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Non-linear second-order periodic systems with non-smooth potential

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Abstract. In this paper we study second order non-linear periodic systems driven by the ordinary vector p-Laplacian with a non-smooth, locally Lipschitz potential function. Our approach is variational and it is based on the non-smooth critical point theory. We prove existence and multiplicity results under general growth conditions on the potential function. Then we establish the existence of non-trivial homoclinic (to zero) solutions. Our theorem appears to be the first such result (even for smooth problems) for systems monitored by the p-Laplacian. In the last section of the paper we examine the scalar non-linear and semilinear problem. Our approach uses a generalized Landesman–Lazer type condition which generalizes previous ones used in the literature. Also for the semilinear case the problem is at resonance at any eigenvalue.

Keywords. Ordinary vector p-Laplacian; non-smooth critical point theory; locally Lipschitz function; Clarke subdifferential; non-smooth Palais–Smale condition; homoclinic solution; problem at resonance; Poincaré–Wirtinger inequality; Landesman–Lazer type condition.

1. Introduction

In a recent paper [28], we proved existence and multiplicity results for non-linear second-order periodic systems driven by the one-dimensional p-Laplacian and having a non-smooth potential. Our results there extended to the recent works of Tang [31,32], who examined semilinear (i.e. $p = 2$) systems with smooth potential. In this paper we continue the study of non-linear, non-smooth periodic systems. We prove new existence theorems under more general growth conditions on the non-smooth potential. In [28] all the results assumed a strict sub-p growth (i.e. strictly sublinear potential in the semilinear ($p = 2$) case). Here the growth conditions are more general. Also we obtain new multiplicity results and we also establish the existence of non-trivial homoclinic solutions. Our approach is variational and it is based on the non-smooth critical point theory of Chang [4]. Extensions of this theory were obtained recently by Kourogenis and Papageorgiou [17] and Kourogenis *et al* [18].

Problems with non-differentiable potential which is only locally Lipschitz in the state variable $x \in \mathbb{R}^N$, are known as 'hemivariational inequalities' and have applications in mechanics and engineering. For details in this direction we refer to the book of Naniewicz and Panagiotopoulos [27].

In the last decade there has been an increasing interest for problems involving the one-dimensional p-Laplacian or generalizations of it. We refer to the works of Dang and

Oppenheimer [6], Del Pino *et al* [7], Fabry and Fayyad [8], Gasinski and Papageorgiou [9], Guo [11], Halidias and Papageorgiou [12], Kyritsi *et al* [19], Manasevich and Mawhin [21], Mawhin [22,23] and the references therein.

2. Mathematical preliminaries

As we have already mentioned our approach is variational, based on the non-smooth critical point theory. For the convenience of the reader, in this section we recall the basic facts from this theory. It is based on the Clarke subdifferential theory for locally Lipschitz functions. Let X be a Banach space and $\varphi\colon X \to \mathbb{R}$. We say that φ is locally Lipschitz, if for every bounded open set $U \subseteq X$, there exists a constant $k_U > 0$ such that $|\varphi(y) - \varphi(z)| \leq k_U ||y - z||$ for all $y, z \in U$. It is a well-known fact from convex analysis that a proper, convex and lower semicontinuous function $\psi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\mathrm{dom}\, \psi = \{x \in X\colon \psi(x) < +\infty\}$. In particular an $\mathbb{R}$-valued, convex and lower semicontinuous function is locally Lipschitz. In analogy with the directional derivative of a convex function, for a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we define the generalized directional at derivative $x \in X$ in the direction $h \in X$, by

$$\varphi^0(x; h) = \limsup_{\substack{x' \longrightarrow x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that the function $h \to \varphi^0(y; h)$ is sublinear, continuous and so by the Hahn–Banach theorem it is the support function of a non-empty, convex and w^*-compact set

$$\partial\varphi(x) = \{x^* \in X^*\colon (x^*, h) \leq \varphi^0(x; h) \quad \text{for all} \quad h \in X\}.$$

