,l_2}\|_{L^2\to L^2} \leq C(1+|k_1-k_2|)^{-2N_0}(1+4^{-j}|l_1-l_2|)^{-2N_0}. \tag{2.11}$$

Then for $N_0 > n$, $T_j = \sum_{m\in Z^{2n}} T_{j,m}$ defines an operator which is continuous from L^2 to L^2 and $\|T_j\|_{L^2\to L^2} \leq C4^{jn}$.

Proof. First, we consider a finite sum $S_j = S_{j,N} = \sum_{|m|\leq N} T_{j,m}$. Since $S^*_j S_j$ is a self-adjoint operator, we have $\|S_j\|^2 = \|S^*_j S_j\| = \|(S^*_j S_j)^M\|^{1/M}$ for all integer M. But we have

$$\begin{aligned}(S^*_j S_j)^M = &\sum_{k_1,l_1}\sum_{k_2,l_2}\cdots\sum_{k_{2M-1},l_{2M-1}} \\ &\times \sum_{k_{2M},l_{2M}} T^*_{j,k_1,l_1}T_{j,k_2,l_2}\cdots T^*_{j,k_{2M-1},l_{2M-1}}T_{j,k_{2M},l_{2M}}.\end{aligned} \tag{2.12}$$

We maximize $\|(S^*_j S_j)^M\|$ by

$$\sum_{k_1,l_1}\sum_{k_2,l_2}\cdots\sum_{k_{2M-1},l_{2M-1}}\sum_{k_{2M},l_{2M}}\|T^*_{j,k_1,l_1}T_{j,k_2,l_2}\cdots T^*_{j,k_{2M-1},l_{2M-1}}T_{j,k_{2M},l_{2M}}\|. \tag{2.13}$$

First, we re-group all the operators two by two, and apply the continuity of $\|T^*_{j,m}T_{j,m'}\|$. We get

$$\begin{aligned}&\|T^*_{j,k_1,l_1}T_{j,k_2,l_2}\cdots T^*_{j,k_{2M-1},l_{2M-1}}T_{j,k_{2M},l_{2M}}\| \\ &\quad\leq C^M(1+|k_1-k_2|)^{-2N_0}(1+4^{-j}|l_1-l_2|)^{-2N_0} \\ &\qquad\times\cdots\times(1+|k_{2M-1}-k_{2M}|)^{-2N_0}(1+4^{-j}|l_{2M-1}-l_{2M}|)^{-2N_0}.\end{aligned}$$

Then we maximize $\|T^*_{j,k_1,l_1}\|$ and $\|T_{j,k_{2M},l_{2M}}\|$ by the constant C, then re-group the remaining operators two by two. Applying the continuity of $\|T_{j,m}T^*_{j,m'}\|$, we get

$$\begin{aligned}
&\|T^*_{j,k_1,l_1}T_{j,k_2,l_2}\cdots T^*_{j,k_{2M-1},l_{2M-1}}T_{j,k_{2M},l_{2M}}\|\\
&\quad\le C^{M+2}(1+4^{-j}|k_2-k_3|)^{-2N_0}(1+|l_2-l_3|)^{-2N_0}\\
&\qquad\times\cdots\times(1+4^{-j}|k_{2M-2}-k_{2M-1}|)^{-2N_0}(1+|l_{2M-2}-l_{2M-1}|)^{-2N_0}.
\end{aligned}$$

Combining the above two cases, we have

$$\begin{aligned}
&\|T^*_{j,k_1,l_1}T_{j,k_2,l_2}\cdots T^*_{j,k_{2M-1},l_{2M-1}}T_{j,k_{2M},l_{2M}}\|\le C^{M+1}(1+|k_1-k_2|)^{-N_0}\\
&\quad\times(1+4^{-j}|k_2-k_3|)^{-N_0}\cdots(1+|k_{2M-1}-k_{2M}|)^{-N_0}\\
&\quad\times(1+4^{-j}|l_1-l_2|)^{-N_0}(1+|l_2-l_3|)^{-N_0}\\
&\quad\times\cdots\times(1+4^{-j}|l_{2M-1}-l_{2M}|)^{-N_0}.
\end{aligned}$$

Summing in order $k_1,\ldots,k_{2M-1}$ and $l_1,\ldots,l_{2M-1}$, one gets $C^{M+1}4^{jn(2M-1)}$; then summing k_{2M} and l_{2M}, one gets

$$\|S\|^{2M}\le CN^{2n}C^{M+1}4^{jn(2M-1)}\quad\text{or}\quad\|S\|\le(CN^{2n}C^{M+1}4^{jn(2M-1)})^{\frac{1}{2M}}.\tag{2.14}$$

Letting $M\to\infty$, we get $\|S\|\le C4^{jn}$.

Further, we adopt Journé's methods to pass to the general case. According to the above result, $\forall f(x)\in L^2$, we have

$$\left\|\sum_{|m|\le N}\lambda_m T_m f(x)\right\|_{L^2}\le C\|f(x)\|_{L^2},\quad\forall N\in N,\ |\lambda_m|\le 1.\tag{2.15}$$

Let $\epsilon(N)=\sup_{\tilde N\ge N}\|\sum_{N\le|m|\le\tilde N}\lambda_m T_m f(x)\|_{L^2}$. To prove that $\sum_m T_m f(x)$ converges to a function in L^2, it is sufficient to prove that $\lim_{N\to\infty}\epsilon(N)=0$. It is evident that, $\forall N\le N'$, we have $\epsilon(N)\ge\epsilon(N')$. If $\epsilon(N)$ does not approach zero, then there exists $\delta>0$ and $N>0$ such that $\epsilon(N')\ge\delta,\forall N'\ge N$. Then we can choose $m_N^1<m_N^2<\cdots<m_N^{2k}<m_N^{2k+1}<\cdots$ such that for $Z_k=\sum_{m_N^{2k}\le|m|\le m_N^{2k+1}}T_m f(x)$, we have

$$\|Z_k\|_{L^2}\ge\delta.\tag{2.16}$$

For $\theta=(\theta_1,\ldots,\theta_k)\in\{-1,1\}^k$, let $Z(\theta,k)=\sum_{i=1}^k\theta_i Z_i$. According to (2.15), we have $\|Z(\theta,k)\|\le C\|f(x)\|_{L^2}$. Since $\sum_{i=1}^k\|Z_i\|_{L^2}^2\le 2^{-k}\sum_{\theta\in\{-1,1\}^k}\|Z(\theta,k)\|^2$, we have $\sum_{i=1}^k\|Z_i\|_{L^2}\le C\|f\|_{L^2}$, which contradicts (2.16)!

3. Unconditional bases for $\tilde S^m_{0,\delta}$ ($\delta\ge 0$)

In this section, we use the usual $2n$ dimension wavelet basis in phase space to characterize $S^m_{0,0}=\tilde S^m_{0,0}$ and use wavelet basis which comes from tensor product of wavelet basis in n dimension to characterize $\tilde S^m_{0,\delta}$ ($\delta>0$) .

