Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 3, August 2005, pp. 251–257.

Homeomorphisms and the homology of non-orientable surfaces

SIDDHARTHA GADGIL and DISHANT PANCHOLI*

Stat Math Unit, Indian Statistical Institute, Bangalore 560 059, India
*School of Mathematics, Tata Institute of Fundamental Research, Mumbai 400 005, India
E-mail: gadgil@isibang.ac.in; dishant@math.tifr.res.in

MS received 8 February 2005; revised 3 May 2005

Abstract. We show that, for a closed non-orientable surface F, an automorphism of $H_1(F, \mathbb{Z})$ is induced by a homeomorphism of F if and only if it preserves the (mod 2) intersection pairing. We shall also prove the corresponding result for punctured surfaces.

Keywords. Non-orientable surfaces; Dehn twist; mapping class groups; crosscap slide.

1. Introduction

Let F be a closed, non-orientable surface. A homeomorphism $f: F \to F$ induces an automorphism on homology $f_*: H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$. Further, any automorphism $\varphi: H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ in turn induces an automorphism with $\mathbb{Z}/2\mathbb{Z}$-coefficients $\bar{\varphi}: H_1(F, \mathbb{Z}/2\mathbb{Z}) \to H_1(F, \mathbb{Z}/2\mathbb{Z})$. If $\varphi = f_*$ for a homeomorphism f, then $\bar{\varphi}$ also preserves the (mod 2) intersection pairing on homology.

Our main result is that, for an automorphism $\varphi: H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$, if the induced automorphism $\bar{\varphi}: H_1(F, \mathbb{Z}/2\mathbb{Z}) \to H_1(F, \mathbb{Z}/2\mathbb{Z})$ preserves the (mod 2) intersection pairing, then φ is induced by a homeomorphism of F.

Theorem 1.1. *Let $\varphi: H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ be an automorphism. If the induced automorphism $\bar{\varphi}: H_1(F, \mathbb{Z}/2\mathbb{Z}) \to H_1(F, \mathbb{Z}/2\mathbb{Z})$ preserves the* (mod 2) *intersection pairing, then φ is induced by a homeomorphism of F.*

We have a natural homomorphism $\mathrm{Aut}(H_1(F, \mathbb{Z})) \to \mathrm{Aut}(H_1(F, \mathbb{Z}/2\mathbb{Z}))$. Let $\mathcal{K}$ denote the kernel of this homomorphism, so that we have an exact sequence

$$1 \to \mathcal{K} \to \mathrm{Aut}(H_1(F, \mathbb{Z})) \to \mathrm{Aut}(H_1(F, \mathbb{Z}/2\mathbb{Z})) \to 1.$$

Observe that elements of $\mathcal{K}$ automatically preserve the intersection pairing. We shall show that every element of $\mathcal{K}$ is induced by a homeomorphism of F. Further, we shall show that an element of $\mathrm{Aut}(H_1(F, \mathbb{Z}/2\mathbb{Z}))$ is induced by a homeomorphism of F if and only if it preserves the intersection pairing. Theorem 1.1 follows immediately from these results.

Theorem 1.2. *Suppose $\varphi: H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ is an automorphism which induces the identity on $H_1(F, \mathbb{Z}/2\mathbb{Z})$. Then φ is induced by a homeomorphism of F.*

Theorem 1.3. *Let F_1 and F_2 be closed, non-orientable surfaces. Suppose that ψ: $H_1(F_1, \mathbb{Z}/2\mathbb{Z}) \to H_1(F_2, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism which preserves the intersection pairing. Then ψ is induced by a homeomorphism $f: F_1 \to F_2$.*

We also consider the case of a compact non-orientable surface F with boundary. In this case an automorphism of $H_1(F, \mathbb{Z})$ induced by a homeomorphism of F permutes (up to sign) the elements representing the boundary components. We shall show that all automorphisms of $H_1(F, \mathbb{Z})$ which satisfy this additional condition are induced by homeomorphisms. Other results regarding the homeomorphisms of non-orientable surfaces have been obtained by many authors, for instance [1–3].