The set $\partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of φ at $x \in X$. If $\varphi, \psi\colon X \to \mathbb{R}$ are both locally Lipschitz functions, then for all $x \in X$ and all $\lambda \in \mathbb{R}$ we have $\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$. Moreover, if φ is also convex, then the subdifferential $\partial\varphi$ coincides with the subdifferential in the sense of convex analysis. Recall that the convex subdifferential of φ is defined by $\partial\varphi(x) = \{x^* \in X^* \colon (x^*, y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}$. Also if $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$ for all $x \in X$.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, a point $x \in X$ is said to be a ‘critical point’ of φ, if $0 \in \partial\varphi(x)$. If $\varphi \in C^1(X)$, then as we saw above, $\partial\varphi(x) = \{\varphi'(x)\}$ and so this definition of critical point coincides with the classical (smooth) one. It is easy to see that if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(x)$. From the smooth critical point theory, we know that a basic tool is a compactness-type condition, known as the ‘Palais–Smale condition’ (PS-condition for short). In the present non-smooth setting this condition takes the following form: ‘A locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ satisfies the non-smooth PS-condition, if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) = \inf[\|x_n^*\|\colon x_n^* \in \partial\varphi(x_n)] \to 0$ as $n \to \infty$, has a strongly convergent subsequence’. A version of the theory based on a weaker condition known as the ‘non-smooth C-condition’ can be found in Kourogenis and Papageorgiou [17].

A $\lambda \in \mathbb{R}$ is said to be an 'eigenvalue' of minus the p-Laplacian with periodic boundary conditions, if the problem

$$\left\{ \begin{array}{l} -(||x'(t)||^{p-2}x'(t))' = \lambda||x(t)||^{p-2}x(t) \quad \text{a.e on } T = [0,b] \\ x(0) = x(b), x'(0) = x'(b), \quad 1 < p < \infty \end{array} \right\},$$

has a non-trivial solution $x \in C^1(T, \mathbb{R}^N)$, known as corresponding to λ 'eigenfunction'. Let S denote the set of these eigenvalues. Evidently $0 \in S$ and if $\lambda \notin S$, then for every $h \in L^1(T, \mathbb{R}^N)$ the periodic problem

$$\left\{ \begin{array}{l} -(||x'(t)||^{p-2}x'(t))' = \lambda||x(t)||^{p-2}x(t) + h(t) \quad \text{a.e on } T = [0,b] \\ x(0) = x(b), \quad x'(0) = x'(b) \end{array} \right\},$$

has at least one solution. Each element of S is non-negative and 0 is the smallest (first) eigenvalue. If $N = 1$ (scalar case), by direct integration of the equation we obtain all the eigenvalues which are

$$\lambda_n = \left(\frac{2n\pi_p}{b}\right)^p, \quad \text{where } \pi_p = 2(p-1)^{1/p}\frac{(\pi/p)}{\sin(\pi/p)}.$$

When $p = 2$ (semilinear case), then $\pi_2 = \pi$ and we recover the well-known eigenvalues of the 'scalar periodic negative Laplacian' which are $\lambda_n = (n\omega)^2$ with $\omega = 2\pi/b$. In the case $N > 1$ (vector case), $\{\lambda_n\}_{n\geq 1} \subseteq S$ but S contains more elements (see [22]).

3. Existence theorem

In this section we prove an existence theorem for non-smooth periodic systems driven by the ordinary vector p-Laplacian, which will be used in our investigation of homoclinic orbits in §5. It concerns the following non-linear and non-smooth periodic system:

$$\left\{ \begin{array}{l} -(||x'(t)||^{p-2}x'(t))' + g(t)\|x(t)\|^{p-2}x(t) \in \partial j(t, x(t)) \text{ a.e on } T = [0,b] \\ x(0) = x(b), x'(0) = x'(b), \ 1 < p < \infty \end{array} \right\}. \tag{1}$$

Our hypotheses on the data of (1) are the following:

H(g): $g \in C(T)$, $g(0) = g(b)$ and for all $t \in T$, $g(t) \geq c > 0$.
H(j)$_1$: j: $T \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a functional such that $j(\cdot, 0) \in L^\infty(T)$, $\int_0^b j(t,0)\mathrm{d}t \geq 0$ and

(i) for all $x \in \mathbb{R}^N$, $t \longmapsto j(t,x)$ is measurable;
(ii) for almost all $t \in T$, the function $x \longmapsto j(t,x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \leq a_1(t) + c_1(t)\|x\|^{r-1},$$

$1 \leq r < +\infty$ with $a_1, c_1 \in L^\infty(T)$;
(iv) there exists $M > 0$ such that for almost all $t \in T$ and all $x \in \mathbb{R}^N$ with $\|x\| \geq M$ we have

$$\mu j(t,x) \leq -j^0(t,x;-x) \quad \text{with } \mu > p;$$

(v) $\limsup_{\|x\|\to\infty} \frac{pj(t,x)}{\|x\|^p} \leq 0$ uniformly for almost all $t \in T$;

(vi) there exists $x_* \in \mathbb{R}^N$, $\|x_*\| \geq M$ such that $\int_0^b j(t, x_*)\mathrm{d}t > 0$.

