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Some remarks on good sets

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Abstract. It is shown that (1) if a good set has finitely many related components, then they are full, (2) loops correspond one-to-one to extreme points of a convex set. Some other properties of good sets are discussed.

Keywords. Good set; full set; related component; loop; relatively full set.

Introduction and preliminaries

In this note we make some remarks on good sets in n-fold Cartesian product as defined in [2]. We need the following definitions:

Let $X_1, X_2, \ldots, X_n$ be non-empty sets and let $\Omega = X_1 \times X_2 \times \cdots \times X_n$ be their Cartesian product. For each i, $1 \leq i \leq n$, Π_i will denote the canonical projection of Ω onto X_i. A subset $S \subset \Omega$ is said to be *good*, if every complex valued function f on S is of the form:

$$f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), (x_1, x_2, \ldots, x_n) \in S,$$

for suitable functions $u_1, u_2, \ldots, u_n$ on $X_1, X_2, \ldots, X_n$ respectively ([3], p. 181).

A subset S of Ω is said to be *full*, if S is a maximal good set in $\Pi_1 S \times \Pi_2 S \times \cdots \times \Pi_n S$ ([3], p. 183).

Two points x, y in a good set S are said to be *related*, denoted by xRy, if there exists a finite subset of S, which is full and contains both x and y. R is an equivalence relation, whose equivalence classes we call as the *related components* of S. Note that related components of S are full subsets of S ([3], p. 190).

Remark 1. Here we prove:

Theorem 1. *If a full set F has finitely many related components*: $F = \cup_{i=1}^{k} R_i$, *then* $k = 1$.

To prove this we need some preliminary results.

Let S be a good set, $S = \cup R_\alpha$ where R_α are its related components. Define an equivalence relation E_i on $\Pi_i S$ as follows: $x_i E_i y_i$ if there exists a finite sequence $R_1, R_2, \ldots, R_k$ such that $x_i \in \Pi_i R_1$, $y_i \in \Pi_i R_k$ and $\Pi_i R_j \cap \Pi_i R_{j+1} \neq \emptyset$ for $1 \leq j \leq k-1$ ([3], p. 189). For $x_i \in \Pi_i S$, $[x_i]$ denotes the E_i-equivalence class of x_i. If an element in $\Pi_i R_\alpha$ is E_i-equivalent to an element in $\Pi_i R_\beta$, we will say that R_α and R_β are E_i-equivalent. Let C be a cross-section of R_α's. Let $\mathcal{F}_i$ be the set of all E_i-equivalence classes for $1 \leq i \leq n$. Define ϕ: $C \to \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ by $f(x_1, \ldots, x_n) = ([x_1], \ldots, [x_n])$.

Lemma 1. *$\phi(C)$ is good. S is full if and only if $\phi(C)$ is full.*

Proof. The ϕ: $C \mapsto \phi(C)$ is one-to-one: If $(x_1, \ldots, x_n) \neq (y_1, \ldots, y_n)$ are in C then they belong to two different related components. If $([x_1], \ldots, [x_n]) = ([y_1], \ldots, [y_n])$ then these two related components, say R_1 and R_2, have all the coordinates equivalent which is not possible. To prove this, let us define a function h which is equal to zero everywhere in S except on R_1 where it is a non-zero constant. There exists functions u_i defined on $\Pi_i S$, $1 \leq i \leq n$, such that $u_1 + \cdots + u_n = h$ on S. As the function h is constant on each R_α, the u_i are constants on $\Pi_i R_\alpha$ ([3], p. 185, Corollary 2). Let $c_1, \ldots, c_n$ be these constants $\Pi_i R_2$, $1 \leq i \leq n$. As all the coordinates of R_1 and R_2 are equivalent we get the same constants $c_1, \ldots c_n$ on the coordinates of R_1. But $c_1 + \cdots + c_n = 0$ and h is non-zero on R_1. This contradicts the fact that $h = u_1 + u_2 + \cdots + u_n$. Therefore ϕ is one-to-one.