2. Preliminaries

Let F be a closed, non-orientable surface with $\chi(F) = 2 - n$ and let $\hat{F}$ be obtained from F by deleting the interior of a disc. Then F is the connected sum of n projective planes $\mathcal{P}_i$ and $\hat{F}$ is the boundary-connected sum of n corresponding Möbius bands $\mathcal{M}_i$. Let γ_i denote the central circle of $\mathcal{M}_i$ and let $\alpha_i = [\gamma_i] \in H_1(\hat{F}, \mathbb{Z})$ be the corresponding elements in homology. Then $H_1(\hat{F}, \mathbb{Z}) \cong \mathbb{Z}^n$ with basis α_i and $H_1(F, \mathbb{Z})$ is the quotient $H_1(\hat{F}, \mathbb{Z})/\langle 2\Sigma_i \alpha_i \rangle$.

We shall need the following elementary algebraic lemma.

Lemma 2.1. *Any automorphism φ: $H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ lifts to an automorphism $\tilde{\varphi}$: $H_1(\hat{F}, \mathbb{Z}) \to H_1(\hat{F}, \mathbb{Z})$ such that $\tilde{\varphi}(\sum_i \alpha_i) = \sum_i \alpha_i$.*

Proof. Consider the basis of $H_1(\hat{F}, \mathbb{Z})$ given by $e_1 = \alpha_1, \dots, e_{n-1} = \alpha_{n-1}$, $e_n = \alpha_1 + \cdots + \alpha_n$ and let $[e_j]$ be the corresponding generators of $H_1(F, \mathbb{Z})$. Observe that $[e_n]$ is the unique element of order 2 in $H_1(F, \mathbb{Z})$, and hence $\varphi([e_n]) = [e_n]$. Thus, we can define $\tilde{\varphi}(e_n) = e_n$. For $1 \leq j \leq n-1$, pick an arbitrary lift h_j of $\varphi(e_j)$ and set $\tilde{\varphi}(e_j) = h_j$.

Observe that $H_1(\hat{F}, \mathbb{Z})/\langle e_n \rangle \cong H_1(F, \mathbb{Z})/\langle [e_n] \rangle$. Further, as $\tilde{\varphi}(e_n) = e_n$ we have an induced map on $H_1(\hat{F}, \mathbb{Z})/\langle e_n \rangle$ which agrees with the quotient map induced by φ on $H_1(F, \mathbb{Z})/\langle [e_n] \rangle$ (which exists as $\varphi([e_n]) = [e_n]$) under the natural identification of these groups. As φ is an isomorphism, so is the induced quotient map on $H_1(F, \mathbb{Z})/\langle [e_n] \rangle$, and hence the map induced by $\tilde{\varphi}$ on $H_1(\hat{F}, \mathbb{Z})/\langle e_n \rangle$.

Thus, $\tilde{\varphi}$ induces an isomorphism on the quotient $H_1(\hat{F}, \mathbb{Z})/\langle e_n \rangle$ as well as the kernel $\langle e_n \rangle$ of the quotient map. By the five lemma, $\tilde{\varphi}$ is an isomorphism. □

Henceforth, given an automorphism φ as above, we shall assume that a lift has been chosen as in the lemma. Observe that a homeomorphism of $\hat{F}$ induces a homeomorphism of F. Hence it suffices to construct a homeomorphism of $\hat{F}$ inducing $\tilde{\varphi}$. Note that the intersection pairing is preserved by $\tilde{\varphi}$ as it only depends on the induced map on homology with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

3. Automorphisms in $\mathcal{K}$

In this section we prove Theorem 1.2. Let φ: $H_1(F, \mathbb{Z}) \to H_1(F, \mathbb{Z})$ be as in the hypothesis. As in Lemma 2.1, we can lift φ to an automorphism of $H_1(\hat{F}, \mathbb{Z})$ fixing $\sum_i \alpha_i$. We shall denote this lift also by φ. We shall construct a homeomorphism of $\hat{F}$ inducing this automorphism.

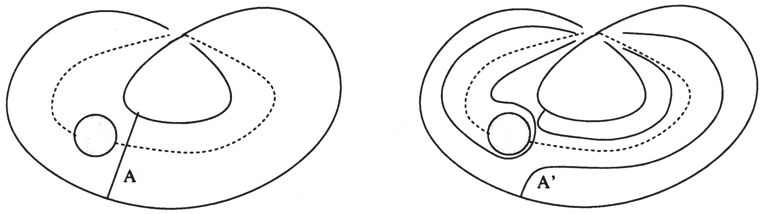

Figure 1. Cross-cap slide.