Theorem 1. *If hypotheses* H(g), H(j)$_1$ *hold, then problem* (1) *has at least one non-trivial solution* $x \in C^1(T, \mathbb{R}^N)$ *with* $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,r'}(T, \mathbb{R}^N)$.

Proof. Let φ: $W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \to \mathbb{R}$ be the locally Lipschitz function defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_0^b g(t)\|x(t)\|^p\mathrm{d}t - \int_0^b j(t, x(t))\mathrm{d}t.$$

First we show that φ satisfies the non-smooth PS-condition. To this end let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$ be a sequence such that $|\varphi(x_n)| \leq M_1$ for all $n \geq 1$ and some $M_1 > 0$ and $m(x_n) \to 0$.

Since $\partial\varphi(x_n) \subseteq W^{1,p}_{\text{per}}(T, \mathbb{R}^N)^*$ is w-compact, the norm functional in a Banach space is weakly lower semicontinuous and $W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$ is embedded compactly in $C_{\text{per}}(T, \mathbb{R}^N)$, from the Weierstrass theorem we know that we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|, n \geq 1$. We have $x_n^* = A(x_n) + g|x_n|^{p-2}x_n - u_n$ with A: $W^{1,p}_{\text{per}}(T, \mathbb{R}^N) \to W^{1,p}_{\text{per}}(T, \mathbb{R}^N)^*$ being the non-linear operator defined by

$$\langle A(x), y\rangle = \int_0^b \|x'(t)\|^{p-2}(x'(t), y'(t))_{\mathbb{R}^N}\mathrm{d}t, \quad \text{for all } x, y \in W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$$

and $u_n \in L^{r'}(T, \mathbb{R}^N)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see [5], pp. 47 and 83). It is easy to check that A is monotone, demicontinuous; thus maximal monotone (see [14], p. 309).

Combining hypothesis H(j)$_1$(iii) with the Lebourg mean value theorem (see [20] or p. 41 of [5]), we see that for almost all $t \in T$ and all $x \in \mathbb{R}^N$,

$$|j(t, x)| \leq \hat{\alpha}_1(t) + \hat{c}_1(t)\|x\|^r \quad \text{with } \hat{\alpha}_1, \hat{c}_1 \in L^\infty(T)_+.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T, \mathbb{R}^N)$, we have

$$\mu\varphi(x_n) + \left\langle x_n^*, -x_n\right\rangle \leq \mu M_1 + \varepsilon_n\|x_n\| \quad \text{with } \varepsilon_n \downarrow 0$$

$$\begin{aligned}
&\left(\frac{\mu}{p} - 1\right)\|x_n'\|_p^p + \left(\frac{\mu}{p} - 1\right)\int_0^b g(t)\|x_n(t)\|^p\mathrm{d}t \\
&\quad - \int_0^b [(u_n(t), -x_n(t))_{\mathbb{R}^N} + \mu j(t, x_n(t))] \\
&\leq \mu M_1 + \varepsilon_n\|x_n\| \\
\Rightarrow &\left(\frac{\mu}{p} - 1\right)\left(\|x_n'\|_p^p + \int_0^b g(t)\|x_n(t)\|^p\mathrm{d}t\right) \\
&\quad + \int_0^b [-j^0(t, x_n(t); -x_n(t)) - \mu j(t, x_n(t))]\mathrm{d}t \\
&\leq \mu M_1 + \varepsilon_n\|x_n\|.
\end{aligned}$$

Using hypotheses H(j)$_1$(iii) and (iv), we obtain

$$\begin{aligned}
&\int_0^b [-j^0(t,x_n(t);-x_n(t)) - \mu j(t,x_n(t))]\mathrm{d}t \\
&= \int_{\{\|x_n\|<M\}} [-j^0(t,x_n(t);-x_n(t)) - \mu j(t,x_n(t))]\mathrm{d}t \\
&\quad + \int_{\{\|x_n\|\geq M\}} [-j^0(t,x_n(t);-x_n(t)) - \mu j(t,x_n(t))]\mathrm{d}t \\
&\geq -c_2 \quad \text{for some } c_2>0 \text{ and all } n\geq 1.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
&\left(\frac{\mu}{p}-1\right)\left(\|x_n'\|_p^p + \int_0^b g(t)\|x_n(t)\|^p\mathrm{d}t\right) \leq \mu M_1 + \varepsilon_n\|x_n\| + c_2 \\
&\qquad \text{for some } c_2>0 \text{ and } \varepsilon_n \downarrow 0, \\
&\Rightarrow \left(\frac{\mu}{p}-1\right)\left(\|x_n'\|_p^p + c\|x_n\|_p^p\right) \leq c_3 + \varepsilon_n\|x_n\| \\
&\qquad \text{with } c_3 = \mu M_1 + c_2 > 0, \\
&\Rightarrow \|x_n\|^p \leq c_4 + \varepsilon_n'\|x_n\| \quad \text{for some } c_4>0 \text{ and } \varepsilon_n'\downarrow 0.
\end{aligned}$$