Let $\Lambda_{0,0} = \Lambda_{2n}$ and $\forall\lambda = (\epsilon, \epsilon', j, k, l) \in \Lambda_{0,0}$, let $\Phi_\lambda(x,\xi) = \Phi^{\epsilon}_{j,k}(x)\Phi^{\epsilon'}_{j,l}(\xi)$. Then $\{\Phi_\lambda(x,\xi)\}_{\lambda\in\Lambda_{0,0}}$ is an orthonormal basis in $L^2(R^n \times R^n)$. For $a_\lambda = \langle\sigma(x,\xi), \Phi_\lambda(x,\xi)\rangle$, the following equality is true in the sense of distribution:

$$\sigma(x,\xi) = \sum_{\lambda\in\Lambda_{0,0}} a_\lambda \Phi_\lambda(x,\xi). \tag{3.1}$$

Hence, we know that $\{a_\lambda\}_{\lambda\in\Lambda_{0,0}}$ becomes a new representation for symbol $\sigma(x,\xi)$. We say that $\{a_\lambda\}_{\lambda\in\Lambda_{0,0}} \in N^m_{0,0}$, if

$$|a_\lambda| \le C_N 2^{-jN}(1 + |2^{-j}l|)^m, \quad \forall N > 0, \lambda \in \Lambda_{0,0}. \tag{3.2}$$

For $\delta > 0$, let $\Lambda_{0,\delta} = \Lambda_n \times \Lambda_n$; and for $\lambda = (\epsilon, j, k, \epsilon', j', k') \in \Lambda_{0,\delta}$, let $\Phi_\lambda(x,\xi) = \Phi^{\epsilon}_{j,k}(x)\Phi^{\epsilon'}_{j',k'}(\xi)$. Then $\{\Phi_\lambda(x,\xi)\}_{\lambda\in\Lambda_{0,\delta}}$ is an orthonormal wavelet basis in $L^2(R^n \times R^n)$. For $a_\lambda = \langle\sigma(x,\xi), \Phi_\lambda(x,\xi)\rangle$, it is clear that $\{a_\lambda\}_{\lambda\in\Lambda_{0,\delta}}$ becomes a new representation for symbol. For $\delta > 0$, we write $\{a_\lambda\}_{\lambda\in\Lambda_{0,\delta}} \in N^m_{0,\delta}$, if

$$|a_\lambda| \le C_{\alpha,\beta} 2^{-(\frac{n}{2}+\alpha)j} 2^{-(\frac{n}{2}+\beta)j'} (1 + |2^{-j'}k'|)^{m+\delta\alpha}, \quad \forall\alpha, \beta \ge 0. \tag{3.3}$$

On basis of the above notation, for $\delta \ge 0$, we have the following.

Theorem 5. *The following two conditions are equivalent*:

$$\sigma(x,\xi) \in \tilde{S}^m_{0,\delta}, \tag{3.4}$$

$$\{a_\lambda\}_{\lambda\in\Lambda_{0,\delta}} \in N^m_{0,\delta}. \tag{3.5}$$

Proof. First step. We consider the case where $\delta = 0$ and we prove that $\sigma(x,\xi) \in \tilde{S}^m_{0,0}$ implies that $\{a_\lambda\}_{\lambda\in\Lambda_{0,0}} \in N^m_{0,0}$. We consider three cases: (i) $\epsilon' \neq 0$, (ii) $\epsilon' = 0, \epsilon \neq 0$ and (iii) $\epsilon = \epsilon' = 0$. For arbitrary $\epsilon \in \{0,1\}^n\backslash\{0\}$ and $N > 0$, let $I^N_\epsilon f(x)$ be the Nth integration of $f(x)$ for the τ_ϵ-coordinate. For Case (i) and for sufficiently large $N' > 2N + n + |m|$, we have

$$\begin{aligned}
|a_\lambda| &= |\langle\sigma(x,\xi), \Phi^{\epsilon}_{j,k}(x)\Phi^{\epsilon'}_{j,l}(\xi)\rangle| \\
&= 2^{j(n-N)}|\langle\partial^N_{\xi_{\tau_{\epsilon'}}}\sigma(x,\xi), \Phi^{\epsilon}(2^jx - k)(I^N_{\epsilon'}\Phi)^{\epsilon'}(2^j\xi - l)\rangle| \\
&\le 2^{j(n-N)}\int |\langle\partial^N_{\xi_{\tau_{\epsilon'}}}\sigma(x,\xi), \Phi^{\epsilon}(2^jx - k)\rangle||(I^N_{\epsilon'}\Phi)^{\epsilon'}(2^j\xi - l)|\mathrm{d}\xi \\
&\le C2^{-jN}\int \frac{(1+|\xi|)^m}{(1+|2^j\xi - l|)^{N'}}\mathrm{d}\xi.
\end{aligned}$$

Then applying Lemma 3 to $(1+|\xi|)^m$, we have

$$\begin{aligned}
|a_\lambda| &\le C2^{-jN}(1 + |2^{-j}l|)^m \int (1 + |2^j\xi - l|)^{|m|-N'}\mathrm{d}\xi \\
&\le C2^{-j(n+N)}(1 + |2^{-j}l|)^m.
\end{aligned}$$

For Case (ii), by Lemma 2, we have

$$|a_\lambda| = \int |\langle\sigma(x,\xi), \Phi^{\epsilon}_{j,k}(x)\rangle||\Phi^{\epsilon'}_{j,l}(\xi)|\mathrm{d}\xi \le C2^{-jN}\int \frac{(1+|\xi|)^m}{(1+|2^j\xi - l|)^{N'}}\mathrm{d}\xi.$$

Then applying Lemma 3 to $(1+|\xi|)^m$, we have

$$|a_\lambda| \le C2^{-jN}(1+|2^{-j}l|)^m \int (1+|2^j\xi - l|)^{|m|-N'}\mathrm{d}\xi$$
$$\le C2^{-j(n+N)}(1+|2^{-j}l|)^m.$$

For Case (iii), by Lemmas 2 and 3, we have

$$|a_\lambda| = \int |\langle \sigma(x,\xi), \Phi^0(x-k)\rangle||\Phi^0(\xi-l)|\mathrm{d}\xi \le C\int \frac{(1+|\xi|)^m}{(1+|\xi-l|)^{N'}}\mathrm{d}\xi$$
$$\le C(1+|l|)^m \int (1+|\xi-l|)^{|m|-N'}\mathrm{d}\xi \le C(1+|l|)^m.$$

Second step. We consider the case where $\delta = 0$ and we prove that $\{a_\lambda\}_{\lambda\in\Lambda_{0,0}} \in N^m_{0,0}$ implies that $\sigma(x,\xi) \in S^m_{0,0}$. For arbitrary $\alpha, \beta \in N^n$, we choose $N > n + |\alpha| + |\beta|$ and $N' > n + |m|$. We have

$$\left|\partial_x^\alpha \partial_\xi^\beta \sum_{\lambda\in\Lambda_{0,0}} a_\lambda \Phi_\lambda(x,\xi)\right| \le \sum_{j\ge 0} 2^{j(n+|\alpha|+|\beta|-N)} \sum_k |(\partial_x^\alpha \Phi^\epsilon)(2^j x - k)|$$
$$\times \sum_l (1+|2^{-j}l|)^m |(\partial_\xi^\beta \Phi^{\epsilon'})(2^j\xi - l)|$$
$$\le \sum_{j\ge 0} 2^{j(n+|\alpha|+|\beta|-N)} \sum_l \frac{(1+|2^{-j}l|)^m}{(1+|2^j\xi - l|)^{-N'}}.$$

Then applying Lemma 3 to $(1+|2^{-j}l|)^m$, we have

$$\left|\partial_x^\alpha \partial_\xi^\beta \sum_{\lambda\in\Lambda_{0,0}} a_\lambda \Phi_\lambda(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^m \sum_{j\ge 0} 2^{j(n+|\alpha|+|\beta|-N)}$$
$$\le C_{\alpha,\beta}(1+|\xi|)^m.$$

Third step. We consider the case where $\delta > 0$ and we prove that (3.4) implies (3.5). We distinguish four cases.