Our strategy is to use *elementary automorphisms* e_{ij}, $1 \leq i, j \leq n$, which are induced by homeomorphisms g_{ij}. Observe that, for $1 \leq i, j \leq n$, the automorphism φ is induced by a homeomorphism if and only if $e_{ij} \circ \varphi$ is induced by a homeomorphism (as e_{ij} is induced by a homeomorphism). Thus we can replace φ with $e_{ij} \circ \varphi$. We call this an *elementary move*. For φ preserving the intersection pairing, we shall find a sequence of elementary moves such that on performing these moves we obtain the identity automorphism, which is obviously induced by a homeomorphism (namely the identity). This will prove the result.

Lemma 3.1. *There are homeomorphisms g_{ij} of $\hat{F}$ so that if e_{ij} is the induced automorphism on $H_1(\hat{F}, \mathbb{Z})$, then $e_{ij}(\alpha_i) = \alpha_i + 2\alpha_j$, $e_{ij}(\alpha_j) = -\alpha_j$ and $e_{ij}(\alpha_k) = \alpha_k$ for $k \neq i, j$.*

Proof. We shall use *cross-cap slides* [3, 4] of the surface F. Namely, suppose α is an orientation reversing simple closed curve on a surface S' and D is a small disc centered around a point on α. Let S be the surface obtained by replacing D by a Möbius band. Consider a homeomorphism f' of S' which is the identity outside a neighbourhood of α and which is obtained by dragging D once around α so that D is mapped to itself. By construction this extends to a homeomorphism f of S, which we call a cross-cap slide. In figure 1, the arc A in the Möbius band $\mathcal{M}$ on the left-hand side is mapped to the arc A' in the Möbius band $\mathcal{M}'$ on the right-hand side and the homeomorphism is the identity in a neighbourhood of the boundary.

We define g_{ij} as the *cross-cap slide* of $\mathcal{M}_j$ around the curve γ_i. Note that the Möbius band $\mathcal{M}_j$ is mapped to itself, but, as γ_i is orientation reversing, the map on the Möbius band takes α_j to $-\alpha_j$. Further for any k different from i and j, the cross-cap slide fixes γ_j, hence α_j. Finally, in figure 1 (where we regard $\mathcal{M}$ as a neighbourhood of α_j), if B is a curve in the boundary of $\mathcal{M}$ joining the endpoints of A, then $[A \cup B] = \alpha_i$ and $[A' \cup B] = g_{ij}(\alpha_i)$. It is easy to see that $[A \cup B] - [A' \cup A] = [A \cup A']$ is homologous to the boundary $2\alpha_j$ of the cross-cap. Thus, $e_{ij}(\alpha_i) = \alpha_i + 2\alpha_j$. □

Lemma 3.2. *There exists a sequence of elementary moves e_{ij} taking φ to the identity.*

Proof. Let φ be represented by a matrix $A = (a_{ij})$ with respect to the basis α_i. Then $A \equiv I \,(\mathrm{mod}\, 2)$. As φ fixes $\sum_i \alpha_i$, for every i, $\sum_j a_{ij} = 1$. Observe that on performing the elementary move e_{ij}, the ith column A_{*i} of A is replaced by $A_{*i} + 2A_{*j}$, the jth column is replaced by $-A_{*j}$ and the other columns of A are unchanged.

We first use the elementary moves e_{ij} to reduce the first row A_{1*} to $[1, 0, 0, \ldots, 0]$. To do this, we define a complexity $C_1(A)$ of A as $|a_{11}| + |a_{12}| + \cdots + |a_{1n}|$.

Observe that if a_{1k} and a_{1l} are both non-zero, have different signs and $|a_{1k}| > |a_{1l}|$, e_{kl} reduces the complexity $C_1(A)$. As a_{11} is odd and a_{1j} is even for $j > 1$, we know that $a_{11} \neq a_{1j}$ for every $j > 1$. Further, as $\sum_j a_{1j} = 1$, unless a_{11} is 1 and $a_{1j} = 0$ for

$j \neq 1$, there exists a $j > 1$ such that a_{11} and a_{1j} are of opposite signs (and both non-zero). Thus we can reduce complexity by performing an elementary operation. By iterating this finitely many times, we reduce the first row to $[1, 0, 0, \ldots, 0]$.