From the last inequality it follows that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is bounded and so by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ and $x_n \to x$ in $C_{\mathrm{per}}(T,\mathbb{R}^N)$. We have

$$\begin{aligned}
|\left\langle x_n^*, x_n - x\right\rangle| = {} & |\langle A(x_n), x_n - x\rangle \\
& - \int_0^b g(t)\|x_n(t)\|^{p-2}(x_n(t), x_n(t)-x(t))_{\mathbb{R}^N}\mathrm{d}t \\
& - \int_0^b (u_n(t), x_n(t)-x(t))_{\mathbb{R}^N}\mathrm{d}t| \leq \varepsilon_n\|x_n - x\| \\
\Rightarrow {} & \lim\langle A(x_n), x_n - x\rangle = 0.
\end{aligned}$$

Because A is maximal monotone, it is a generalized pseudomonotone (see [14], p. 365) and so we have $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Rightarrow \|x_n'\|_p \to \|x'\|_p$. Because $x_n' \xrightarrow{w} x'$ in $L^p(T,\mathbb{R}^N)$ and the latter is uniformly convex, from the Kadec–Klee property (see [14], p. 28), we have $x_n' \to x'$ in $L^p(T,\mathbb{R}^N)$, hence $x_n \to x$ in $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$. So φ satisfies the non-smooth PS-condition.

Because of hypothesis H(j)$_1$(v), given $\varepsilon>0$ we can find $\delta>0$ such that for almost all $t\in T$ and all $x\in\mathbb{R}^N$ with $\|x\|\leq\delta$ we have $j(t,x)\leq\frac{\varepsilon}{p}\|x\|^p$. On the other hand, hypothesis H(j)$_1$(iii) and the Lebourg mean value theorem, imply that for almost all $t\in T$ and all $x\in\mathbb{R}^N$ with $\|x\|\geq\delta$ we have $j(t,x)\leq c_5\|x\|^r$ for some $c_5>0$. So finally for almost all $t\in T$ and all $x\in\mathbb{R}^N$ we can write that $j(t,x)\leq\frac{\varepsilon}{p}\|x\|^p + c_6\|x\|^s$ for some $c_6>0$ and with $s>\max\{r,p\}$. Therefore for every $x\in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ we have

$$\begin{aligned}
\varphi(x) &= \frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_0^b g(t)\|x(t)\|^p\mathrm{d}t - \int_0^b j(t,x(t))\mathrm{d}t \\
&\geq \frac{1}{p}\|x'\|_p^p + \frac{c}{p}\|x\|_p^p - \frac{\varepsilon}{p}\|x\|_p^p - c_7\|x\|_\infty^s \quad \text{for some } c_7>0.
\end{aligned}$$

Because $W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$ is embedded continuously in $C(T,\mathbb{R}^N)$, we have

$$\varphi(x) \geq \frac{1}{p}(\|x'\|_p^p + (c-\varepsilon)\|x\|_p^p) - c_8\|x\|^s \quad \text{for some } c_8 > 0.$$

Taking $\varepsilon < c$ we obtain that

$$\varphi(x) \geq c_9\|x\|^p - c_8\|x\|^s \quad \text{for some } c_9 > 0.$$

Recall that $s > p$. So we can find $\rho > 0$ small so that $\inf[\varphi(x)\colon \|x\| = \rho] = \xi > 0$.

