Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 1, February 2005, pp. 93–102.

On two functionals connected to the Laplacian in a class of doubly connected domains in space-forms

M H C ANISA and A R AITHAL

Department of Mathematics, University of Mumbai, Mumbai 400 098, India
E-mail: anisa@sankhya.mu.ac.in; aithal@math.mu.ac.in

MS received 7 September 2004; revised 15 December 2004

Abstract. Let B_1 be a ball of radius r_1 in $S^n(\mathbb{H}^n)$, and let B_0 be a smaller ball of radius r_0 such that $\overline{B_0} \subset B_1$. For S^n we consider $r_1 < \pi$. Let u be a solution of the problem $-\Delta u = 1$ in $\Omega := B_1 \setminus \overline{B_0}$ vanishing on the boundary. It is shown that the associated functional $J(\Omega)$ is minimal if and only if the balls are concentric. It is also shown that the first Dirichlet eigenvalue of the Laplacian on Ω is maximal if and only if the balls are concentric.

Keywords. Eigenvalue problem; Laplacian; maximum principles.

1. Introduction

Let (M, g) be a Riemannian manifold and let D denote the Levi–Civita connection of (M, g). For a smooth vector field X on M the divergence $\mathrm{div}(X)$ is defined as $\mathrm{trace}(DX)$. For a smooth function $f\colon M \longrightarrow \mathbb{R}$, the gradient ∇f is defined by $g(\nabla f(p), v) = \mathrm{d}f(p)(v)$ $(p \in M,\ v \in T_pM)$ and the Laplace–Beltrami operator Δ is defined by $\Delta f = \mathrm{div}(\nabla f)$. Further, $\nabla^2 f$ denotes the Hessian of f. Throughout this paper, ω and $\mathrm{d}V$ denote the volume element of (M, g).

Let $\Omega \subset M$ be a domain such that $\bar{\Omega}$ is a smooth compact submanifold of M. The Sobolev space $H^1(\Omega)$ is defined as the closure of $\mathcal{C}^\infty(\bar{\Omega})$ (the space of real valued smooth functions on $\bar{\Omega}$) with respect to the Sobolev norm

$$\|f\|_{H^1(\Omega)} = \left(\int_\Omega \{f^2 + \|\nabla f\|^2\}\,\mathrm{d}V\right)^{1/2} \quad (f \in \mathcal{C}^\infty(\bar{\Omega})).$$

The closure of $\mathcal{C}_0^\infty(\Omega)$ (the space of real valued smooth functions on Ω having compact support in Ω) in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. The Sobolev space $H^2(\Omega)$ is defined as the closure of $\mathcal{C}^\infty(\bar{\Omega})$ with respect to the Sobolev norm

$$\|f\|_{H^2(\Omega)} = \left(\int_\Omega \{f^2 + \|\nabla f\|^2 + \|\nabla^2 f\|^2\}\,\mathrm{d}V\right)^{1/2} \quad (f \in \mathcal{C}^\infty(\bar{\Omega})).$$

These spaces are Hilbert spaces with the corresponding norms.

Consider the Dirichlet boundary value problem on Ω:

$$\left.\begin{aligned} -\Delta u &= 1 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.1}$$

Let $u \in H_0^1(\Omega)$ be the unique weak solution of problem (1.1). By Theorem 4.8, p. 105 of [1], $u \in \mathcal{C}^\infty(\bar{\Omega})$.

Consider the following eigenvalue problem on Ω:

$$\left.\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

The eigenvalues of the positive Laplace–Beltrami operator $-\Delta = -\mathrm{div}(\nabla f)$ are strictly positive. The eigenfunctions corresponding to the first eigenvalue λ_1 are proportional to each other. They belong to $\mathcal{C}^\infty(\bar{\Omega})$ and they are either strictly positive or strictly negative on Ω. Moreover,

$$\lambda_1 = \inf\{\, \|\nabla\phi\|^2_{L^2(\Omega)} \mid \phi \in H_0^1(\Omega),\ \|\phi\|^2_{L^2(\Omega)} = 1\}$$

(cf. [1], Theorem 4.4, p. 102). Let $y := y(\Omega) \in \mathcal{C}^\infty(\bar{\Omega})$ be the unique solution of problem (1.1). Let $y_1 := y_1(\Omega)$ be the unique solution of problem (1.2), corresponding to the first eigenvalue $\lambda_1 := \lambda_1(\Omega)$, characterized by

$$y_1 > 0 \quad \text{on } \Omega \quad \text{and} \quad \int_\Omega y_1^2 \,\mathrm{d}V = 1.$$

The aim of this paper is to prove the main results of [3] for simply connected spherical and hyperbolic space-forms.