Next, suppose $i > 1$ and the rows $A_{1*}, A_{2*}, \ldots, A_{(i-1)*}$ are the unit vectors $e_1, e_2, \ldots, e_{(i-1)*}$. We shall transform the ith row to $[0, 0, \ldots, 1, 0, \ldots, 0]$ without changing the earlier rows.

First we shall transform the row A_{i*} to a row of the form $[*, *, \ldots, 1, 0, \ldots, 0]$ (i.e., with the first $i-1$ entries arbitrary) by performing elementary moves e_{ij}. To do this, we define a complexity $C_i(A) = \sum_j |a_{ij}|$, $j \geq i$.

Observe that, for $k \geq i$, the elementary operation e_{1k} changes the sign of a_{ik}, does not alter a_{im} for $m \neq k, m \geq i$ and does not change first $i-1$ rows. By such operations we can ensure that $a_{ii} > 0$ and $a_{ij} \leq 0$ for $j > i$ without changing the complexity.

As before, $a_{ii} \neq a_{ij}$ for $j > i$ (as a_{ii} is odd and a_{ij} is even) and (using operations e_{1k} if necessary) a_{ii} and a_{ij} have different signs. Hence, unless $a_{ij} = 0$ for $j > i$ we can reduce the complexity using either e_{ij} or e_{ji}, without altering the first i rows. Thus we can reduce A_{i*} to a vector of the form $[*, \ldots, *, m, 0, \ldots, 0]$.

Now A is a block lower triangular matrix with a_{ii} as a diagonal entry. As A is invertible it follows that $m = a_{ii} = \pm 1$.

We define another complexity $C'_i(A) = \sum_{i \geq j} |a_{ij}|$. As $\sum_j a_{ij} = 1$ and $a_{ii} = \pm 1$, unless A_{i*} is a unit vector we can find as before an operation e_{ji}, $j < i$, which reduces this complexity (without changing the first $(i-1)$ rows). Hence after finitely many steps the ith row is reduced to a unit vector. By applying these moves for $i = 2, 3, \ldots, n$, we are done. $\square$

4. Automorphisms of $H_1(F, \mathbb{Z}/2\mathbb{Z})$

We now prove Theorem 1.3. We shall proceed by induction on n. In the case when $n = 1$ the result is obvious. We henceforth assume that n is greater than 1.

We first make some observations. For a surface S, any element α of $H_1(S, \mathbb{Z}/2\mathbb{Z})$ can be represented by a simple closed curve. The curve α is orientation reversing if and only if $\alpha \cdot \alpha = 1$. The surface is non-orientable if and only if there exist $\alpha \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ with $\alpha \cdot \alpha = 1$.

As before, let F_1 be the connected sum of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$, where $\mathcal{P}_i$ denotes a projective plane and $\mathcal{M}_i$ denotes the corresponding Möbius band. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$ be as before.

Let ψ be as in the hypothesis. Let $\beta_i = \psi(\alpha_i)$ and let C be a simple close curve that represents β_1. As $\beta_1 \cdot \beta_1 = \alpha_1 \cdot \alpha_1 = 1$, C is orientation reversing (as is γ_1). Hence regular neighbourhoods of C and γ_1 are Möbius bands.

Let $\hat{F}'_1 = F_1 - \mathrm{int}(\mathcal{N}(\gamma_1))$ and $\hat{F}'_2 = F_2 - \mathrm{int}(\mathcal{N}(C))$. Let $F'_1 = \hat{F}'_1 \cup D^2$ and $F'_2 = \hat{F}'_2 \cup D^2$ be closed surfaces obtained by capping off $\hat{F}_i$.

Observe that the surface F'_1 is non-orientable as $n \geq 2$ and γ_2 is an orientation reversing curve on it. Now since ψ preserves the intersection pairing it takes orthonormal basis of $H_1(F_1, \mathbb{Z}/2\mathbb{Z})$ to orthonormal basis of $H_1(F_2, \mathbb{Z}/2\mathbb{Z})$. It follows that $\beta_j \cdot \beta_j = 1$ for every j. Further, by a Mayer–Vietoris argument, $H_1(F_i, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus H_1(F'_i, \mathbb{Z}/2\mathbb{Z})$, with the decomposition being orthogonal and the component $\mathbb{Z}/2\mathbb{Z}$ in $H_1(F_1, \mathbb{Z}/2\mathbb{Z})$ (respectively $H_1(F_2, \mathbb{Z}/2\mathbb{Z})$) is spanned by α_1 (respectively β_1). As ψ preserves the intersection pairing, it follows that ψ induces an isomorphism $\psi\colon H_1(F_1, \mathbb{Z}/2\mathbb{Z}) \to H_1(F_2, \mathbb{Z}/2\mathbb{Z})$.