On $\mathbb{R}_+ \setminus \{0\}$, the function $r \to 1/r^\mu$ is continuous convex, thus it is locally Lipschitz. From ([5], p. 48) we have that $r \to (1/r^\mu)j(t,rx)$ is locally Lipschitz on $\mathbb{R}_+ \setminus \{0\}$ for almost all $t \in T$ (hypothesis H(j)$_1$(ii)) and we have

$$\partial_r\left(\frac{1}{r^\mu}j(t,rx)\right) \subseteq -\frac{\mu}{r^{\mu+1}}j(t,rx) + \frac{1}{r^\mu}(\partial_x j(t,rx),x)_{\mathbb{R}^N}.$$

Using Lebourg's mean value theorem, we can find $\lambda \in (1,r)$ such that

$$\begin{aligned}
&\frac{1}{r^\mu}j(t,rx) - j(t,x) \in \left(-\frac{\mu}{\lambda^{\mu+1}}j(t,\lambda x) + \frac{1}{\lambda^\mu}(\partial_x j(t,\lambda x),x)_{\mathbb{R}^N}\right)(r-1),\\
&\Rightarrow \frac{1}{r^\mu}j(t,rx) - j(t,x) = \frac{r-1}{\lambda^{\mu+1}}(-\mu j(t,\lambda x) + (\partial_x j(t,\lambda x),\lambda x)_{\mathbb{R}^N})\\
&\qquad\qquad \geq \frac{r-1}{\lambda^{\mu+1}}(-\mu j(t,\lambda x) - j^0(t,\lambda x;-\lambda x)) \geq 0\\
&\qquad\qquad \text{(see hypothesis H(j)}_1\text{(iv))}\\
&\Rightarrow r^\mu j(t,x) \leq j(t,rx) \quad \text{for almost all } t \in T, \text{ all } \|x\| \geq M \text{ and all } r \geq 1.
\end{aligned}$$

Choosing $x_* \in \mathbb{R}^N$ as postulated by hypothesis H(j)$_1$(vi), for $\lambda \geq 1$ large we have

$$\begin{aligned}
\varphi(\lambda x_*) &= \int_0^b g(t)\|\lambda x_*\|^p \mathrm{d}t - \int_0^b j(t,\lambda x_*)\mathrm{d}t\\
&\leq \lambda^p\|g\|_\infty\|x_*\|^p b - \lambda^\mu\int_0^b j(t,x_*)\mathrm{d}t\\
&\Rightarrow \varphi(\lambda x_*) \to -\infty \text{ as } \lambda \to +\infty \text{ (recall that } \mu > p).
\end{aligned}$$

Thus we can find $\lambda > 0$ large so that $\|\lambda x_*\| > p$ and $\varphi(\lambda x_*) < \xi$. Also note that $\varphi(0) \leq 0$ (recall that $\int_0^b j(t,0)\mathrm{d}t \geq 0$). Therefore we can apply the non-smooth mountain pass theorem (see [4] or [17]) and obtain $x \in W^{1,p}_{\mathrm{per}}(T,\mathbb{R}^N)$, $x \neq 0$ such that $0 \in \partial\varphi(x)$.

We have $0 \in \partial\varphi(x) \subseteq A(x) - \partial I_j(x)$ and so

$$A(x) = u - g\|x\|^{p-2}x \quad \text{with } u \in L^{r'}(T,\mathbb{R}^N), u(t) \in \partial j(t,x(t)) \text{ a.e. on } T. \tag{2}$$

Let $\theta \in C_0^\infty((0,b),\mathbb{R}^N)$. We have

$$\langle A(x),\theta\rangle = \int_0^b (u(t),\theta(t))_{\mathbb{R}^N}\,\mathrm{d}t - \int_0^b g(t)\|x(t)\|^{p-2}(x(t),\theta(t))_{\mathbb{R}^N}\,\mathrm{d}t$$

$$\Rightarrow \int_0^b \|x'(t)\|^{p-2}(x'(t),\theta'(t))_{\mathbb{R}^N}\,\mathrm{d}t = \int_0^b (u(t),\theta(t))_{\mathbb{R}^N}\,\mathrm{d}t - \int_0^b g(t)\|x(t)\|^{p-2}(x(t),\theta(t))_{\mathbb{R}^N}\,\mathrm{d}t.$$

Since $(\|x'\|^{p-2}x')' \in W^{-1,q}(T,\mathbb{R}^N)$ (see [1], p. 50), we have

$$\langle -(\|x'\|^{p-2}x')',\theta\rangle_0 = \langle u - g\|x\|^{p-2}x,\theta\rangle_0$$

with $\langle\cdot,\cdot\rangle_0$ denoting the duality brackets for the pair $(W_0^{1,p}(T,\mathbb{R}^N), W^{-1,q}(T,\mathbb{R}^N) = W_0^{1,p}(T,\mathbb{R}^N)^*)$. Since $C_0^\infty((0,b),\mathbb{R}^N)$ is dense in $W_0^{1,p}(T,\mathbb{R}^N)$, it follows that

$$-(\|x'(t)\|^{p-2}x'(t))' + g(t)\|x(t)\|^{p-2}x(t) = u(t) \quad \text{a.e. on } T,$$
$$u \in L^{r'}(T,\mathbb{R}^N). \tag{3}$$