Consider the unit sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ with induced Riemannian metric $\langle\, ,\, \rangle$ from the Euclidean space $\mathbb{R}^{n+1}$. Also consider the hyperbolic space $\mathbb{H}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$ with the Riemannian metric induced from the quadratic form $(x, y) := \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$, where $x = (x_1, x_2, \ldots, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_{n+1})$.

Fix $0 < r_0 < r_1$. We choose $r_1 < \pi$ for the case of S^n. Let B_1 be any ball of radius r_1 in $S^n(\mathbb{H}^n)$ and B_0 be any ball of radius r_0 such that $\overline{B_0} \subset B_1$. Consider the family $\mathcal{F} = \{B_1 \setminus \overline{B_0}\}$ of domains in $S^n(\mathbb{H}^n)$. We study the extrema of the following functionals:

$$J(\Omega) = -\int_\Omega \{\|\nabla y(\Omega)\|^2 - 2y(\Omega)\}\,\mathrm{d}V, \tag{1}$$

$$J_1(\Omega) = -\int_\Omega \{\|\nabla y_1(\Omega)\|^2 - 2\lambda_1(\Omega)[y_1(\Omega)]^2\}\,\mathrm{d}V \tag{2}$$

on $\mathcal{F}$, associated to problems (1.1) and (1.2) respectively. Note here that the functionals J and J_1 are nothing but negative of the energy functional $\int_\Omega \|\nabla y(\Omega)\|^2\,\mathrm{d}V$ and the Dirichlet eigenvalue λ_1, respectively.

We state our main results: Put $\Omega_0 = B(p, r_1) \setminus \overline{B(p, r_0)}$ for any fixed $p \in S^n(\mathbb{H}^n)$.

Theorem 1. *The functional $J(\Omega)$ on $\mathcal{F}$ assumes minimum at Ω if and only if $\Omega = \Omega_0$, i.e., when the balls are concentric.*

Theorem 2. *The functional $J_1(\Omega)$ on $\mathcal{F}$ assumes maximum at Ω if and only if $\Omega = \Omega_0$, i.e., when the balls are concentric.*

In §§2 and 3, following [5], we develop the ‘shape calculus’ for Riemannian manifolds for the stationary problem (1.1) and the eigenvalue problem (1.2) respectively. In §4, we prove Theorems 1 and 2 for S^n, and make the necessary remarks to carry out the proofs of Theorems 1 and 2 for $\mathbb{H}^n$.

2. Shape calculus for the stationary problem

Let V be a smooth vector field on M having compact support. Let $\Phi\colon \mathbb{R}\times M \longrightarrow M$ be the smooth flow for V. For each $t \in \mathbb{R}$, denote $\Phi(t,x)$ by $\Phi_t(x)$ $(x \in M)$. Let Ω be an open subset of M such that $\bar{\Omega}$ is a smooth compact submanifold of M. Put $\Omega_t := \Phi_t(\Omega)$ $(t \in \mathbb{R})$.

Let $\mathcal{D}$ be a domain in M such that $\operatorname{supp} V \subset \mathcal{D}$. Fix $f \in \mathcal{C}^\infty(\mathcal{D})$. Consider the Dirichlet boundary value problem on Ω_t:

$$\left.\begin{aligned} \Delta u = f \quad &\text{on } \ \Omega_t, \\ u = 0 \quad &\text{on } \ \partial\Omega_t. \end{aligned}\right\} \tag{2.1}$$

Let $y_t \in \mathcal{C}^\infty(\bar{\Omega}_t)$ be the unique solution of problem (2.1) (cf. [1], Theorem 4.8, p. 105). Throughout this section $y := y(\Omega)$ denotes the unique solution of (2.1) for $t = 0$.

Denote $y_t \circ \Phi_t|_\Omega$ by y^t $(t \in \mathbb{R})$.

PROPOSITION 2.1

The map $t \longmapsto y^t$ is a $\mathcal{C}^1$-curve in $H^2(\Omega) \cap H_0^1(\Omega)$ from a neighbourhood of 0 *in* $\mathbb{R}$.