(1) If $\epsilon = \epsilon' = 0$, then $a^{\epsilon,\epsilon'}_{j,k,j',k'} = a^{0,0}_{0,k,0,k'} = \langle \sigma(x,\xi), \Phi^0(x-k)\Phi^0(\xi-k')\rangle$. Since $|\langle \sigma(x,\xi), \Phi^0(x-k)\rangle| \le C(1+|\xi|)^m$, we apply Lemma 3, and get

$$|a^{0,0}_{0,k,0,k'}| \le C(1+|k'|)^m.$$

(2) If $\epsilon = 0$, $\epsilon' \ne 0$, then

$$a^{\epsilon,\epsilon'}_{j,k,j',k'} = a^{0,\epsilon'}_{0,k,j',k'} = \langle \sigma(x,\xi), \Phi^0(x-k)\Phi^\epsilon_{j',k'}(\xi)\rangle$$
$$= 2^{-j'|\beta|}\langle \partial^\beta_{\xi_{\tau_\epsilon}} \sigma(x,\xi), \Phi^0(x-k)(I^\beta_\epsilon \Phi^\epsilon)_{j',k'}(\xi)\rangle.$$

Since $|\langle \partial^\beta_{\xi_{\tau_\epsilon}} \sigma(x,\xi), \Phi^0(x-k)\rangle| \le C(1+|\xi|)^m$, we apply Lemma 3, and get

$$|a_\lambda| \le C_N 2^{-j'N}(1+|2^{-j'}k'|)^m.$$

(3) If $\epsilon \neq 0, \epsilon' = 0$, then

$$a^{\epsilon,\epsilon'}_{j,k,j',k'} = a^{\epsilon,0}_{j,k,0,k'} = \langle \sigma(x,\xi), \Phi^{\epsilon}_{j,k}(x)\Phi^0(\xi - k')\rangle.$$

Hence by Lemmas 2 and 3, we get

$$|a^{\epsilon,0}_{j,k,0,k'}| \leq C2^{-(\frac{n}{2}+\alpha)j}(1+|k'|)^{m+\delta\alpha}.$$

(4) If $|\epsilon||\epsilon'| \neq 0$, then

$$\begin{aligned} a^{\epsilon,\epsilon'}_{j,k,j',k'} &= \langle \sigma(x,\xi), \Phi^{\epsilon}_{j,k}(x)\Phi^{\epsilon'}_{j',k'}(\xi)\rangle \\ &= 2^{-j'|\beta|}\langle \partial^{\beta}_{\xi_{\tau_\epsilon}}\sigma(x,\xi), \Phi^{\epsilon}_{j,k}(x)(I^{\beta}_{\epsilon'}\Phi^{\epsilon'})_{j',k'}(\xi)\rangle. \end{aligned}$$

Hence by Lemmas 2 and 3, we get

$$|a_\lambda| \leq C_{\alpha,\beta}2^{-(\frac{n}{2}+\alpha)j}2^{-(\frac{n}{2}+|\beta|)j'}(1+|2^{-j'}k'|)^{m+\delta\alpha}.$$

Final step. We consider the case where $\delta > 0$ and we prove that (3.5) implies (3.4). Let $\sigma(x,\xi) = \sum_{\lambda\in\Lambda_{0,\delta}} a_\lambda \Phi_\lambda(x,\xi)$, then we have

$$\partial^{\beta}_{\xi}\sigma(x,\xi) = \sum_{\lambda\in\Lambda_{0,\delta}} 2^{j'|\beta|}a_\lambda\Phi^{\epsilon}_{j,k}(x)(\partial^{\beta}_{\xi}\Phi^{\epsilon'})_{j',k'}(\xi).$$

By Lemma 2, we have

$$\begin{aligned} \|\partial^{\beta}_{\xi}\sigma_2(x,\xi)\|_{B^{\alpha,\infty}_{\infty}} &= \sup_{\epsilon,j,k}|2^{(\frac{n}{2}+\alpha)j}\sum_{\epsilon',j',k'}2^{j'|\beta|}a_\lambda(\partial^{\beta}_{\xi}\Phi^{\epsilon'})_{j',k'}(\xi)| \\ &\leq \sup_{k}\sum_{\epsilon',j',k'}2^{j'|\beta|}|a^{0,\epsilon'}_{0,k,j',k'}||(\partial^{\beta}_{\xi}\Phi^{\epsilon'})_{j',k'}(\xi)| \\ &\quad + \sup_{\epsilon\neq 0,j,k}2^{(\frac{n}{2}+\alpha)j}\sum_{\epsilon',j',k'}2^{j'|\beta|}|a^{\epsilon,\epsilon'}_{j,k,j',k'}||(\partial^{\beta}_{\xi}\Phi^{\epsilon'})_{j',k'}(\xi)| \\ &\leq C(1+|\xi|)^m + C(1+|\xi|)^{m+\delta\alpha} \leq C(1+|\xi|)^{m+\delta\alpha}. \end{aligned}$$

Hence we get $\sigma(x,\xi) \in \tilde{S}^m_{0,\delta}$.

4. Wavelet characterization for $\tilde{S}^m_{\rho,\delta}(\rho > 0)$

We use the wavelet basis which comes from the tensor product of wavelet basis in n-dimension. Let $\Lambda_{\rho,\delta} = \Lambda_n \times \Lambda_n$. For $\lambda = (\epsilon, j, k, \epsilon', j', k') \in \Lambda_{\rho,\delta}$, let $\Phi_\lambda(x,\xi) = \Phi^{\epsilon}_{j,k}(x)\Phi^{\epsilon'}_{j',k'}(\xi)$. Then $\{\Phi_\lambda(x,\xi)\}_{\lambda\in\Lambda_{\rho,\delta}}$ is an orthogonal normal wavelet basis in $L^2(R^n \times R^n)$. For $a_\lambda = \langle\sigma(x,\xi), \Phi_\lambda(x,\xi)\rangle$, it is clear that $\{a_\lambda\}_{\lambda\in\Lambda_{\rho,\delta}}$ becomes a new representation for symbol. For $\lambda = (\epsilon, j, k, \epsilon', j', k')$, if $\epsilon' = 0$, then $j' = 0$ and we write $\lambda = (\epsilon, j, k, k')$ and $a_\lambda = a^{\epsilon}_{j,k,k'}$. Let $\tau^\beta = \tau^\beta_+$ be the operator acting on k'. We say that $\{a_\lambda\}_{\lambda\in\Lambda_{\rho,\delta}} \in N^m_{\rho,\delta}$, if a_λ satisfies the following properties:

(i) The absolute value of a_λ satisfies:

$$|a_\lambda| \le \begin{cases} C_{\alpha,\beta} 2^{-(\frac{n}{2}+\alpha)j} 2^{-(\frac{n}{2}+\beta)j'} (1+|2^{-j'}k'|)^{m+\delta\alpha-\rho\beta}, \forall \alpha, \beta \ge 0, & \text{if } \epsilon' \neq 0; \\ C_\alpha 2^{-(\frac{n}{2}+\alpha)j} (1+|k'|)^{m+\delta\alpha}, \forall \alpha \ge 0, & \text{if } \epsilon' = 0. \end{cases} \tag{4.1}$$

(ii) In addition, for $\epsilon' = 0$, a_λ also satisfies

$$|\tau^\beta a^\epsilon_{j,k,k'}| \le C_{\alpha,\beta} 2^{-(\frac{n}{2}+\alpha)j} (1+|k'|)^{m+\delta\alpha-\rho|\beta|}, \forall \alpha \ge 0, \beta \in N^n, \text{ if } \epsilon' = 0. \tag{4.2}$$

Then we have the following theorem.