Next we show that $\phi(C)$ is good. Take a function h on $\phi(C)$. This defines in a natural manner a function on C. Denote it also by h. Define g on S by taking it as a constant on each R_α, i.e., $g(y_1, \ldots, y_n) = h(x_1, \ldots, x_n)$ for all $(y_1, \ldots, y_n) \in R_\alpha$ where $(x_1, \ldots, x_n)$ is in $C \cap R_\alpha$. There exists $u_1, \ldots, u_n$ such that $u_1 + \cdots + u_n = g$ on S as S is good. Since g is constant on each R_α, u_i is constant on $\Pi_i R_\alpha$. Define $v_i([x_i]) = u_i(x_i)$ for all x_i, $1 \leq i \leq n$. These functions are well-defined. Further,

$$\begin{aligned} v_1([x_1]) + \cdots + v_n([x_n]) &= u_1(x_1) + \cdots + u_n(x_n) \\ &= g(x_1, \ldots, x_n) = h([x_1], \ldots, [x_n]). \end{aligned}$$

This shows $\phi(C)$ is good.

Suppose S is full. If $\phi(C)$ is not full, then given the zero function on $\phi(C)$ there exist two distinct sets of functions $\{v_i\}$ and $\{v_i'\}$ defined on the ith coordinate space of $\phi(C)$ for each i whose sum is equal to zero which also satisfy

$$v_i([x_i^0]) = v_i'([x_i^0]) = 0$$

for some $[x_i^0] \in \Pi_i(\phi(C))$ for $1 \leq i \leq n-1$. Define $\{u_i\}$ and $\{u_i'\}$ on the ith coordinate space $\Pi_i S$ by

$$u_i(x_i) = v_i([x_i]) \text{ and } u_i'(x_i) = v_i'([x_i])$$

for $1 \leq i \leq n$. Then the sum of u_i as well as u_i' is equal to zero but they are different solutions (with the same boundary conditions) because v_i and v_i' are different. This contradicts the fact that S is full.

Conversely, if $\phi(C)$ is full, then we prove S is also full. For this, take the zero function on S. Suppose there are two distinct sets of functions $\{u_i\}$ and $\{u_i'\}$ with

$$u_1 + \cdots + u_n = u_1' + \cdots + u_n' = 0$$

on S with $u_i(x_i^0) = u_i'(x_i^0) = 0$ for some $x_i^0 \in \Pi_i(S)$, $1 \leq i \leq n-1$. All the functions u_i and u_i' are constant on each $\Pi_i R_\alpha$ so also on each E_i equivalence class. Define $v_i([x_i]) = u_i(x_i)$ and $v_i'([x_i]) = u_i'(x_i), \forall i$. Then $\{v_i\}$ and $\{v_i'\}$ are distinct solutions of the zero function on $\phi(C)$ which also satisfy $v_i([x_i^0]) = v_i'([x_i^0]) = 0$ for $1 \leq i \leq n-1$. Since $\phi(C)$ is full, the functions v_i and v_i' are the same which implies u_i and u_i' are equal. So S is full. $\square$

DEFINITION

For a finite good set S, we call the cardinality of $\cup_{i=1}^{n} \Pi_i S$ as the *number of coordinates* of S and denote it by $N(S)$. The cardinality of $\Pi_i S$ is called the *number of i-th coordinates* of S.

A finite good set S is full if and only if $N(S) - (n-1) = |S|$. If S is good then $N(S) - (n-1) \geq |S|$. For a finite set S if $|S| > N(S) - (n-1)$ then S is not good ([2], p. 80).

Proof of Theorem 1. Suppose $F = \cup_{i=1}^{k} R_i$ is full. We want to show that there is a finite, full subset S of F which intersects each R_i. If $k > 1$ this will be a contradiction to the fact that R_i's are related components. Since F is full, by lemma 1, $\phi(C)$ is full. It has k points and dimension n. So the total number of coordinates in $\phi(C)$ is $k + (n-1)$. Let these coordinates be labeled as $\alpha_1, \ldots, \alpha_{k+n-1}$ in some order. There are k points each having n entries and each of these nk entries should be one of these $k+n-1$ coordinates of $\phi(C)$. So we get a partition of nk as $nk = l_1 + l_2 + \cdots + l_{(k+(n-1))}$ where l_i denotes the number of times α_i is repeated. When a coordinate, say $[x_j] \in \Pi_j \phi(C)$, is repeated in $\phi(C)$, it means the corresponding two related components of F are E_j-equivalent. If $[x_j]$ occurs l times in $\phi(C)$ then l number of related components of F are E_j-equivalent. For this, it is necessary to have at least $l-1$ different pairs of related components (R_α, R_β) such that $\Pi_j R_\alpha \cap \Pi_j R_\beta \neq \emptyset$:

Suppose l related components are E_j-equivalent. Consider a graph whose vertices are these related components and whose edges are pairs of related components among these which have at least one common jth coordinate. This graph is connected because the related components are E_j-equivalent. The number of vertices is l so there should be at least $l-1$ edges in it.

In this way we get totally (at least) $l_1 - 1 + l_2 - 1 + \cdots + l_{(k+(n-1))} - 1 = nk - (k + (n-1))$ pairs of related components (R_α, R_β) such that for some i, $\Pi_i R_\alpha \cap \Pi_i R_\beta \neq \emptyset$. For each such pair (R_α, R_β) take one point from each of the two related components R_α and R_β such that the chosen points have the same ith coordinate. All these points together form a finite subset of F. The intersection of this set with each R_i is also finite and non-empty. (Note that since F is full, each R_α has a common coordinate with some other related component.) Take the finite full set $F_i \subset R_i$ which contains this intersection. (Any finite subset of a related component is contained in a finite full set.) Let $S = \cup_{i=1}^{k} F_i$. Then S is a finite subset of F.

To show that S is full we have to find the number of coordinates of S and the number of points in S. Let A_i denote the number of coordinates of F_i. Then, since F_i is full, the number of points in F_i is $A_i - (n-1)$. So the number of points in S is $|S| = A - k(n-1)$ where $A = A_1 + \cdots + A_k$. Now the number of coordinates of S is no more than A. In this counting, if F_α and F_β have a common coordinate, then this common coordinate will be counted once each in A_α and A_β. But we know that there are at least $nk - (k + (n-1))$ such pairs F_α, F_β. So the number of coordinates of S is at most

$$A - (nk - (k + (n-1))) = A - (n-1)(k-1) = |S| + (n-1).$$

But the number of coordinates of S cannot be lesser than this: if it is the case S will not be good. This shows the number of coordinates of S is equal to $|S| + (n-1)$. So S is full. If $k > 1$, this is a contradiction as noted at the beginning of the proof. □

Remark 2. Here we show the connection between loops and extreme points of a convex set. We need the following definitions.

Given any finitely many symbols $t_1, t_2, \ldots, t_k$ with *repetitions* allowed and given any finitely many integers $n_1, n_2, \ldots, n_k$, we say that the formal sum $n_1t_1 + n_2t_2 + \cdots + n_kt_k$ *vanishes*, if for every t_j the sum of the coefficients of t_j is equal to zero ([3], p. 183).

DEFINITION

Let $\Omega = X_1 \times X_2 \times \cdots \times X_n$. A non-empty finite subset $L = \{x_1, x_2, \ldots, x_k\}$ of Ω is called a *loop*, if there exist non-zero integers $n_1, n_2, \ldots, n_k$ such that the sum $\sum_{j=1}^{k} n_j x_j$ vanishes in the sense that the formal sum vanishes coordinate-wise and no strictly smaller non-empty subset of L has this property ([3], p. 183).

Lemma 2. *Let $L = \{x_1, \ldots, x_k\}$ be a loop. Then there is a unique* (*except for the sign*) *set of integers $n_1, \ldots, n_k$ with $gcd(n_1, \ldots, n_k) = 1$ such that the formal sum $\sum_{j=1}^{k} n_j x_j$ vanishes.*