Proof. By problem (2.1), for each $t \in \mathbb{R}$, y_t satisfies the equation

$$\int_{\Omega_t} g(\nabla y_t, \nabla\psi)\, \mathrm{d}V = -\int_{\Omega_t} f\psi\, \mathrm{d}V \quad \forall\, \psi \in \mathcal{C}_0^\infty(\Omega_t). \tag{3}$$

There exists smooth function $\gamma_t\colon M \longrightarrow (0,\infty)$ such that $\Phi_t^*\omega = \gamma_t\omega$ (here, $\omega := \mathrm{d}V$, the volume element of (M,g)). Put $B_t := (D\Phi_t)^{-1}$, $B_t^* =$ transpose of B_t (i.e., $g(B_t(x)v, w) = g(v, B_t^*(x)w)$ $\ \forall v \in T_x\Omega_t$, $w \in T_{x'}\Omega$, where $x' := \Phi_t^{-1}(x)$) and $A_t := \gamma_t B_t B_t^*$. By the change of variable $\Phi_t\colon \Omega \longrightarrow \Omega_t$, eq. (3) can be re-written as

$$\int_\Omega -\operatorname{div}(A_t\nabla(y_t\circ\Phi_t))\ \psi\circ\Phi_t\ \mathrm{d}V = -\int_\Omega f\circ\Phi_t\psi\circ\Phi_t\gamma_t\ \mathrm{d}V.$$

Therefore, $y^t := y_t \circ \Phi_t\colon \Omega \longrightarrow \mathbb{R}$ satisfies

$$\left.\begin{aligned} -\operatorname{div}(A_t\nabla y^t) + f\circ\Phi_t\gamma_t = 0 \quad &\text{on } \ \Omega, \\ y^t = 0 \quad &\text{on } \ \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

Define $F\colon \mathbb{R} \times H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$ by $F(t,u) = -\operatorname{div}(A_t\nabla u) + f\circ\phi_t\gamma_t$. Then F is a $\mathcal{C}^1$-map. Further $D_2F|_{(0,y)}(0,u) = -\operatorname{div}(\nabla u)$ (recall $y = y(\Omega)$). By the standard theory of Dirichlet boundary value problem on compact Riemannian manifolds ([1], Theorem 4.8, p. 105 and [2], Theorem 7.32, p. 259),

$$D_2F|_{(0,y)}\colon H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$$

is an isomorphism. By (2.2), $F(t,y^t) = 0\ \forall t$. Proposition 2.1 now follows by the implicit function theorem. □

DEFINITION

$\dot{y}(\Omega,V) := \left(\frac{\mathrm{d}}{\mathrm{d}t}y^t\right)\Big|_{t=0} \in H_0^1(\Omega)$ is called the (strong) *material derivative of* y in the direction of V.

Consider $\Omega' \subset\subset \Omega$.

PROPOSITION 2.2

The map $t \longmapsto y_t|_{\Omega'}$ *is a* $\mathcal{C}^1$*-curve in* $H^1(\Omega')$ *from a neighbourhood of* 0 *in* $\mathbb{R}$ *and* $\mathrm{d}/\mathrm{d}t|_{t=0}\,(y_t|_{\Omega'}) = \{\dot{y}(\Omega, V) - g(\nabla y, V)\}|_{\Omega'}$.

Proof. There exists $\delta > 0$ such that $\Omega' \subset \Phi_t(\Omega)$ $\forall\,|t| < \delta$. Then $y_t|_{\Omega'} = y^t \circ \Phi_{-t}|_{\Omega'}$ $\forall\,|t| < \delta$. Proposition 2.2 now follows from Proposition 2.1 and Proposition 2.38, p. 71 of [5]. □

DEFINITION

$y'(\Omega, V) := \dot{y}(\Omega, V) - g(\nabla y, V) \in H^1(\Omega)$ is called the *shape derivative of* y in the direction of V.

Consider the domain functional $J(\Omega_t)$ defined by $J(\Omega_t) := \int_{\Omega_t} y_t\, \mathrm{d}V$ $(t \in \mathbb{R})$.

DEFINITION

The *Eulerian derivative* $\mathrm{d}J(\Omega, V)$ of $J(\Omega_t)$ at $t = 0$ is defined as

$$\mathrm{d}J(\Omega, V) := \lim_{t \longrightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

PROPOSITION 2.3

The function $J(\Omega_t)$ *is differentiable at* $t = 0$ *and* $\mathrm{d}J(\Omega, V) = \int_\Omega y'\, \mathrm{d}V$.

Proof. Let $L_V\omega$ denote the Lie derivative of ω with respect to V, and $i_V\omega$ denote the interior multiplication of ω with respect to V. Then

$$\frac{\mathrm{d}}{\mathrm{dt}}(\Phi_t^*\omega)|_{t=0} =: L_V\omega = (d\, i_V + i_V\, d)\,\omega = \mathrm{d}(i_V\omega) = \mathrm{div}(V)\,\omega.$$