Theorem 6. *The following two conditions are equivalent*:

$$\sigma(x,\xi) \in \tilde{S}^m_{\rho,\delta}, \tag{4.3}$$

$$\{a_\lambda\}_{\lambda \in \Lambda_{\rho,\delta}} \in N^m_{\rho,\delta}. \tag{4.4}$$

Proof. From symbol to number array. To prove (4.1), we consider first the case where $\epsilon' \neq 0$. For arbitrary α and $\beta \in N^n$, for sufficiently large $N' > n + |m| + \delta|\alpha| - \rho|\beta|$, we have

$$\begin{aligned} |a_\lambda| &= |\langle \sigma(x,\xi), \Phi^\epsilon_{j,k}(x) \Phi^{\epsilon'}_{j',k'}(\xi)\rangle| \\ &= 2^{\frac{nj}{2}} 2^{j'(\frac{n}{2}-|\beta|)} |\langle \partial^\beta_{\xi_{\tau_{\epsilon'}}} \sigma(x,\xi), \Phi^\epsilon(2^j x - k)(I^\beta_{\epsilon'} \Phi^{\epsilon'})(2^{j'}\xi - k')\rangle|. \end{aligned} \tag{4.5}$$

By Lemma 2, we get

$$|a_\lambda| \le C 2^{-j(\frac{n}{2}+|\alpha|)} 2^{j'(\frac{n}{2}-|\beta|)} \int \frac{(1+|\xi|)^{m+\delta\alpha-\rho|\beta|}}{(1+|2^{j'}\xi - k'|)^{N'}} \mathrm{d}\xi.$$

Then applying Lemma 3 to $(1+|\xi|)^{m+\delta\alpha-\rho|\beta|}$, we have

$$\begin{aligned} |a_\lambda| &\le C 2^{j(\frac{n}{2}-|\alpha|)} 2^{j'(\frac{n}{2}-|\beta|)} (1+|2^{-j'}k'|)^{m+\delta\alpha-\rho|\beta|} \\ &\quad \times \int (1+|2^{j'}\xi - k'|)^{|m+\delta\alpha-\rho|\beta||-N'} \mathrm{d}\xi \\ &\le C 2^{-j(\frac{n}{2}+|\alpha|)} 2^{-j'(\frac{n}{2}+|\beta|)} (1+|2^{-j'}k'|)^{m+\delta\alpha-\rho|\beta|}. \end{aligned}$$

For $\epsilon' = 0$, for arbitrary α and for sufficiently large $N' > n + |m| + \delta|\alpha|$, by Lemma 2, we have

$$\begin{aligned} |a_\lambda| &= |\langle \sigma(x,\xi), \Phi^\epsilon_{j,k}(x) \Phi^0_{0,k'}(\xi)\rangle| \\ &\le C 2^{-j(\frac{n}{2}+|\alpha|)} \int \frac{(1+|\xi|)^{m+\delta\alpha-\rho|\beta|}}{(1+|\xi - k'|)^{N'}} \, \mathrm{d}\xi. \end{aligned}$$

Then applying Lemma 3 to $(1+|\xi|)^{m+\delta\alpha}$, we have

$$\begin{aligned} |a_\lambda| &\le C 2^{j(\frac{n}{2}-|\alpha|)} (1+|k'|)^{|m+\delta\alpha|} \int (1+|\xi - k'|)^{|m+\delta\alpha|-N'} \, \mathrm{d}\xi \\ &\le C 2^{-j(\frac{n}{2}+|\alpha|)} (1+|k'|)^{m+\delta\alpha}. \end{aligned}$$

To prove (4.2), $\forall \alpha \in N^n$, let $\tau^\alpha_\pm f(x) = \prod_{i=1}^n \tau^{\alpha_i}_{\pm e_i} f(x)$. Hence we have

$$\begin{aligned}\tau^\beta a^\epsilon_{j,k,k'} &= \langle \sigma(x,\xi), \Phi^\epsilon_{j,k}(x)\tau^\beta \Phi^0(\xi - k')\rangle \\ &= \langle \tau^\beta_- \sigma(x,\xi), \Phi^\epsilon_{j,k}(x)\Phi^0(\xi - k')\rangle.\end{aligned}$$

For $\xi \in R^n$, $\beta \in N^n$, there exists a $\xi' \in B(\xi, 1+|\beta|)$ such that $\tau^\beta_- \sigma(x,\xi) = \partial^\beta_\xi \sigma(x,\xi')$. Hence we have

$$\tau^\beta a^\epsilon_{j,k,k'} = \langle \partial^\beta_\xi \sigma(x,\xi'), \Phi^\epsilon_{j,k}(x)\Phi^0(\xi - k')\rangle.$$

Then applying the same argument as above, we get the desired conclusion (4.2).

From wavelet representation to symbol representation. We consider three cases: (1) $|\epsilon||\epsilon'| \neq 0$; (2) $\epsilon = 0, \epsilon' \neq 0$; (3) $\epsilon' = 0$. We calculate the derivation of the following three symbols:

$$\begin{aligned}\sigma_1(x,\xi) &= \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ |\epsilon||\epsilon'|\neq 0}} a_\lambda \Phi_\lambda(x,\xi),\\ \sigma_2(x,\xi) &= \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ \epsilon=0,\epsilon'\neq 0}} a_\lambda \Phi_\lambda(x,\xi),\\ \sigma_3(x,\xi) &= \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ \epsilon'=0}} a_\lambda \Phi_\lambda(x,\xi).\end{aligned}$$

We prove that $\sigma_1(x,\xi), \sigma_2(x,\xi) \in S^m_{\rho,\delta}$ and $\sigma_3(x,\xi) \in \tilde{S}^m_{\rho,\delta}$. As for $\sigma_1(x,\xi)$, for arbitrary $\alpha, \beta \in N^n$, we choose $s > |\alpha|, t > |\beta|, \delta(s-\alpha) \leq \rho(t-\beta)$ and $N' > n + |m + s\delta - t\rho|$. Then we have

$$\begin{aligned}|\partial^\alpha_x \partial^\beta_\xi \sigma_1(x,\xi)| &\leq \sum_{j,j'\geq 0} 2^{j(|\alpha|-s)} 2^{j'(|\beta|-t)} \sum_{\epsilon,k} |(\partial^\alpha_x \Phi^\epsilon)(2^j x - k)| \\ &\quad \times \sum_{\epsilon',k'} (1 + |2^{-j'}k'|)^{m+s\delta-t\rho} |(\partial^\beta_\xi \Phi^{\epsilon'})(2^{j'}\xi - k')| \\ &\leq \sum_{j,j'\geq 0} 2^{j(|\alpha|-s)} 2^{j'(|\beta|-t)} \sum_{k'} \frac{(1+|2^{-j'}k'|)^{m+s\delta-t\rho}}{(1+|2^{j'}\xi - k'|)^{-N'}}.\end{aligned}$$