Proof. Suppose there are two sets of integers $\{n_j\}$ and $\{m_j\}$ with these properties. Also assume there is a p for which $|n_p| \neq |m_p|$. Then $\sum_{j=1}^{k} n_j x_j = 0$ and $\sum_{j=1}^{k} m_j x_j = 0$ imply $\sum_{j=1}^{k} (m_p n_j - n_p m_j) x_j = 0$, where the co-efficient of x_p vanishes. As L is a loop any proper subset of L is not a loop. So $m_p n_j = n_p m_j$ for $1 \leq j \leq n$. Then $gcd(m_p n_1, \ldots, m_p n_k) = gcd(n_p m_1, \ldots, n_p m_k)$, i.e., $|m_p| = |n_p|$. We have to prove that either for all j, $m_j = n_j$ or for all $j, m_j = -n_j$. Suppose $m_j = n_j$ for some l number of j's for $0 < l < n$ and $m_j = -n_j$ for the remaining $n - l$ number of j's. Then adding the equations $\sum_{j=1}^{k} n_j x_j = 0$ and $\sum_{j=1}^{k} m_j x_j = 0$ we get a smaller formal sum (containing only l terms) to be zero which is a contradiction to the minimality of the loop $\sum_{j=1}^{k} n_j x_j = 0$. □

Lemma 3. *Let $L = \{x_1, \ldots, x_k\}$ be a loop. Let n_j be as in Lemma* 2 *above. Suppose the formal sum $\sum_{j=1}^{k} r_j x_j = 0$ for some real numbers r_j. Then there exists a real number α such that $r_j = \alpha n_j$ for each j.*

Proof. If we assume that r_j's are rationals, then the result is easy to prove.

To prove the general case, note that the formal sum $\sum_{j=1}^{k} r_j x_j = 0$ gives a set of N homogeneous equations – one for each coordinate in $\cup_{i=1}^{n} \Pi_i L$ where $N = |\cup_{i=1}^{n} \Pi_i L|$. The matrix corresponding to this set of equations gives a linear map from R^k to R^N. This matrix consists only of 0's and 1's and so can be thought of as a linear map from Q^k to Q^N the kernel of which is one-dimensional (by the result for the rational case). It means that the rank of the matrix is $k - 1$. This is also the rank of the matrix, when the linear map is considered from R^k to R^N. The null space of this matrix is one-dimensional, i.e., there exists some α such that $r_j = \alpha n_j$ for $1 \leq j \leq k$. □

This also shows that $\sum_{j=1}^{m} r_j x_j = 0$ is not possible for $m < k$ even for real r_j's. The above proof in fact shows that a finite set $\{x_1, x_2, \ldots, x_k\}$ of points in $X_1 \times X_2 \times \cdots \times X_n$ is a loop if and only if there is a finite set of non-zero real numbers $r_1, r_2, \ldots, r_k$ such that the formal sum $\sum_{j=1}^{k} r_j x_j$ vanishes and no proper subset of $\{x_1, x_2, \ldots, x_k\}$ has this property. If we assume that $\sum_{j=1}^{k} |r_j| = 1$ then $|\alpha| = (\sum_{j=1}^{k} |n_j|)^{-1}$.

Let $S \subset X_1 \times \cdots \times X_n$ be a finite set, not necessarily good. Let $C(S)$ be the set of all functions on S. The norm in $C(S)$ is defined by $\|f\| = \max_{x \in S} |f(x)|$. Let $U(S) = \{f \in C(S) | f(x_1, \ldots, x_n) = u_1(x_1) + \cdots + u_n(x_n)$ where u_i is a function on $\Pi_i(S)\}$. $U(S)$ is a subspace of $C(S)$. Let $M(S)$ denote the space of all signed measures on S with the total variation norm (which is just the L_1 norm). Then $M(S) = (C(S))^*$. Take the subspace $(U(S))^\perp \subset M(S)$. This is the set of all signed measures μ with $\mu(f) = 0, \forall f \in U(S)$. Note that $\mu \in (U(S))^\perp$ if and only if all the one-dimensional marginals of μ vanish. Consider

$$A = \{\mu \in (U(S))^\perp : \|\mu\| \leq 1\}.$$

This set is convex.

DEFINITION

By a *weak loop* we mean a finite set $\{x_1, x_2, \ldots, x_l\} \in X_1 \times X_2 \times \cdots \times X_n$ for which there exist real numbers $r_1, r_2, \ldots, r_l$, with at least one r_i non-zero, such that the formal sum $\sum_{i=1}^{l} r_i x_i = 0$, coordinate-wise.