Hence, by Propositions 2.1 and 2.2 we get

$$\begin{aligned}\mathrm{d}J(\Omega, V) &= \lim_{t \longrightarrow 0} \int_\Omega \left\{ \frac{y^t\ \Phi_t^*\omega - y\omega}{t} \right\} = \int_\Omega \left(\frac{\mathrm{d}}{\mathrm{dt}} \{y^t\ \Phi_t^*\omega\} \right)_{|t=0} \\ &= \int_\Omega \{\dot{y} + y\,\mathrm{div}(V)\}\, \mathrm{d}V = \int_\Omega \{y' + g(\nabla y, V) + y\,\mathrm{div}(V)\}\, \mathrm{d}V \\ &= \int_\Omega y'\, \mathrm{d}V + \int_\Omega \mathrm{d}(y\, i_V\omega) = \int_\Omega y'\, \mathrm{d}V.\end{aligned}$$ □

PROPOSITION 2.4

The shape derivative $y' = y'(\Omega, V)$ *is the weak solution of the Dirichlet boundary value problem*

$$\left.\begin{aligned}\Delta v &= 0 \quad \text{on } \Omega, \\ v|_{\partial\Omega} &= -\tfrac{\partial y}{\partial n}\, g(V, n)\end{aligned}\right\} \tag{2.3}$$

in the space $H^1(\Omega)$. (*Here,* n *is the outward unit normal field on* $\partial\Omega$).

Proof. Consider $\psi \in \mathcal{C}_0^\infty(\Omega)$ having support in a domain $\Omega' \subset\subset \Omega$. There exists $\delta > 0$ such that $\Omega' \subset \Omega_t \ \forall\ |t| < \delta$. By problem (2.1),

$$\int_{\Omega'} g(\nabla y_t, \nabla\psi)\ \mathrm{d}V = -\int_{\Omega'} f\ \psi\ \mathrm{d}V \quad \text{for } |t| < \delta. \tag{4}$$

By Proposition 2.2, differentiation of LHS of eq. (4) with respect to t at $t = 0$ can be carried out under the integral sign. So we get

$$\int_{\Omega'} g(\nabla y', \nabla\psi)\ \mathrm{d}V = 0.$$

Thus y' satisfies $\Delta y' = 0$ weakly on Ω.

Now $\dot{y}, y \in H^2(\Omega) \cap H_0^1(\Omega)$, and $y' = \dot{y} - g(\nabla y, V) \in H^1(\Omega)$. So by Proposition 2.39, p. 88 of [2], we get

$$y'|_{\partial\Omega} = \dot{y}|_{\partial\Omega} - g(\nabla y, V)|_{\partial\Omega} \quad \text{and} \quad \dot{y}|_{\partial\Omega} = 0.$$

Also, $y \in \mathcal{C}^\infty(\bar{\Omega})$ and $y = 0$ on $\partial\Omega$ by (2.1). So, $g(\nabla y, V)|_{\partial\Omega} = \frac{\partial y}{\partial n}\, g(V, n)$. Thus, $y'|_{\partial\Omega} = -\frac{\partial y}{\partial n}\, g(V, n)$. □

3. Shape calculus for the eigenvalue problem

Let (M, g), V, Φ_t, Ω, Ω_t, γ_t, A_t be as in §2. Consider problem (1.2) posed in Ω_t:

$$\left.\begin{aligned} -\Delta u &= \lambda u \quad \text{on } \ \Omega_t, \\ u &= 0 \qquad \text{on } \ \partial\Omega_t. \end{aligned}\right\} \tag{3.1}$$

Let $\lambda_1(t) := \lambda_1(\Omega_t)$ and $y_1(t) := y_1(\Omega_t)$ be as in §1. We denote $y_1(\Omega)$ by y_1 and $\lambda_1(\Omega)$ by λ_1 throughout this section.

Denote $y_1(t) \circ \Phi_t|_\Omega$ by y_1^t $(t \in \mathbb{R})$.

PROPOSITION 3.1

The map $t \longmapsto \left(\lambda_1(t)\,,\ y_1^t\right)$ *is a* $\mathcal{C}^1$*-curve in* $\mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$ *from a neighbourhood of* 0 *in* $\mathbb{R}$.

Proof. By problem (3.1), for each $t \in \mathbb{R}$, $y_1(t)$ satisfies the equation

$$\int_{\Omega_t} g(\nabla y_1(t), \nabla\psi)\ \mathrm{d}V = \int_{\Omega_t} \lambda_1(t)\, y_1(t)\, \psi\ \mathrm{d}V \quad \forall\, \psi \in H_0^1(\Omega_t). \tag{5}$$

As in the proof of Proposition 2.1, eq. (5) can be re-written as

$$-\int_\Omega \mathrm{div}(A_t \nabla y_1^t)\, \psi\ \mathrm{d}V = \int_\Omega \lambda_1(t)\, y_1^t\, \gamma_t\, \psi\ \mathrm{d}V \quad \forall\, \psi \in H_0^1(\Omega). \tag{6}$$