Applying Lemma 3 to $(1+|2^{-j'}k'|)^{m+s\delta-t\rho}$, we have

$$\begin{aligned}|\partial^\alpha_x \partial^\beta_\xi \sigma_1(x,\xi)| &\leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|} \sum_{j,j'\geq 0} 2^{j(|\alpha|-s)} 2^{j'(|\beta|-t)} \\ &\leq C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}.\end{aligned}$$

As for $\sigma_2(x,\xi)$, for arbitrary $\alpha,\beta\in N^n$, we choose $t>|\beta|$ and $N'>n+|m-t\rho|$. Then we have

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta\sigma_2(x,\xi)| &\le \sum_{j'\ge 0}2^{j'(|\beta|-t)}\sum_k|(\partial_x^\alpha\Phi^0)(x-k)|\\
&\quad\times\sum_{\epsilon',k'}(1+|2^{-j'}k'|)^{m-t\rho}|(\partial_\xi^\beta\Phi^{\epsilon'})(2^{j'}\xi-k')|\\
&\le \sum_{j'\ge 0}2^{j'(|\beta|-t)}\sum_{k'}\frac{(1+|2^{-j'}k'|)^{m+s\delta-t\rho}}{(1+|2^{j'}\xi-k'|)^{-N'}}.
\end{aligned}
$$

Applying Lemma 3 to $(1+|2^{-j'}k'|)^{m-t\rho}$, we have

$$
\begin{aligned}
|\partial_x^\alpha\partial_\xi^\beta\sigma_2(x,\xi)| &\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|}\sum_{j'\ge 0}2^{j(|\alpha|-s)}2^{j'(|\beta|-t)}\\
&\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|}.
\end{aligned}
$$

As for $\sigma_3(x,\xi)$, for arbitrary $\beta\in N^n$, we have

$$
\begin{aligned}
\partial_\xi^\beta\sigma_3(x,\xi) &= \sum_{\epsilon,j,k,k'}a^\epsilon_{j,k,k'}\Phi^\epsilon_{j,k}(x)(\partial_\xi^\beta\Phi^0)(\xi-k')\\
&= \sum_{\epsilon,j,k,k'}\tau^\beta a^\epsilon_{j,k,k'}\Phi^\epsilon_{j,k}(x)S^\beta(\partial_\xi^\beta\Phi^0)(\xi-k').
\end{aligned}
$$

For all $\alpha\in N$, we choose $N>n+|m+\delta\alpha-\rho|\beta||$. Applying Lemmas 1 and 2, we have

$$
\begin{aligned}
\|\partial_\xi^\beta\sigma_3(x,\xi)\|_{B_\infty^{\alpha,\infty}} &\le C_{\alpha,\beta}\left\|2^{j(\frac{n}{2}+\alpha)}\sum_{k'}\tau^\beta a^\epsilon_{j,k,k'}S^\beta(\partial_\xi^\beta\Phi^0)(\xi-k')\right\|_\infty\\
&\le C\left\|\sum_{k'}(1+|k'|)^{m+\delta\alpha-\rho|\beta|}(1+|\xi-k'|)^{-N}\right\|\\
&\le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|}.
\end{aligned}
$$

That is to say, $\sigma_3(x,\xi)\in\tilde{S}^m_{\rho,\delta}$.

5. Kernel-distribution

In this section, we consider the kernel-distribution property of symbols and prove Theorem 2. By Theorem 7, the kernel-distribution of the symbol operator $\sigma(x,D)$ can be written as

$$
k(x,z)=(2\pi)^{-n}\sum_{(\epsilon,j,k;\epsilon',j',k')\in\Lambda_n}a^{\epsilon,\epsilon'}_{j,k,j',k'}\Phi^\epsilon(2^jx-k)\hat{\Phi}^{\epsilon'}(2^{-j'}z)\,\mathrm{e}^{i2^{-j'}k'z},
$$

where

$$
2^{\frac{n}{2}(j'-j)}a^{\epsilon,\epsilon'}_{j,k,j',k'}\in N^m_{\rho,\delta}.
$$

We decompose $k(x,z)$ into three parts:

$$k_1(x,z) = (2\pi)^{-n} \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ |\epsilon||\epsilon'|\neq 0}} a^{\epsilon,\epsilon'}_{j,k,j',k'} \Phi^\epsilon(2^j x - k)\hat{\Phi}^{\epsilon'}(2^{-j'}z)\, \mathrm{e}^{i2^{-j'}k'z},$$

$$k_2(x,z) = (2\pi)^{-n} \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ \epsilon=0,\epsilon'\neq 0}} a^{0,\epsilon'}_{0,k,j',k'} \Phi^0(x - k)\hat{\Phi}^{\epsilon'}(2^{-j'}z)\, \mathrm{e}^{i2^{-j'}k'z},$$

$$k_3(x,z) = (2\pi)^{-n} \sum_{\substack{\lambda\in\Lambda_{\rho,\delta}\\ \epsilon'=0}} a^{\epsilon}_{j,k,k'}\, \mathrm{e}^{ik'z} \Phi^\epsilon(2^j x - k)\hat{\Phi}^{0}(z).$$

Hence, by Meyer's wavelet property, we know that: (i) if $|z| \leq \frac{\pi}{3}$, then $k_1(x,z) = k_2(x,z) = 0$; and (ii) if $|z| \geq \frac{4\pi}{3}$, then $k_3(x,z) = 0$. Now we prove that

$$|\partial^\alpha_x \partial^\beta_z k_1(x,z)| + |\partial^\alpha_x \partial^\beta_z k_2(x,z)| \leq C_{\alpha,\beta,N}(1+|z|)^{-N}, \quad \forall N > 0$$

and

$$\|\partial^\beta_z k_3(x,z)\|_{B^{\alpha,\infty}_\infty} \leq C_{\alpha,\beta,N}|z|^{-N}, \quad \forall N \geq 0$$

and

$$n + m + \delta\alpha + \max(1,\rho)\beta < N\rho.$$

First, we consider $k_1(x,z)$. For $\alpha, \beta \in N^n$, $\forall s_1$ and t_1, we have

$$\begin{aligned} I_1 = |\partial^\alpha_x \partial^\beta_z k(x,z)| &\leq C \sum_{j,j'\geq 0} 2^{(|\alpha|-s_1)j} 2^{-(n+t_1)j'} \\ &\quad \times \sum_{k'} (1 + |2^{-j'}k'|)^{m+s_1\delta - t_1\rho + |\beta|} \\ &\quad \times \sum_{\epsilon\neq 0,k} |\partial^\alpha_x \Phi^\epsilon(2^j x - k)| \sum_{\epsilon'\neq 0, |\gamma|\leq|\beta|} |\partial^\gamma_z \hat{\Phi}^{\epsilon'}(2^{-j'}z)|. \end{aligned}$$