Theorem 2. *The extreme points of A are given by μ_L where $L = \{x_1, \ldots, x_k\}$ is a loop. In this case*

$$\mu_L(x_j) = n_j(|n_1| + \cdots + |n_k|)^{-1},$$

where $(n_1, \ldots, n_k)$ are given by $\sum_{j=1}^{k} n_j x_j = 0$, and for all other $x \in S$, $\mu_L(x) = 0$.

Proof. First we note that $\mu_L \in (U(S))^\perp$: It is enough to show that $\mu_L(u_i) = 0$ where u_i is a function on $\Pi_i S$, and this is easily verified from the form of μ_L and the fact that L is a loop. To show that μ_L is an extreme point of A suppose $\mu_L = a\lambda + b\nu$ where $a + b = 1, a > 0, b > 0$ and $\lambda, \nu \in A$. Then $\|\lambda\|, \|\nu\| = 1$ and $\lambda, \nu \in (U(S))^\perp$. Restricting all these three measures to L, we have $\mu_L = \mu_L|_L = a\lambda|_L + b\nu|_L$, and since $\|\mu_L\| = 1$, we have $\|\lambda|_L\| = \|\nu|_L\| = 1$. This shows that λ and ν are supported on L. Denote $\lambda(x_j) = \alpha_j, \nu(x_j) = \beta_j, 1 \leq j \leq k$. Since λ, ν are in $U(S)^\perp$, their marginals vanish, which is equivalent to saying that the formal sum $\sum_{j=1}^{k} \alpha_j x_j = 0$, $\sum_{i=1}^{k} \beta_j x_j = 0$. Since $\sum_{j=1}^{k} |\alpha_j| = 1$, $\sum_{j=1}^{k} |\beta_j| = 1$. By the above lemma $\lambda = +\nu$ or $\lambda = -\nu$, and since $a, b > 0$ we see that $\mu_L = \lambda = \nu$, and μ_L is an extreme point of A.

Conversely, take an extreme point μ of A. Then $\|\mu\| = 1$. We show that support of μ is a weak loop. Let $\{x_1, \ldots, x_k\}$ be the support of μ. Since $\mu \in (U(S))^\perp$, for any i, if u_i is a function on $\Pi_i(S)$ then $\mu(u_i) = 0$, i.e., $\sum_{j=1}^{k} \mu(x_j) u_i(x_j) = 0$. Take $u_i(x_{1i}) = 1$ and $u_i(x) = 0$ for all other $x \in \Pi_i(S)$. Then $\mu(u_i) = \sum \mu(x_j) = 0$ where the sum runs over all x_j for which $x_{ji} = x_{1i}$. With similar arguments for other x_{ji} we see that the formal sum $\sum_{j=1}^{k} \mu(x_j) x_j = 0$.

Now we prove that this weak loop has to be a loop. Call $\mu(x_j)$ as m_j. Then we have $\sum_{j=1}^{k} m_j x_j = 0$. Suppose this is not a loop. We prove that the measure μ is not an extreme point of A. Any weak loop contains a loop. Let $\sum n_j x_j = 0$ be this loop. Here the sum runs over a proper subset of $\{1, \ldots, k\}$. Then taking $n_j = 0$ whenever necessary $\sum_{j=1}^{k} m_j x_j = \sum_{j=1}^{k} n_j x_j + \sum_{j=1}^{k} (m_j - n_j) x_j$ is a sum of two weak loops. Note that the two weak loops on the right side are not multiples of each other. Let μ_1, μ_2 be the measures corresponding to $\sum_{j=1}^{k} n_j x_j$ and $\sum_{j=1}^{k} (m_j - n_j) x_j$ respectively. That is, $\mu_1(x_j) = n_j$, $\mu_2(x_j) = m_j - n_j$ for $1 \leq j \leq k$ and for any other x, $\mu_1(x) = \mu_2(x) = 0$. Clearly $\mu_1, \mu_2 \in (U(S))^\perp$. Then $\mu_1 + \mu_2 = \mu$ and $\|\mu\| = 1 \leq \|\mu_1\| + \|\mu_2\|$. If for each j the coefficients n_j and $m_j - n_j$

of x_j on the right-hand side have the same sign, then $|m_j| = |n_j| + |m_j - n_j|$ for each j so that $\|\mu\| = 1 = \|\mu_1\| + \|\mu_2\|$. Then we can write

$$\sum_{j=1}^{k} m_j x_j = \|\mu_1\| \left(\sum_{j=1}^{k} n_j x_j (\|\mu_1\|)^{-1} \right) + \|\mu_2\| \left(\sum_{j=1}^{k} (m_j - n_j) x_j (\|\mu_2\|)^{-1} \right)$$

which shows μ is not an extreme point of A.