Therefore, $t \longmapsto \left(\lambda_1(t)\,,\ y_1^t\right)$ satisfies

$$\left.\begin{aligned} \mathrm{div}(A_t \nabla y_1^t) + \lambda_1(t)\, y_1^t\, \gamma_t = 0 \quad \text{on } \ \Omega, \\ \textstyle\int_\Omega \left(y_1^t\right)^2 \gamma_t\ \mathrm{d}V = 1. \end{aligned}\right\} \tag{3.2}$$

Let $X := \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$. Define F: $\mathbb{R} \times X \longrightarrow L^2(\Omega) \times \mathbb{R}$ by $F(t, \mu, u) = \big(\mathrm{div}(A_t \nabla u) + \mu u \gamma_t, \int_\Omega u^2 \gamma_t \,\mathrm{d}V - 1\big)$. Then F is a $\mathcal{C}^1$-map. Further $D_2 F|_{(0, \lambda_1, y_1)}$ $(0, \mu, u) = \big(\Delta u + \lambda_1 u + \mu y_1\ ,\ 2\int_\Omega y_1 u \,\mathrm{d}V\big)$.

Claim. $D_2 F|_{(0, \lambda_1, y_1)}$: $\mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega) \times \mathbb{R}$ is an isomorphism.

Let $(v, b) \in L^2(\Omega) \times \mathbb{R}$ be arbitrary. Consider the following problem:

$$\left.\begin{aligned} \Delta u + \lambda_1 u + \mu y_1 &= v \quad \text{on } \Omega, \\ 2\int_\Omega y_1 u \,\mathrm{d}V &= b. \end{aligned}\right\} \tag{3.3}$$

Now by Fredholm alternative, $\Delta u + \lambda_1 u = v - \mu y_1$ has a solution in $H^2(\Omega) \cap H_0^1(\Omega)$ if and only if $v - \mu y_1 \perp y_1$ in $L^2(\Omega)$. So, for $\mu_0 := \int_\Omega v y_1 \,\mathrm{d}V$ there exists $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\Delta u_1 + \lambda_1 u_1 + \mu_0 y_1 = v$. Moreover, the solutions of $\Delta u + \lambda_1 u + \mu_0 y_1 = v$ are of the form $u = u_1 + a y_1, a \in \mathbb{R}$. Given $b \in \mathbb{R}$ there exists a unique $a_0 := b/2 - \int_\Omega y_1 u_1 \,\mathrm{d}V \in \mathbb{R}$ such that $2\int_\Omega y_1 u \,\mathrm{d}V = b$. Put $u_0 = u_1 + a_0 y_1$. Thus for $(v, b) \in L^2(\Omega) \times \mathbb{R}$ there exists a unique $(\mu_0, u_0) \in \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$ such that $D_2 F|_{(0, \lambda_1, y_1)}(0, \mu_0, u_0) = (v, b)$. This proves the claim.

By (3.2), $F(t, \lambda_1(t), y_1^t) = 0\ \forall t$. Proposition 3.1 now follows by the implicit function theorem. □

DEFINITION

$\dot{y}_1(\Omega, V) := ((\mathrm{d}/\mathrm{d}t) y_1^t)|_{t=0} \in H_0^1(\Omega)$ is called the (strong) *material derivative of* y_1 in the direction of V.

Consider $\Omega' \subset\subset \Omega$.

PROPOSITION 3.2

The map $t \longmapsto y_1(t)|_{\Omega'}$ *is a* $\mathcal{C}^1$*-curve in* $H^1(\Omega')$ *from a neighbourhood of* 0 *in* $\mathbb{R}$ *and* $((\mathrm{d}/\mathrm{d}t)[y_1(t)|_{\Omega'}])|_{t=0} = (\dot{y}_1 - g(\nabla y_1, V))\,|_{\Omega'} \in H^1(\Omega')$. *Further,* y_1' *satisfies* $y_1' = \dot{y}_1 - g(\nabla y_1, V)$ *in* $H^1(\Omega)$ *and* $y_1'|_{\partial\Omega} = -\frac{\partial y_1}{\partial n}\, g(V, n)$.

Proof. There exists $\delta > 0$ such that $\Omega' \subset \Phi_t(\Omega)\ \forall\, |t| < \delta$. The first part of Proposition 3.2 follows from Proposition 3.1 and Proposition 2.38, p. 71 of [5]. Now as $\dot{y}_1 \in H^1(\Omega)$ and $\nabla y_1 \in \mathcal{C}^\infty(\bar{\Omega})$, we get $y_1' = \dot{y}_1 - g(\nabla y_1, V) \in H^1(\Omega)$. Hence, $y_1'|_{\partial\Omega} = \dot{y}_1|_{\partial\Omega} - g(\nabla y_1, V)|_{\partial\Omega} = -\frac{\partial y_1}{\partial n}\, g(V, n)$. □

DEFINITION

The *shape derivative of* y_1 in the direction of V is the element $y_1' = y_1'(\Omega, V) \in H^1(\Omega)$ defined by $y_1' = \dot{y}_1 - g(\nabla y_1, V)$.