By choosing $t_1\rho > m + s_1\delta + |\beta| + n$, we have

$$\sum_{k'} (1 + |2^{-j'}k'|)^{m+s_1\delta - t_1\rho + |\beta|} \leq C2^{nj'}.$$

Note that $\sum_{\epsilon,k} |\partial^\alpha_x \Phi^\epsilon(2^j x - k)| \leq C$; hence we have

$$I_1 \leq C \sum_{j,j'\geq 0} 2^{(|\alpha|-s_1)j} 2^{-t_1 j'} \sum_{\epsilon'\neq 0} |\hat{\Phi}^{\epsilon'}(2^{-j'}z)|.$$

Since $\Phi^\epsilon(x)(\epsilon\neq 0)$ are Meyer's wavelets, there exists $0 < M' < M$ such that $\forall\alpha \in N^n$, we have $\operatorname{supp}\partial^\alpha_z \hat{\Phi}^\epsilon(z) \subset B(0, 2^M)\backslash B(0, 2^{M'})$. Hence, there exists at most a finite number j' such that $\sum_{\epsilon'\neq 0} |\hat{\Phi}^{\epsilon'}(2^{-j'}z)| \neq 0$ and $2^{-j'} \sim C(1+|z|)^{-1}$. By choosing $s_1 > |\alpha|$ and $t_1 > \max\{\frac{m+s_1\delta+|\beta|+n}{\rho}, N\}$, we get

$$I_1 \leq C_{\alpha,\beta,N}(1+|z|)^{-N}.$$

Secondly, we consider $k_2(x,z)$. For $\alpha, \beta \in N^n$, s_2 and t_2, we have

$$I_2 = |\partial_x^\alpha \partial_z^\beta k(x,z)| \leq C \sum_{j' \geq 0} 2^{-(n+t_2)j'} \sum_{k'} (1 + |2^{-j'}k'|)^{m - t_2\rho + |\beta|}$$
$$\times \sum_k |\partial_x^\alpha \Phi^0(x-k)| \sum_{\epsilon' \neq 0, |\gamma| \leq |\beta|} |\partial_z^\gamma \hat{\Phi}^{\epsilon'}(2^{-j'}z)|.$$

We choose $t_2 > \max\{\frac{m+|\beta|+n}{\rho}, N\}$ and applying the same proof as above, we get

$$I_2 \leq C_{\alpha,\beta,N}(1+|z|)^{-N}.$$

Finally, we consider $k_3(x,z)$. We know that

$$\sum_{k'} a^\epsilon_{j,k,k'} \, \mathrm{e}^{ik'z} = \prod_{i=1}^n (1 - \mathrm{e}^{iz_i})^{-\beta_i} \sum_{k'} (\tau^\beta a^\epsilon_{j,k,k'}) \, \mathrm{e}^{ik'z}$$

and

$$\partial_z^\gamma \sum_{k'} (\tau^\beta a^\epsilon_{j,k,k'}) \, \mathrm{e}^{ik'z} = C_\gamma \sum_{k'} k'^\gamma (\tau^\beta a^\epsilon_{j,k,k'}) \, \mathrm{e}^{ik'z}.$$

Hence,

$$\partial_z^\gamma k_3(x,z) = C \sum_{(\epsilon,j,k)\in\Lambda_n} \Phi^\epsilon(2^j x - k) \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} C_\gamma^{\gamma_1,\gamma_2}$$
$$\times \left(\partial_z^{\gamma_1} \prod_{i=1}^n (1-\mathrm{e}^{iz_i})^{-\beta_i}\right) \left(\sum_{k'} k'^{\gamma_2} (\tau^\beta a^\epsilon_{j,k,k'}) \, \mathrm{e}^{ik'z}\right) \partial_z^{\gamma_3} \hat{\Phi}^0(z).$$

By Lemma 2 and by the estimation of $\tau^\beta a^\epsilon_{j,k,k'}$, we choose a convenient $\beta \in N^n$ such that $m + \delta\alpha - \rho|\beta| + |\gamma_2| + n < 0$ and get the desired conclusion.

6. L^2-continuity for symbol operator

First, we prove a useful lemma. For $i = 1, 2$, let $\Phi^i(x)$ be real-valued functions which belong to $S(R^n)$. For $j \geq 0$ and $m = (k,l) \in Z_{2n}$, let

$$T_{j,m} f(x) = \int \mathrm{e}^{ixy} \Phi^1_{j,k}(x) \Phi^2_{j,l}(y) f(y) \, \mathrm{d}y = \int K_{j,m}(x,y) f(y) \, \mathrm{d}y, \quad (6.1)$$

where

$$K_{j,m}(x,y) = \mathrm{e}^{ixy} \Phi^1_{j,k}(x) \Phi^2_{j,l}(y). \quad (6.2)$$

The kernel-distribution of the conjugate operator $T^*_{j,m}$ is

$$K^*_{j,m}(x,y) = \mathrm{e}^{-ixy} \Phi^2_{j,l}(x) \Phi^1_{j,k}(y). \quad (6.3)$$

Then we have the following lemma.

Lemma 5. *$\forall m=(k,l), m'=(k',l')\in Z_{2n}$, there exists a sufficiently large $N_0>n$ such that $T_{j,m}$ satisfies the following two conditions*:

$$\|T_{j,k,l}T^*_{j,k',l'}\|_{L^2\to L^2}\le C(1+4^{-j}|k-k'|)^{-2N_0}(1+|l-l'|)^{-2N_0}, \tag{6.4}$$

$$\|T^*_{j,k,l}T_{j,k',l'}\|_{L^2\to L^2}\le C(1+|k-k'|)^{-2N_0}(1+4^{-j}|l-l'|)^{-2N_0}. \tag{6.5}$$

Proof. The kernel-distribution of $T^*_{j,k,l}T_{j,k',l'}$ is

$$K^{1,j}_{m,m'}(y,z)=\Phi_{1,k,k'}(2^{-j}(y-z))\Phi^2_{j,l}(y)\Phi^2_{j,l'}(z),$$

where

$$\Phi_{1,k,k'}(z)=\int\Phi^1(x-k)\Phi^1(x-k')\,\mathrm{e}^{-ixz}\,\mathrm{d}x.$$

Since

$$|\Phi_{1,k,k'}(2^{-j}(y-z))|\le C(1+|k-k'|)^N(1+2^{-j}|y-z|)^{-N},$$

we get the desired conclusion for the norm of $T^*_{j,k,l}T_{j,k',l'}$.

Further, the kernel-distribution of $T_{j,k,l}T^*_{j,k',l'}$ is

$$K^{2,j}_{m,m'}(y,z)=\Phi_{2,l,l'}(2^{-j}(y-z))\Phi^1_{j,k}(y)\Phi^1_{j,k'}(z),$$

where

$$\Phi_{2,l,l'}(z)=\int\Phi^2(x-l)\Phi^2(x-l')\,\mathrm{e}^{ixz}\,\mathrm{d}x.$$

Since

$$|\Phi_{2,l,l'}(2^{-j}(y-z))|\le C(1+|l-l'|)^N(1+2^{-j}|y-z|)^{-N},$$

we get the desired conclusion for the norm of $T^*_{j,k,l}T_{j,k',l'}$.