From (3) it follows that $(\|x'(\cdot)\|^{p-2}x'(\cdot)) \in W^{1,r'}(T,\mathbb{R}^N)$. Because the map $z \to \|z\|^{p-2}z$ is a homeomorphism on $\mathbb{R}^N$ onto itself and $W^{1,r'}(T,\mathbb{R}^N) \subseteq C(T,\mathbb{R}^N)$, we infer that $x' \in C(T,\mathbb{R}^N)$, hence $x \in C^1(T,\mathbb{R}^N)$.

Next if $y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$, from Green's inequality (integration by parts), we have

$$\begin{aligned}\langle A(x),y\rangle &= \int_0^b \|x'(t)\|^{p-2}(x'(t),y'(t))_{\mathbb{R}^N}\,\mathrm{d}t\\ &= \|x'(b)\|^{p-2}(x'(b),y(b))_{\mathbb{R}^N} - \|x'(0)\|^{p-2}(x'(0),y(0))_{\mathbb{R}^N}\\ &\quad - \int_0^b ((\|x'(t)\|^{p-2}x'(t))', y(t))_{\mathbb{R}^N}\,\mathrm{d}t.\end{aligned}$$

Using (2) and (3), we obtain

$$\|x'(0)\|^{p-2}(x'(0),y(0))_{\mathbb{R}^N} = \|x'(b)\|^{p-2}(x'(b),y(b))_{\mathbb{R}^N}$$
$$\text{for all } y \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N),$$
$$\Rightarrow \|x'(0)\|^{p-2}x'(0) = \|x'(b)\|^{p-2}x'(b),$$
$$\Rightarrow x'(0) = x'(b).$$

Also since $x \in W^{1,p}_{\text{per}}(T,\mathbb{R}^N)$, $x(0) = x(b)$. Therefore $x \in C^1(T,\mathbb{R}^N)$ is the desired solution of (1). QED

Remark. The following function is a non-smooth potential satisfying hypotheses H(j)$_2$ (and does not satisfy the conditions imposed by Tang [31,32] (for $p = 2$) and Papageorgiou and Papageorgiou [28]). Again for simplicity we drop the t-dependence. We have

$$j(x) = \begin{cases} -\|x\|, & \text{if } \|x\| \le 1\\ \frac{1}{\mu}\|x\|^\mu - \|x\|\ln\|x\| + c, & \text{if } \|x\| > 1\end{cases}, \ p < \mu \quad \text{and} \quad c = \frac{-\mu-1}{\mu} < 0,$$

$$\Rightarrow -j^0(x;-x) = \begin{cases} -\|x\|, & \text{if } \|x\| \le 1\\ \|x\|^\mu - \|x\|\ln\|x\| - \|x\|, & \text{if } \|x\| > 1\end{cases}.$$

Since $c < 0$, we have $-j^0(x, -x) - \mu j(x) \geq (\mu - 1)\|x\| \ln \|x\| - \|x\| \geq 0$ if $\|x\| \geq 1$. Thus hypotheses $\mathrm{H(j)}_1$ hold.

4. Multiplicity theorems

In this section we prove a multiplicity result. It concerns an eigenvalue version of problem (1).

$$\left\{ \begin{array}{l} -(\|x'(t)\|^{p-2}x'(t))' + g(t)\|x(t)\|^{p-2}x(t) \in \lambda \partial j(t, x(t)) \text{ a.e on } T = [0, b] \\ x(0) = x(b), x'(0) = x'(b), \lambda \in \mathbb{R}. \end{array} \right\}. \tag{4}$$

We prove a multiplicity result for a whole semiaxis of values of the parameter $\lambda \in \mathbb{R}$. Our hypotheses on the non-smooth potential are the following:

$\mathrm{H(j)}_2$: $j : T \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a functional such that $j(\cdot, 0) \in L^\infty(T)$ and

(i) for all $x \in \mathbb{R}^N$, $t \longmapsto j(t, x)$ is measurable;
(ii) for almost all $t \in T$, the function $x \longmapsto j(t, x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t, x)$, we have

$\|u\| \leq c_1(t)(1 + \|x\|^{r-1})$,

$1 \leq r < p$ with $c_1 \in L^\infty(T)$,

(iv) $\int_0^b j(t, 0)\mathrm{d}t = 0$ and there exists $x_0 \in L^r(T, \mathbb{R}^N)$ such that $\int_0^b j(t, x_0(t))\mathrm{d}t > 0$;
(v) $\limsup\limits_{\|x\| \to \infty} \frac{pj(t,x)}{\|x\|^p} < 0$ uniformly for almost all $t \in T$.