Hence if C_2 is a curve in F_2' representing β_2 in $H_1(F_2, \mathbb{Z}/2\mathbb{Z})$, then C_2 is orientation reversing and hence F_2' is non-orientable. Also, we have seen that the map ψ induces an isomorphism from $H_1(F_1', \mathbb{Z}/2\mathbb{Z})$ to $H_1(F_2', \mathbb{Z}/2\mathbb{Z})$. By the induction hypothesis such a map is induced by a homoeomorphism $f'\colon F_1' \to F_2'$.

Note that F_1 (respectively F_2) is obtained from F_1' (respectively F_2') by deleting the interior of a disc D_1 (respectively D_2) and gluing in $\mathcal{N}(\gamma_1)$ (respectively $\mathcal{N}(C)$). We can modify f' so that $f'(D_1) = D_2$. On $F_1 - \operatorname{int}(D_1)$ we define $f = f'$. This restricts to a homeomorphism mapping $\partial\mathcal{N}(\gamma_1)$ to $\partial\mathcal{N}(C)$, which extends to a homeomorphism mapping $\mathcal{N}(\gamma_1)$ to $\mathcal{N}(C)$. As $f|_{\mathcal{N}(\gamma_1)}\colon \mathcal{N}(\gamma_1) \to \mathcal{N}(C)$ is a homeomorphism, it maps the generator α_1 of $H_1(\mathcal{N}(\gamma_1), \mathbb{Z}) = \mathbb{Z}$ to a generator $\pm\beta$ of $H_1(\mathcal{N}(C), \mathbb{Z}) = \mathbb{Z}$. Thus with mod 2 coefficients, $f_* = \varphi$ as required. □

5. An algebraic corollary

We shall deduce from Theorem 1.1 and a theorem of Lickorish [3] a purely algebraic corollary. While this has a straightforward algebraic proof (and is presumably well-known), it may still be of interest to see its relation to topology.

Let $V = (\mathbb{Z}/2\mathbb{Z})^n$ be a vector space over $\mathbb{Z}/2\mathbb{Z}$ and let $\{e_j\}$ be the standard basis of V. Consider the standard inner product $\langle (x_i), (y_i) \rangle = \sum_i x_i y_i$. Let $\mathcal{O}$ be the group of automorphisms of V that preserve the inner product. We shall show that $\mathcal{O}$ is generated by certain involutions.

Namely, let $1 \leq i_1 < i_2 < \cdots < i_{2k} \leq n$ be $2k$ integers between 1 and n. We define an element $R = R(i_1, \ldots, i_{2k})$ to be the transformation defined by

$$R(e_{i_j}) = \sum_{l \neq j} e_{i_l},$$

$$R(e_j) = e_j,\ j \neq i_1, i_2, \ldots i_{2k}.$$

Theorem 5.1. *The group $\mathcal{O}$ is generated by the involutions $R(i_1, \ldots, i_{2k})$.*

Proof. We identify V with $H_1(F, \mathbb{Z}/2\mathbb{Z})$ for a non-orientable surface F and identify the basis elements e_i with α_i. Under this identification, the bilinear pairing on V corresponds to the intersection pairing. We shall see that the transformations $R(i_1, \ldots, i_{2k})$ correspond to the action of Dehn twists on $H_1(F, \mathbb{Z}/2\mathbb{Z})$, where we identify the generators e_i with α_i. First note that any element γ of $H_1(F, \mathbb{Z}/2\mathbb{Z})$ can be expressed as $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_m}$. Observe that a simple closed curve C representing γ is orientation preserving if and only if $\gamma \cdot \gamma = 0$, which is equivalent to m being even.