Proof of Theorem 3. Let

$$\tilde K^{\epsilon,\epsilon'}_j(x,y)=\mathrm{e}^{ixy}\sum_{k,l}a^{\epsilon,\epsilon'}_{j,k,l}\Phi^\epsilon_{j,k}(x)\Phi^{\epsilon'}_{j,l}(y)$$

be the kernel-distribution of $\tilde T^{\epsilon,\epsilon'}_j$. By Lemmas 4 and 5, we have

$$\|\tilde T^{\epsilon,\epsilon'}_j\|_{L^2\to L^2}\le C4^{jn}\sup_{k,l}|a^{\epsilon,\epsilon'}_{j,k,l}|.$$

Let

$$\tilde K(x,y)=\mathrm{e}^{ixy}\sum_{\epsilon,\epsilon',j,k,l}a^{\epsilon,\epsilon'}_{j,k,l}\Phi^\epsilon_{j,k}(x)\Phi^{\epsilon'}_{j,l}(y)$$

be the kernel-distribution of $\tilde T$; then we have

$$\|\tilde T\|_{L^2\to L^2}\le C\sum_j 4^{jn}\sup_{\epsilon,\epsilon',k,l}|a^{\epsilon,\epsilon'}_{j,k,l}|.$$

Let $Ff(x)$ be the Fourier transform of $f(x)$; then we have $\sigma(x, D)f(x) = \tilde{T}Ff(x)$, i.e. $\sigma(x, D)$ is continuous from L^2 to L^2.

Now we prove part (ii) of Theorem 3. Let $\Phi^1(x)$ be a regular mother wavelet, and $\mathrm{supp}\Phi^1(x) \subset B(0, 2^M)$ where M is an integer. Let $\tilde{\Phi}(x)$ be the Fourier transform of the function $(\Phi^1(x))^2$; then there exists C_1 such that for $|x| \le 2C_1$, we have $\tilde{\Phi}(x) \ge C_1$. For $j \ge 0$, let τ_j be the set of l satisfying $2^{-M-2}l \in Z^n$ and $|l| \le C_2 4^j$ where C_2 satisfies $|\sum_{l\in\tau_j} \mathrm{e}^{i4^{-j}lx}| \ge C4^{jn}$ for $|x| \le C_1$. Further, for $j \ge 0$, $2^{-M-2}k$ and $2^{-M-2}l \in Z^n$, let $a_{j,k,l} = \mathrm{e}^{-i4^{-j}kl}$; otherwise, $a_{j,k,l} = 0$. For $j \ge 0, l \in \tau_j$, let $a_{j,l} = 2^{-jn}$; otherwise, $a_{j,l} = 0$.

To show that the result in Theorem 3 is sharp, we construct a special function and a special operator. Let $f_j(x) = \sum_l a_{j,l}\Phi^1_{j,l}(x)$ and let

$$K_j(x, y) = \mathrm{e}^{ixy} \sum_{k,l} a_{j,k,l}\Phi^1_{j,k}(x)\Phi^1_{j,l}(y)$$

be the kernel-distribution of the operator $\tilde{T}_j$. We have $\|f_j\|_{L^2} \sim C$ and

$$\begin{aligned} I_j &= \|\tilde{T}_j f_j(x)\|^2_{L^2} \\ &= \sum_k \int \left| \int \sum_l a_{j,k,l}a_{j,l}(\Phi^1_{j,l}(y))^2 \,\mathrm{e}^{ixy}\,\mathrm{d}y\right|^2 (\Phi^1_{j,k}(x))^2 \,\mathrm{d}x. \end{aligned}$$

According to the definition of $\tilde{\Phi}(x)$ and $a_{j,k,l}$, we have

$$\begin{aligned} I_j &= \sum_k \int \left| \sum_l a_{j,k,l}a_{j,l}\,\mathrm{e}^{i2^{-j}lx} \right| \tilde{\Phi}(2^{-j}x)|^2 (\Phi^1_{j,k}(x))^2 \,\mathrm{d}x \\ &= \sum_k \int \left| \sum_l a_{j,l}\,\mathrm{e}^{i2^{-j}l(x-2^{-j}k)} \right| \tilde{\Phi}(2^{-j}x)|^2 (\Phi^1_{j,k}(x))^2 \,\mathrm{d}x. \end{aligned}$$

By changing variables $2^jx - k \to x$ and by the definition of $a_{j,l}$, we have

$$I_j = 4^{-jn} \sum_k \int \left| \sum_{l\in\tau_j} \mathrm{e}^{i4^{-j}lx} \right|^2 |\tilde{\Phi}(4^{-j}x + 4^{-j}k)|^2 |\Phi^1(x)|^2 \,\mathrm{d}x.$$

For $|k| \le C_1 4^j$ and $|x| \le C_1$, we have $|\tilde{\Phi}(4^{-j}x + 4^{-j}k)|^2 \ge C_1$. Hence, we have

$$I_j \ge C \int \left| \sum_{l\in\tau_j} \mathrm{e}^{i4^{-j}lx} \right|^2 |\Phi^1(x)|^2 \,\mathrm{d}x \ge C4^{2jn}.$$

Let $K_j(x, \xi)$ be the symbol of the operator $\sigma_j(x, D)$; then we have

$$\|\sigma_j(x, D)\|_{L^2\to L^2} \ge C4^{jn} \quad \text{and} \quad \|K_j(x, \xi)\|_{B^{s,\infty}_\infty} = 2^{j(s+n)}.$$

That is to say, for $0 < s < n$, there exists a symbol $\sigma(x, \xi) \in B^{s,\infty}_\infty$ but $\sigma(x, D)$ is not continuous from L^2 to L^2.

7. L^p-continuity

We begin with a lemma about the characterization of symbol.

Lemma 6. *If $\sigma(x,\xi)$ satisfies condition* (1.6), *then*

$$\sum_{j,\epsilon,\epsilon'} 2^{nj} \sup_k \sum_l |a^{\epsilon,\epsilon'}_{j,k,l}| < \infty. \tag{7.1}$$

In addition, for $0 < s < n$, the following two conditions are equivalent:

$$\sum_j 2^{j(n+s)} \omega(j) < \infty, \tag{7.2}$$

$$\sum_{j,\epsilon,\epsilon'} 2^{sj} \sup_k \sum_l |a^{\epsilon,\epsilon'}_{j,k,l}| < \infty. \tag{7.3}$$

Proof. From wavelet representation to symbol. That is to say, we prove that (7.3) implies (7.2). For $j \geq 1$, $e \in I_{2n}$, we have

$$\begin{aligned}\sigma_{j,e}(x,\xi) = &\sum_{j'\geq j} \sum_{(\epsilon,\epsilon',k,l)} a^{\epsilon,\epsilon'}_{j',k,l} \tau^n_{2^{-j}e} \Phi^{\epsilon,\epsilon'}_{j',k,l}(x,\xi) \\ &+ \sum_{j'<j} \sum_{(\epsilon,\epsilon',k,l)} a^{\epsilon,\epsilon'}_{j',k,l} \tau^n_{2^{-j}e} \Phi^{\epsilon,\epsilon'}_{j',k,l}(x,\xi).\end{aligned}$$