Theorem 2. *If hypotheses* H(g) *and* $\mathrm{H(j)}_2$ *hold, then there exists* $\lambda_* > 0$ *such that for all* $\lambda \geq \lambda_*$ *problem* (4) *has at least two non-trivial solutions* $x_1, x_2 \in C^1(T, \mathbb{R}^N)$ *such that* $\|x_k'(\cdot)\|^{p-2}x_k'(\cdot) \in W^{1,r'}(T, \mathbb{R}^N)$, $k = 1, 2$.

Proof. For every $\lambda \in \mathbb{R}$ we consider the locally Lipschitz functional φ_λ: $W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\varphi_\lambda(x) = \frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_0^b g(t)\|x(t)\|^p \mathrm{d}t - \lambda \int_0^b j(t, x(t))\mathrm{d}t.$$

First we show that φ_λ satisfies the PS-condition. For this purpose, we consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)$ such that $|\varphi_\lambda(x_n)| \leq M_1$ for all $n \geq 1$ and some $M_1 > 0$ and $m(x_n) \to 0$. As before we can find $x_n^* \in \partial\varphi_\lambda(x_n)$ such that $m(x_n) = \|x_n^*\|$ for all $n \geq 1$. For every $n \geq 1$ we have $x_n^* = A(x_n) + g\|x_n\|^{p-2}x_n - \lambda u_n$ where $A : W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N) \to W^{1,p}_{\mathrm{per}}(T, \mathbb{R}^N)^*$ is as in the proof of Theorem 1 and $u_n \in L^{r'}(T, \mathbb{R}^N)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T. From hypothesis $\mathrm{H(j)}_2$(iii) and the Lebourg mean value theorem, we obtain that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, $|j(t, x)| \leq c_2(t)(1 + \|x\|^r)$ with $c_2 \in L^\infty(T)$. For every $n \geq 1$, we have

$$\begin{aligned} \varphi_\lambda(x_n) &= \frac{1}{p}\|x_n'\|_p^p + \frac{1}{p}\int_0^b g(t)\|x_n(t)\|^p \mathrm{d}t - \lambda \int_0^b j(t, x_n(t))\mathrm{d}t \\ &\geq \frac{1}{p}\|x_n'\|_p^p + \frac{c}{p}\|x_n\|_p^p - \lambda\|c_2\|_\infty b - \lambda c_3\|x_n\|_p^r, \end{aligned}$$

for some $c_3 > 0$.

Using Young's inequality with $\varepsilon > 0$, we obtain $\lambda c_3 \|x_n\|_p^r \leq M_2(\varepsilon, \lambda) + \frac{\lambda\varepsilon}{p} \|x_n\|_p^p$ for some $M_2(\varepsilon, \lambda) > 0$ (recall that $r < p$). Therefore we obtain

$$\frac{1}{p}\|x_n'\|_p^p + \frac{1}{p}(c - \lambda\varepsilon)\|x_n\|_p^p - c_4(\varepsilon, \lambda) \leq \varphi_\lambda(x_n) \leq M_1 \tag{5}$$

for all $n \geq 1$ and some $c_4(\varepsilon, \lambda) > 0$.

Choose $\varepsilon > 0$ so that $\lambda\varepsilon < c$. From (5) it follows that $\{x_n\}_{n\geq 1} \subseteq W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ is bounded. Arguing as in the last part of the proof of Theorem 1, we conclude that φ_λ satisfies the non-smooth PS-condition. In fact from (5) we infer that φ_λ is coercive. Also exploiting the compact embedding of $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ into $C(T, \mathbb{R}^N)$ (Sobolev embedding theorem), we can check easily that φ_λ is sequentially weakly lower semicontinuous. So from the Weierstrass theorem it follows that there exists $x_1 \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ such that $\varphi_\lambda(x_1) = \inf \varphi_\lambda$ and $0 \in \partial\varphi_\lambda(x_1)$.