Now let C be an orientation preserving curve on F and consider the Dehn twist τ about C. Let $\gamma = [C] \in H_1(F, \mathbb{Z}/2\mathbb{Z})$ be the element represented by C. By the above (as $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_m}$ and m is even), we can express γ as $\gamma = \alpha_{i_1} + \cdots + \alpha_{i_{2k}}$. If α is another element of $H_1(F, \mathbb{Z}/2\mathbb{Z})$ and $\alpha \cdot \gamma$ is the (mod 2) intersection number, then (with mod 2 coefficients) $\tau_*(\alpha) = \alpha + \gamma$. It is easy to see that $\tau_* = R(i_1, \ldots, i_{2k})$. Note that $\tau_*^2(\alpha) = \alpha + 2\gamma = \alpha$, hence $\tau_* = R(i_1, \ldots, i_{2k})$ is an involution as claimed.

Now, by Theorem 1.3, any element $\phi \in \mathcal{O}$ is induced by a homeomorphism f of F. Further, by a theorem of Lickorish [3], f is homotopic to a composition of Dehn twists and cross-cap slides. We have seen that Dehn twists induce the automorphisms $\tau_* = R(i_1, \ldots, i_{2k})$ on V. It is easy to see that cross-cap slides induce the identity on $H_1(F, \mathbb{Z}/2\mathbb{Z})$. Thus ϕ is a composition of elements of the form $\tau_* = R(i_1, \ldots, i_{2k})$ as claimed. □

Remark 5.2. We can alternatively deduce Theorem 1.3 from Theorem 5.1 as the generators of $\mathcal{O}$ can be represented by homeomorphisms (namely Dehn twists).

6. Punctured surfaces

Let F be a compact non-orientable surface with m boundary components and let $\beta_j \in H_1(F, \mathbb{Z})$, $1 \leq j \leq m$, be elements representing the boundary curves. A homeomorphism $f\colon F \to F$ induces an automorphism $\varphi = f_*$ of $H_1(F, \mathbb{Z})$. Furthermore, as boundary components of F are mapped to boundary components by f (possibly reversing orientations), for some permutation σ of $\{1, \ldots, m\}$ and some constants $\epsilon_j = \pm 1$, $\varphi(\beta_j) = \epsilon_j \beta_{\sigma(j)}$, for all j, $1 \leq j \leq m$.

We show that conversely any automorphism φ that preserves the (mod 2) intersection pairing and takes boundary components to boundary components is induced by a homeomorphism.

Theorem 6.1. *Let F be a compact non-orientable surface with m boundary components and let φ be an automorphism of $H_1(F, \mathbb{Z}/2\mathbb{Z})$ that preserves the* (*mod* 2) *intersection pairing. Suppose for some permutation σ of $\{1, \ldots, m\}$ and some constants $\epsilon_j = \pm 1$, we have $\varphi(\beta_j) = \epsilon_j \beta_{\sigma(j)}$, for all $1 \leq j \leq m$. Then φ is induced by a homeomorphism of F.*

Proof. Let $\bar{F}$ be obtained from F by attaching discs to all the boundary components. Then we can assume that F has been obtained from $\bar{F}$ by deleting the interiors of m discs D_1, $\ldots$ D_m, all of which are contained in a disc $E \subset \bar{F}$. Further we can assume that the central curves γ_i, $1 \leq i \leq n$ in a decomposition of $\bar{F}$ into projective planes are disjoint from E, as are all the Dehn twists and cross-cap slides we perform on $\bar{F}$ in the proof of Theorem 1.1. Hence the Dehn twists and cross-cap slides we perform give homeomorphisms of F which are the identity on the boundary components.

Let $\alpha_i = [\gamma_i]$ and let $\bar{\alpha}_i$ be the images of these elements in $H_1(\bar{F}, \mathbb{Z})$. By choosing appropriate orientations, we get that $H_1(F, \mathbb{Z})$ is generated by the elements α_i and β_j with the relation

$$2\sum_i \alpha_i = \sum_j \beta_j. \tag{6.1}$$

Note that as $H_1(\bar{F}, \mathbb{Z}) = H_1(F, \mathbb{Z})/\langle \beta_j \rangle$, it follows by the hypothesis that φ induces an automorpism $\bar{\varphi}$ of $H_1(\bar{F}, \mathbb{Z})$. By Theorem 1.1 (and its proof), this is induced by a composition of Dehn twists and cross-cap slides, hence a homeomorphism $g\colon F \to F$. By composing φ by g_*^{-1}, we can assume that $\bar{\varphi}$ is the identity.