Now we show that the measure which is supported on a weak loop can be written as a sum of two measures μ_1 and μ_2, both in $U(S)^{\perp}$, with $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. We already know that $\sum_{j=1}^{k} m_j x_j$ can be written as

$$\sum_{j=1}^{k} m_j x_j = \sum_{j=1}^{k} n_j x_j + \sum_{j=1}^{k} r_j x_j,$$

where the two weak loops on the right-hand side are not multiples of each other. In this representation we want $|m_j| = |n_j| + |r_j|$ for each j, i.e., n_j and r_j should have the same sign. Suppose for some j_0 this does not happen. Let us assume $n_{j_0} > 0$, $r_{j_0} < 0$ and $|r_{j_0}| > |n_{j_0}|$. Then we can write

$$\sum_{j=1}^{k} m_j x_j = \sum_{j=1}^{k} (n_j - (n_{j_0}/r_{j_0}) r_j) x_j + \sum_{j=1}^{k} (r_j + (n_{j_0}/r_{j_0}) r_j) x_j.$$

The right-hand side is a sum of two weak loops; the first does not contain the term x_{j_0} and the second contains $m_{j_0} x_{j_0}$. If for some j, n_j and r_j have the same sign, then since $(n_{j_0}/r_{j_0}) < 0$, $-(n_{j_0}/r_{j_0}) r_j$ has the same sign as r_j so $n_j - (n_{j_0}/r_{j_0}) r_j$ has the same sign as n_j. Also $r_j + (n_{j_0}/r_{j_0}) r_j$ has the same sign as r_j because $|(n_{j_0}/r_{j_0})| < 1$. This shows that the new coefficients of x_j in the new weak loops have the same sign if they had the same sign in the original weak loops. Also because the original weak loops are not multiples of each other, these two weak loops cannot have all the coefficients equal to 0. In this way we get another representation of the left-hand side as a sum of weak loops with lesser number of j's for which n_j and r_j have opposite signs. Again the two weak loops are not multiples of each other because x_{j_0} is present in the second weak loop but not in the first. Applying the same procedure repeatedly we get the two weak loops with all j having the n_j and r_j of the same sign. This proves μ is not an extreme point of A. $\square$

Remark 3. Here we discuss some properties of a maximal good set contained in a given set.

Let $S \subset X_1 \times \cdots \times X_n$, S not necessarily good. Consider the collection $\mathcal{G}$ of good subsets of S. This collection is closed under arbitrary increasing unions, hence by Zorn's lemma there exists a maximal set M in $\mathcal{G}$. Note that $\Pi_i(M) = \Pi_i(S)$ for $1 \leq i \leq n$. Denote the space of all functions on M as $C(M)$. Call a function f *good* if for each i there is a function u_i on $\Pi_i S$ such that

$$f(x_1, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n)$$

for all $(x_1, \ldots, x_n) \in S$. Let $U(S)$ denote the class of good functions on S.

Theorem 3. *The map $f \mapsto f|M$ is a one-to-one linear map from $U(S)$ onto $C(M)$.*

Proof. Clearly this map is linear. It is also onto. To prove this, take any function g on M. Then $g = u_1 + \cdots + u_n$. Here u_i are defined on $\Pi_i(M)$ which is same as $\Pi_i(S)$. Then f defined on S by $f = u_1 + \cdots + u_n$ has the property that $f|M = g$. Because M is a maximal good set, any $x \in S \backslash M$ forms a loop with some finitely many elements of M. Let this loop be $\{x, y_2, \ldots, y_k\}$ where y_j are from M. Then $n_1 x + \sum_{j=2}^{k} n_j y_j = 0$ for some integers n_j. This loop is unique because if $\{x, z_2, \ldots, z_l\}$ where z_j are from M is another loop, then $m_1 x + \sum_{j=2}^{l} m_j z_j = 0$ for some integers m_j. Multiplying the first equation by m_1 and the second by n_1 and subtracting one from the other we get a weak loop in M which is not possible because M is good.