PROPOSITION 3.3

The shape derivative $y_1' \in H^1(\Omega)$ *satisfies*

$$-\Delta y_1' = \lambda_1 y_1' + \lambda_1' y_1 \quad \textit{on } \Omega$$

in the sense of distributions.

Proof. Let $\psi \in \mathcal{C}_0^\infty(\Omega)$. Let $\Omega' \subset\subset \Omega$ be a domain such that $\operatorname{supp}\psi \subset \Omega'$. As $y_1(t)$ is a solution of problem (1.2) posed in Ω_t, for t sufficiently small we get

$$\int_{\Omega'} g(\nabla y_1(t), \nabla\psi)\,\mathrm{d}V = \int_{\Omega'} \lambda_1(t)\, y_1(t)\,\psi\,\mathrm{d}V. \tag{7}$$

By Propositions 3.1 and 3.2, we can differentiate with respect to t under the integral sign in eq. (7). Thus we have

$$\int_{\Omega'} g(\nabla y_1', \nabla\psi)\,\mathrm{d}V = \int_{\Omega'} (\lambda_1 y_1' + \lambda_1' y_1)\,\psi\,\mathrm{d}V.$$

Hence,

$$-\int_{\Omega} y_1'\,\Delta\psi\,\mathrm{d}V = \int_{\Omega} (\lambda_1 y_1' + \lambda_1' y_1)\,\psi\,\mathrm{d}V \quad \forall\,\psi \in \mathcal{C}_0^\infty(\Omega). \qquad \square$$

PROPOSITION 3.4

$$y_1' \in \mathcal{C}^\infty(\bar{\Omega}).$$

Proof. By Proposition 3.2, $y_1' = \dot{y}_1 - g(\nabla y_1, V)$ on Ω. Hence it is enough to prove that $\dot{y}_1 \in \mathcal{C}^\infty(\bar{\Omega})$. Consider $L := \Delta + \lambda_1$, a linear elliptic operator of order 2. Then $\dot{y}_1 \in H_0^1(\Omega)$ satisfies $L(\dot{y}_1) = L(y_1' + g(\nabla y_1, V)) = -\lambda_1' y_1 + L(g(\nabla y_1, V))$, by Proposition 3.3. From Proposition 3.58, p. 87 of [1], it follows that $\dot{y}_1 \in \mathcal{C}^\infty(\bar{\Omega})$. $\square$

PROPOSITION 3.5

$$\lambda_1' = -\int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n}\right)^2 g(V, n)\,\mathrm{d}S.$$

Proof. We write $\lambda_1' = \lambda_1' \int_\Omega y_1^2\,\mathrm{d}V$. By Proposition 3.3, $\lambda_1' = \int_\Omega \{-\Delta y_1' - \lambda_1 y_1'\}\, y_1\,\mathrm{d}V$. Hence by problem (1.2) and Proposition 3.4, we get

$$\begin{aligned}\lambda_1' = \int_\Omega \{-y_1\,\Delta y_1' + y_1'\,\Delta y_1\}\,\mathrm{d}V &= \int_{\partial\Omega} \left\{ y_1' \frac{\partial y_1}{\partial n} - y_1 \frac{\partial y_1'}{\partial n} \right\} \mathrm{d}S \\ &= \int_{\partial\Omega} y_1' \frac{\partial y_1}{\partial n}\,\mathrm{d}S.\end{aligned}$$

Now the result follows by Proposition 3.2. $\square$

4. Proofs of Theorem 1 and Theorem 2 for S^n

Proof of Theorem 1 *for* S^n. We continue with the notations of §1 such as $r_0, r_1, \mathcal{F}$, and $y(\Omega)$, $J(\Omega)$ for $\Omega \in \mathcal{F}$ for S^n. For $|t| < \pi$, put $p := (0, \ldots, 0, 1)$ and $q(t) = (0, \ldots, 0, \sin t, \cos t) \in S^n$. The Laplace–Beltrami operator Δ of $(S^n, \langle,\rangle)$ is invariant under isometries of S^n. So we need to study the functional J only on domains $\Omega(q(t)) := B(r_1) \setminus \overline{B(q(t), r_0)}$, $0 \le |t| < r_1 - r_0$, where $B(r_1) := B(p, r_1)$.

We define $j\colon (r_0 - r_1, r_1 - r_0) \longrightarrow \mathbb{R}$ by $j(t) = J(\Omega(q(t)))$.