Hence, we have

$$\begin{aligned}I_{s,e} &= \sum_{j\geq 1} 2^{j(n+s)} \sup_{m\in Z^n} \int_{2^{-j}m+2^{-j}Q} \mathrm{d}x \int_{R^n} |\sigma_{j,e}(x,\xi)|\, \mathrm{d}\xi \\ &\leq C \sum_{j\geq 1} 2^{j(n+s)} \sup_{m\in Z^n} \int_{2^{-j}m+2^{-j}Q} \sum_{j'\geq j} \sum_{(\epsilon,\epsilon',k,l)} |a^{\epsilon,\epsilon'}_{j',k,l}||\Phi^{\epsilon}(2^{j'}x-k)|\, \mathrm{d}x \\ &\quad + C \sum_{j\geq 1} 2^{j(n+s)} \sup_{m\in Z^n} \int_{2^{-j}m+2^{-j}Q} \sum_{j'<j} 2^{(j'-j)n} \\ &\quad \times \sum_{(\epsilon,\epsilon',k,l)} |a^{\epsilon,\epsilon'}_{j',k,l}||\Phi^{\epsilon}(2^{j'}x-k)|\, \mathrm{d}x \\ &\leq C \sum_{j\geq 1} \sum_{j'\geq j} 2^{js} \sup_{k\in Z^n} \sup_{\epsilon,\epsilon'} \sum_l |a^{\epsilon,\epsilon'}_{j',k,l}| \\ &\quad + C \sum_{j\geq 1} \sum_{j'<j} 2^{(j'-j)n} 2^{js} \sup_{k\in Z^n} \sup_{\epsilon,\epsilon'} \sum_l |a^{\epsilon,\epsilon'}_{j',k,l}|.\end{aligned}$$

If $0 < s < n$, then

$$I_{s,e} \leq C \sum_{j'} 2^{j's} \sup_{\epsilon,\epsilon',k} \sum_l |a^{\epsilon,\epsilon'}_{j',k,l}|.$$

And further, we have

$$
\begin{aligned}
I' &= \sup_{m\in Z^n}\int_{m+Q} \mathrm{d}x \int_{R^n} |\sigma(x,\xi)|\,\mathrm{d}\xi \\
&\le C \sup_{m\in Z^n}\int_{m+Q}\sum_{j\ge 0}\sum_{(\epsilon,\epsilon',k,l)} |a^{\epsilon,\epsilon'}_{j,k,l}||\Phi^{\epsilon}(2^j x-k)|\,\mathrm{d}x \\
&\le C\sum_{j\ge 0}\sup_{k\in Z^n,\epsilon,\epsilon'}\sum_l |a^{\epsilon,\epsilon'}_{j,k,l}| \\
&\le C\sum_j 2^{js}\sup_{\epsilon,\epsilon',k}\sum_l |a^{\epsilon,\epsilon'}_{j,k,l}|.
\end{aligned}
$$

From symbol to wavelet representation. For $(\epsilon,\epsilon',j,k,l)\in\Lambda_{2n}$, we have

$$|a^{\epsilon,\epsilon'}_{j,k,l}| = |\langle\sigma(x,\xi),\Phi^{\epsilon,\epsilon'}_{j,k,l}(x,\xi)\rangle|.$$

If $|\epsilon|+|\epsilon'|=0$, then $j=0$ and we have

$$
\begin{aligned}
|a^{0,0}_{0,k,l}| &= |\langle\sigma(x,\xi),\Phi^{0,0}(x-k,\xi-l)\rangle| \\
&\le C\sum_{|k-k'|\le 2^M}\int_{k'+Q}\int_{R^n}|\sigma(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi.
\end{aligned}
$$

If $|\epsilon|+|\epsilon'|\ne 0$, according to Lemma 1, we have

$$
\begin{aligned}
|a^{\epsilon,\epsilon'}_{j,k,l}| &= 2^{jn}|\langle\sigma(x,\xi),\tau^n_{-2^{-1}e_{(\epsilon,\epsilon')}}\tilde{\Phi}^{\epsilon,\epsilon'}(2^j x-k,2^j\xi-l)\rangle| \\
&= 2^{jn}|\langle\sigma_{1+j,(\epsilon,\epsilon')}(x,\xi),\tilde{\Phi}^{\epsilon,\epsilon'}(2^j x-k,2^j\xi-l)\rangle|.
\end{aligned}
$$

Hence we get

$$|a^{\epsilon,\epsilon'}_{j,k,l}| \le C\sum_{|k-k'|\le 2^M}2^{jn}\int_{2^{-j}k'+2^{-j}Q}\int_{R^n}|\sigma_{1+j,(\epsilon,\epsilon')}(x,\xi)|\,\mathrm{d}x\,\mathrm{d}\xi.$$

So we get the desired conclusion.

Proof of Theorem 4. Let

$$K^{\epsilon,\epsilon'}_j(x,y) = \sum_{k,l} a^{\epsilon,\epsilon'}_{j,k,l}\Phi^{\epsilon}(2^j x-k)\hat{\Phi}^{\epsilon'}(2^{-j}(x-y))\,\mathrm{e}^{i2^{-j}l(x-y)}$$

be the kernel-distribution of the operator $T^{\epsilon,\epsilon'}_j$. We have

$$|K^{\epsilon,\epsilon'}_j(x,y)| \le C\sum_k\sum_l |a^{\epsilon,\epsilon'}_{j,k,l}||\Phi^{\epsilon}(2^j x-k)||\hat{\Phi}^{\epsilon'}(2^{-j}(x-y))|.$$

That is,

$$\int |K^{\epsilon,\epsilon'}_j(x,y)|\,\mathrm{d}x \le C2^{jn}\sup_k\sum_l |a^{\epsilon,\epsilon'}_{j,k,l}|$$

and

$$\int |K_j^{\epsilon,\epsilon'}(x,y)|\,\mathrm{d}y \le C2^{jn} \sup_k \sum_l |a^{\epsilon,\epsilon'}_{j,k,l}|.$$

Hence, for $1 \le p \le \infty$, $T_j^{\epsilon,\epsilon'}$ is continuous from L^p to L^p.

Let

$$\Gamma = \{(\epsilon,\epsilon',j), \forall k,l \in Z^n, (\epsilon,\epsilon',j,k,l) \in \Lambda_{2n}\}.$$

Hence $\sigma(x,D) = \sum_{(\epsilon,\epsilon',j)\in\Gamma} T_j^{\epsilon,\epsilon'}$ is continuous from L^p to L^p for $1 \le p \le \infty$.

Then we prove part (ii) of Theorem 4. Let M be a sufficiently big integer, let $\Phi^1(x)$ be a regular Daubechies' wavelet with supp $\Phi^1(x) \subset B(0, 2^M)$ and let $\Phi^2(x)$ be Meyer's wavelet. Moreover, let

$$\sigma_j(x,\xi) = \sum_{2^{M+2}k\in Z^n} \Phi^1(2^j x - k)\Phi^2(2^j\xi)$$

and let

$$\sigma(x,\xi) = \sum_{(2+M)j\in N} j^2 2^{-jn} \sigma_j(x,\xi).$$

Then $\sigma(x,\xi)$ satisfies conditions (1.8) and (1.9).

Acknowledgements

This work is supported by the NNSF of China (No. 10001027), the innovation funds of Wuhan University and the subject construction funds of Mathematic and Statistic School, Wuhan University.

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