Given the zero function on M, there is a unique extension of this function to the whole of S which is in $U(S)$ (namely, the zero function): Let $0 = u_1 + \cdots + u_n$ on M. By taking $f = u_1 + \cdots + u_n$ we can extend this function to S. Take a point x in $S \backslash M$ and the loop $\{x, y_2, \ldots, y_k\}$ it makes with the elements of M. Then the formal sum $n_1 x + \sum_{j=2}^{k} n_j y_j = 0$ which gives $n_1 x_i + \sum_{j=2}^{k} n_j y_{ji} = 0$ for each $1 \le i \le n$, where x_i and y_{ji} denote the ith coordinate of x and y_j respectively. This gives $x_i = -\sum_{j=2}^{k} (n_j/n_1) y_{ji}$ formally. Then

$$\sum_{i=1}^{n} u_i(x_i) = \sum_{i=1}^{n} u_i \left(-\sum_{j=2}^{k} (n_j/n_1) y_{ji} \right).$$

We can also write

$$\sum_{i=1}^{n} u_i(x_i) = -\sum_{j=2}^{k} (n_j/n_1) \sum_{i=1}^{n} u_i(y_{ji})$$

because $\sum n_j/n_1 = 0$ when the sum is taken over those j for which y_{ji} fixed and $\neq x_i$ and $\sum -n_j/n_1 = 1$ when the sum is over those j for which $y_{ji} = x_i$. But $\sum_{i=1}^{n} u_i(y_{ji}) = 0$ as y_j are in M. So we get $\sum_{i=1}^{n} u_i(x_i) = 0$. This shows $f = \sum_{i=1}^{n} u_i = 0$ on S. So the zero function on M has a unique extension to a function in $U(S)$. It follows that the map $f \mapsto f|_M$ is one-to-one from $U(S)$ to $C(M)$, and the theorem is proved.

Let $C(S)$ denote the set of all functions on S. Let $U(S)^\perp = \{\mu \in C(S)^* | \mu(f) = 0, \forall f \in U(S)\}$. □

Theorem 4. *The dimension of $U(S)^\perp$ is $|S| - |M|$ when S is finite. A basis for $U(S)^\perp$ is given by the set of μ_L where L is a loop of the form $\{x, y_2, \ldots, y_k\}$ where $x \in S \backslash M$ and $y_2, \ldots, y_k$ are in M.*

Proof. The dimension of $C(S)$ is $|S|$ and that of $U(S)$ is $|M|$ by the previous theorem. The space $U(S)^\perp$ is equivalent to $(C(S)/U(S))^*$. Therefore $\dim(U(S)^\perp) = \dim(C(S)) - \dim(U(S))$, which is equal to $|S| - |M|$. Every $x \in S \backslash M$ makes a unique loop L_x with suitable elements from M. These loops give rise to $|S| - |M|$ measures in $U(S)^\perp$ by the theorem in Remark 2. They are linearly independent: If $\sum c_j \mu_{L_{x_j}} = 0$ then $\sum c_j \mu_{L_{x_j}}(x) = 0, \forall x \in S$. Taking $x = x_i$ we get $\mu_{L_{x_j}}(x_i) = 0, \forall j \neq i$ and $\mu_{L_{x_i}}(x_i) \neq 0$. This gives $c_i = 0$. Therefore, these $|S| - |M|$ measures are linearly independent and so form a basis for $U(S)^\perp$. This proves the theorem. □

DEFINITION

A set $S \subset X_1 \times \cdots \times X_n$ is called relatively full if there exist $x_i^0 \in \Pi_i S$, $1 \le i \le n-1$ such that any $f \in U(S)$ has a unique representation as $f = u_1 + \cdots + u_n$ when we fix the value of $u_i(x_i^0)$, $1 \le i \le n-1$.