Fix t_0 such that $0 \leq t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Fix r_2 such that $r_0 < r_2 < r_1 - t_0$ and consider a smooth function $\rho\colon S^n \longrightarrow \mathbb{R}$ satisfying $\rho = 1$ on $\overline{B(q(t_0), r_2)}$ and $\rho = 0$ on $\partial B(r_1)$. Let V denote the vector field on S^n defined by $V(x) = \rho(x)\,(0, \ldots, 0, x_{n+1}, -x_n)\ \forall x = (x_1, \ldots, x_{n+1}) \in S^n$. Let $\{\Phi_t\}_{t\in\mathbb{R}}$ be the one-parameter family of diffeomorphisms of S^n associated with V. Then for t sufficiently close to 0, $J(\Phi_t(\Omega)) = j(t_0 + t)$. Note that $J(\Phi_t(\Omega)) = \int_{\Omega_t} y_t\ \mathrm{d}V$, hence by Proposition 2.3, j is differentiable at t_0.

Note that j is an even function which is differentiable at 0. Hence $j'(0) = 0$.

Now onwards we fix t_0 such that $0 < t_0 < r_1 - r_0$ and consider $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Let n denote the outward unit normal of Ω on $\partial\Omega$. For $x \in \partial B_0$, put $a = d(p, x)$ and $\alpha =$ the angle at p of the spherical triangle $T := [p, q(t_0), x]$ with vertices p, $q(t_0)$ and x. Then $n(x) = (q(t_0) - \cos r_0\, x)/\sin r_0$ and $\langle V, n\rangle(x) = (\cos a\ \sin t_0 - \sin a\ \cos t_0\ \cos\alpha)/\sin r_0$. Hence, by eq. (19) on p. 30 of [6], we get

$$\langle V, n\rangle(x) = \cos\beta(x), \tag{8}$$

where $\beta(x)$ denotes the angle at $q(t_0)$ of the spherical triangle T defined above.

By Proposition 2.3, $j'(t_0) = \int_\Omega y'\ \mathrm{d}V$. Hence by Proposition 2.4 and problem (1.1),

$$\begin{aligned}\int_\Omega y'\,\mathrm{d}V = -\int_\Omega \left\{y'\Delta y - y\,\Delta y'\right\}\mathrm{d}V &= -\int_{\partial\Omega}\left\{y'\,\frac{\partial y}{\partial n} - y\,\frac{\partial y'}{\partial n}\right\}\mathrm{d}S\\ &= -\int_{\partial\Omega} y'\,\frac{\partial y}{\partial n}\,\mathrm{d}S.\end{aligned}$$

Again by Proposition 2.4 and eq. (8) above, we get

$$j'(t_0) = \int_{x\in\partial B_0}\left(\frac{\partial y}{\partial n}(x)\right)^2 \cos\beta(x)\ \mathrm{d}S. \tag{9}$$

Let H denote the hyperplane in $\mathbb{R}^{n+1}$ through $(0, \ldots, 0)$ having $q'(t_0)$ as a normal vector. Let r_H denote the reflection of S^n about H. Put $\mathcal{O} = \{x \in \Omega \mid \langle x, q'(t_0)\rangle > 0\}$. Then $r_H(\mathcal{O}) \subset B(r_1)$ and $r_H(\overline{B_0}) = \overline{B_0}$. For $x \in \partial B_0 \cap \partial\mathcal{O}$, let x' denote $r_H(x)$. Note that for all $x \in \partial B_0 \cap \partial\mathcal{O}$, $\cos\beta(x) < 0$ and $\cos\beta(x') = -\cos\beta(x)$. Thus eq. (9) can be re-written as

$$j'(t_0) = \int_{x\in\partial B_0\cap\partial\mathcal{O}}\left\{\left(\frac{\partial y}{\partial n}(x)\right)^2 - \left(\frac{\partial y}{\partial n}(x')\right)^2\right\}\ \cos\beta(x)\ \mathrm{d}S. \tag{10}$$

The Laplace–Beltrami operator Δ of S^n is uniformly elliptic on S^n and hence the maximum principle ([4], Theorem 5, p. 61) and the Hopf maximum principle ([4], Theorem 7, p. 65) are applicable on $\bar{\Omega}$. Hence, by arguments analogous to [3] at this stage, we get

$$\left|\frac{\partial y}{\partial n}(x)\right| < \left|\frac{\partial y}{\partial n}(x')\right| \quad \forall\ x \in \partial B_0 \cap \partial\mathcal{O}.$$

Thus from eq. (10), $j'(t_0) > 0$. This completes the proof of Theorem 1 for S^n. $\square$