Similarly, we can use homeomorphisms supported in E (which do not change any α_i) to reduce to the case when the permutation σ is the identity, i.e. $\varphi(\beta_j) = \epsilon_j \beta_j$. As $\bar{\varphi}(\bar{\alpha}_j) = \bar{\alpha}_j$, we get $\varphi(\alpha_i) = \alpha_i + \sum_j c_{ij} \beta_j$ for some integers c_{ij}. We define the complexity of φ to be $C(\varphi) = \sum_{i,j} |c_{ij}|$.

If φ is not the identity, we shall reduce the complexity of φ using homeomorphisms called *boundary slides* [2] similar to cross-cap slides.

Lemma 6.2. *There are homeomorphisms h_{ij} of F such that the induced automorphism of $H_1(F, \mathbb{Z})$ takes α_i to $\alpha_i - \beta_j$, maps β_j to $-\beta_j$ and fixes all other α's and β's.*

Proof. We shall use boundary slides [2] of the surface F. Namely, suppose α is an orientation reversing simple closed curve on a surface S' and D is a small disc centered around a point on α. Let S be the surface obtained by deleting the interior of D. Consider a homeomorphism of S' which is the identity outside a neighbourhood of α and which is obtained by dragging D once around α so that D is mapped to itself. By construction this extends to a homeomorphism of S, which we call a boundary slide.

As in the case of cross-cap slides, the automorphism of $H_1(F, \mathbb{Z})$ induced by the boundary slide of the boundary component corresponding to β_j along the simple closed curve γ_i (representing α_i) is as in the statement of the lemma. $\square$

Now suppose φ is not the identity. Observe that as φ is a homomorphism, $2\sum_i \varphi(\alpha_i) = \sum_j \varphi(\beta_j)$. Using $\varphi(\alpha_i) = \alpha_i + \sum_j c_{ij}\beta_j$, $\varphi(\beta_j) = \epsilon_j\beta_j$ and $2\sum_i \alpha_i = \sum_j \beta_j$, we see that $\sum_j c_{ij}\beta_j = (\epsilon_j - 1)\beta_j$. As the elements β_j, $1 \leq j \leq n$ are independent, it follows that for each j, $\sum_i c_{ij} = \epsilon_j - 1$.

We now consider two cases. Firstly, if some $\epsilon_j = -1$, then observe that postcomposing with h_{ij} takes $\varphi(\alpha_i)$ to $\varphi(\alpha_i) - \varphi(\beta_j) = \varphi(\alpha_i) + \beta_j$. Hence c_{ij} is changed to $c_{ij} + 1$ (and no other c_{kl} is changed). In particular, if $c_{ij} < 0$, the complexity is reduced. But as $\sum_i c_{ij} = \epsilon_j - 1 = -2$, we must have some $c_{ij} < 0$, and hence a move reducing complexity.

Suppose now that each β_j is 1. Then as $\sum_i c_{ij} = \epsilon_j - 1 = 0$, either each $c_{ij} = 0$, in which case we are done, or some $c_{ij} > 0$. Observe that postcomposing with h_{ij} takes $\varphi(\alpha_i)$ to $\varphi(\alpha_i) - \varphi(\beta_j) = \varphi(\alpha_i) - \beta_j$. Hence c_{ij} is changed to $c_{ij} - 1$ (and no other c_{kl} is changed), and hence the complexity is reduced. Thus in finitely many steps, we reduce to the case where φ is the identity. $\square$

Acknowledgements

We would like to thank Shreedhar Inamdar for helpful conversations.

References

[1] Birman J S and Chillingworth D R J, On the homotopy group of a non-orientable surface, *Math. Proc. Cambridge Philos. Soc.* **136** (1972) 437–448

[2] Korkmaz M, Mapping class groups of nonorientable surfaces, *Geometriae Dedicata* **89** (2002) 109–133

[3] Lickorish W B R, Homeomorphisms of non-orientable two-manifold, *Math. Proc. Cambridge Philos. Soc.* **59** (1963) 307–317

[4] Lickorish W B R, Homeomorphisms of non-orientable two-manifold, *Math. Proc. Cambridge Philos. Soc.* **61** (1965) 61–64