It is easy to see that if S is relatively full then for any choice of elements $x_i^0 \in \Pi_i S$, $1 \le i \le n-1$ the solution of $f = u_1 + u_2 + \cdots + u_n$, $f \in U(S)$ is unique if we fix the values of $u_i(x_i^0)$, $1 \le i \le n-1$. Moreover, if the solution is unique, with the prescribed constraints, for the zero function, then it is unique for all functions in $U(S)$.

Theorem 5. *If S is relatively full then any maximal good set $M \subset S$ is full.*

Proof. Take the zero function on M. Fix $u_i(x_i^0) = 0$ for $1 \le i \le n-1$. Let $0 = \sum_{i=1}^n u_i$ on M. It can be uniquely extended to S as a function f in $U(S)$. Then $f = \sum_{i=1}^n u_i$ on S with $u_i(x_i^0) = 0$ for $1 \le i \le n-1$. These u_i's are unique because S is relatively full. Therefore M is full. This proves the theorem.

Any set $S \subset X_1 \times \cdots \times X_n$ can be written uniquely as a disjoint union $S = \cup R_\alpha$ of maximal relatively full sets R_α of S: A one point set of S is relatively full. Union of a chain of relatively full sets is again relatively full. Using Zorn's lemma, there exist maximal relatively full sets in S. As union of two relatively full sets with non-empty intersection is again relatively full, these maximal relatively full sets of S do not intersect each other and their union is S.

1. R_α and R_β for $\alpha \neq \beta$ cannot have $n-1$ coordinates in common.
2. For $n = 2$, $\Pi_i R_\alpha \cap \Pi_i R_\beta = \emptyset$ if $\alpha \neq \beta$.
3. In each R_α, there exists a maximal good set F_α which is also full.
4. Although each F_α is good and full, $\cup F_\alpha \subset S$ need not be good, except when $n = 2$ in which case $\cup F_\alpha$ is a maximal good subset of S. Consider the set $S = \{(0,0,0), (0,0,1), (1,1,0), (1,1,1)\}$ Here $R_1 = F_1 = \{(0,0,0), (0,0,1)\}$ $R_2 = F_2 = \{(1,1,0), (1,1,1)\}$. But $F_1 \cup F_2$ is not good.
5. If bounded functions f on M have bounded solution of
$$f = u_1 + u_2 + \cdots + u_n, \tag{$*$}$$
then clearly bounded functions f in $U(S)$ will have bounded solution of $(*)$.)
6. Suppose $X_1, X_2, \ldots, X_n$ are compact topological spaces. Let $S \subset X_1 \times \ldots \times X_n$ be compact and $M \subset S$ a maximal good set. If continuous functions f on M have continuous solution of $(*)$ then clearly continuous function in $U(S)$ have continuous solution of $(*)$.
7. Let $M \subset S$ is maximal good set and let D be a boundary of M (refer § 4 of [3], for the definition of boundary). Let f be in $U(S)$. Let g_i be defined on $D \cap \Pi_i S = D \cap \Pi_i M$ for $1 \le i \le n$. Then there exists a unique set $v_1, \ldots, v_n$ such that $f = v_1 + \cdots + v_n$ on S and $v_i|_{D \cap \Pi_i S} = g_i$.
8. For any choice of F_α's (F_α as above), $\cup F_\alpha$ is a maximal good set of S if and only if L is a loop in S implies $L \subset R_\alpha$ for some α.

Proof. Suppose $\{x_1, \ldots, x_k\}$ is a loop in S, and $\{x_1, \ldots, x_k\} \cap R_\alpha \neq \emptyset$ for more than one α. Then each such intersect is good because it is part of a loop. Let F_α be a maximal full set in R_α such that $\{x_1, \ldots, x_k\} \cap R_\alpha \subset F_\alpha$. (A good subset of a set is contained in a maximal good subset of that set.) Then $\cup F_\alpha$ contains a loop so is not good. Conversely suppose that any given loop in S is contained in some R_α. Then $\cup F_\alpha$ cannot contain a loop because if it contains a loop then it is in some R_α, hence in some F_α. But F_α cannot contain a loop.

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