Proof of Theorem 2 *for* S^n. We continue with the notations of §1 such as $\lambda_1(\Omega)$, $y_1(\Omega)$ and $J_1(\Omega)$ for $\Omega \in \mathcal{F}$. Let p, $q(t)$ be as in the proof of Theorem 1. Define j_1: $(r_0 -$

$r_1,\ r_1 - r_0) \longrightarrow \mathbb{R}$ by $j_1(t) = J_1(\Omega(q(t)))$. As in the proof of Theorem 1, fix t_0 such that $0 \le t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Then for t sufficiently close to 0 we have $j_1(t_0 + t) = J_1(\Phi_t(\Omega)) = \lambda_1(\Phi_t(\Omega))$. By Proposition 3.1, j_1 is differentiable at $t = t_0$ and $j_1'(t_0) = \lambda_1'(\Omega)$. As $\lambda_1(\Phi_t(\Omega)) = \lambda_1(\Phi_{-t}(\Omega))$, j_1 is an even function which is differentiable at 0. Thus $j_1'(0) = 0$.

Now onwards we fix t_0 such that $0 < t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Then by Proposition 3.5 and eq. (8), we get

$$j_1'(t_0) = \lambda_1'(\Omega) = -\int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n}\right)^2 \langle V, n\rangle \,\mathrm{d}S = -\int_{\partial B_0} \left(\frac{\partial y_1}{\partial n}\right)^2 \cos\beta(x)\,\mathrm{d}S. \tag{11}$$

As in the proof of Theorem 1, eq. (11) can be re-written as

$$j_1'(t_0) = -\int_{x\in\partial B_0\cap\partial\mathcal{O}} \left\{ \left(\frac{\partial y_1}{\partial n}(x)\right)^2 - \left(\frac{\partial y_1}{\partial n}(x')\right)^2 \right\} \cos\beta(x)\,\mathrm{d}S. \tag{12}$$

The Laplace–Beltrami operator Δ of S^n is uniformly elliptic on S^n. So, the Hopf maximum principle ([4], Theorem 7, p. 65) and the generalised maximum principle ([4], Theorem 10, p. 73) are applicable on $\bar{\Omega}$. Hence, by arguments analogous to [3] we get

$$\left|\frac{\partial y_1}{\partial n}(x)\right| < \left|\frac{\partial y_1}{\partial n}(x')\right| \quad \forall\, x \in \partial B_0 \cap \partial\mathcal{O}.$$

It follows from eq. (12) that $j_1'(t_0) < 0$. The proof of Theorem 2 is now complete for S^n. □

Remark on proofs of Theorem 1 *and Theorem* 2 *for* $\mathbb{H}^n$. For $t \in \mathbb{R}$, define $q(t) = (0, \dots, 0, \sinh t, \cosh t) \in \mathbb{H}^n$. Put $p := q(0)$ and $q := q(t_0)$ $(t_0 > 0)$. Define the vector field V on $\mathbb{H}^n$ by $V(x) = \rho(x)\,(0, \dots, 0, x_{n+1}, x_n)$ $\forall x = (x_1, \dots, x_{n+1}) \in \mathbb{H}^n$, where $\rho\colon \mathbb{H}^n \longrightarrow \mathbb{R}$ is as in the proof of Theorem 1 for S^n.

Let n denote the inward unit normal of $B(q, r_0)$ on $\partial B(q, r_0)$. Then,

$$n(x) = (q - \cosh r_0\, x)/\sinh r_0$$

and

$$\langle V, n\rangle\,(x) = (x_{n+1}\ \sinh t_0 - x_n\ \cosh t_0\,)/\sinh r_0 = \cos\beta(x),$$

where $\beta(x)$ denotes the angle at q of the hyperbolic triangle $[p, q, x]$ with vertices p, q and x.

Now Theorems 1 and 2 for the hyperbolic case can be proved using shape calculus of §§2 and 3 as in the case of sphere. □

Acknowledgement

The work of the first author is supported by research scholarship from the National Board for Higher Mathematics NBHM/RAwards.2001/643(1).

References

[1] Aubin T, Nonlinear analysis on manifolds – Monge–Ampere equations (Springer-Verlag) (1982)
[2] Folland G, Introduction to partial differential equations (New Delhi: Prentice-Hall of India Private Limited) (2001)
[3] Kesavan S, On two functionals connected to the Laplacian in a class of doubly connected domains, *Proc. R. Soc. Edinburgh* **133A** (2003) 617–624
[4] Protter M and Weinberger H, Maximum principles in differential equations (New Jersey: Prentice-Hall Inc.) (1967)
[5] Sokolowski J and Zolesio J-P, Introduction to shape optimization – shape sensitivity analysis (Springer-Verlag) (1992)
[6] Todhunter I, Spherical Trigonometry (London: Macmillan and Co. Ltd) (1949)