Next, let $\widehat{\psi}\colon L^r(T, \mathbb{R}^N) \to \mathbb{R}$ be the integral functional defined by $\widehat{\psi}(x) = \int_0^b j(t, x(t))\mathrm{d}t$. By virtue of hypothesis H(j)$_2$(iv), we have $\widehat{\psi}(x_0) > 0$. The Sobolev space $W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ is dense in $L^r(T, \mathbb{R}^N)$. So we can find $y \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ such that $\widehat{\psi}(y) > 0$. Therefore there exists $\lambda_* > 0$ large enough such that for all $\lambda \geq \lambda_*$ we have $\varphi_\lambda(y) = \frac{1}{p}\|y'\|_p^p + \int_0^b g(t)\|y(t)\|^p \mathrm{d}t - \lambda\widehat{\psi}(y) < 0$. Hence $\varphi_\lambda(x_1) \leq \varphi_\lambda(y) < 0 = \varphi_\lambda(0)$, i.e. $x_1 \neq 0$. Since $0 \in \partial\varphi_\lambda(x_1)$ we verify that $x_1 \in C^1(T, \mathbb{R}^N)$, $\|x_1'(\cdot)\|^{p-2}x_1'(\cdot) \in W^{1,r'}(T, \mathbb{R}^N)$ and that it is a non-trivial solution of (4).

Because of hypothesis H(j)$_2$(v) we can find $\theta > 0$ and $\delta > 0$ such that for almost all $t \in T$ and all $\|x\| \leq \delta$, we have $j(t, x) \leq -\frac{\theta}{p}\|x\|^p$. Combining this with the growth condition on j, we obtain that for almost all $t \in T$ and all $x \in \mathbb{R}^N$, $j(t, x) \leq -\frac{\theta}{p}\|x\|^p + c_5\|x\|^s$ for some $c_5 > 0$ and with $s > p$. So we can write that

$$\begin{aligned}\varphi_\lambda(x) &\geq \frac{1}{p}\|x'\|_p^p + \frac{c}{p}\|x\|_p^p + \frac{\theta}{p}\|x\|_p^p - c_6\|x\|_s^s \quad \text{for some } c_6 > 0\\ &\geq c_7\|x\|^p - c_8\|x\|^s \quad \text{for some } c_7, c_8 > 0.\end{aligned}$$

Thus if we choose $0 < \rho < \min\{1, \|x_1\|\}$ small enough, we can have that

$$\inf[\varphi_\lambda(x)\colon \|x\| = \rho] = \gamma > 0.$$

Since $\varphi(0) = 0$, $x_1 \neq 0$ and $0 < \rho < \|x_1\|$, we can apply the non-smooth mountain pass theorem and obtain $x_2 \in W_{\text{per}}^{1,p}(T, \mathbb{R}^N)$ such that $0 = \varphi(0) < \gamma \leq \varphi_\lambda(x_2)$, hence $x_2 \neq 0$, $x_2 \neq x_1$ and $0 \in \partial\varphi_\lambda(x_2)$. As before for $k = 1, 2$ we can check that $x_k \in C^1(T, \mathbb{R}^N)$, $\|x_k(\cdot)\|^{p-2}x_k(\cdot) \in W^{1,r'}(T, \mathbb{R}^N)$ and it solves (4). QED

Remark. The following non-smooth potential satisfies hypotheses H(j)$_2$ (again we drop the t-dependence):

$$j(x) = \begin{cases} -\frac{1}{p}\|x\|^p, & \text{if } \|x\| < 1 \\ \frac{1}{r}\|x\|^r + \cos\|x\| + c, & \text{if } \|x\| \geq 1 \end{cases}, \quad r < p \quad \text{and} \quad c = \frac{1}{p} - \frac{1}{r} - \cos 1.$$

Note that

$$\partial j(x)=\begin{cases}-\|x\|^{p-2}x, & \text{if } \|x\|<1\\ \operatorname{conv}\{-x, x-(\sin 1)x\}, & \text{if } \|x\|=1\\ \|x\|^{r-2}x-\frac{x}{\|x\|}\sin\|x\|, & \text{if } \|x\|>1.\end{cases}$$

5. Homoclinic solutions

In this section we turn our attention to the question of existence of homoclinic solutions (to 0), for the homoclinic problem in $\mathbb{R}^N$ corresponding to (1). Namely, we consider the problem:

$$\left\{\begin{array}{ll} -(||x'(t)||^{p-2}x'(t))'+g(t)\|x(t)\|^{p-2}x(t)\in\partial j(t,x(t)) & \text{a.e on } \mathbb{R}\\ \|x(t)\|\to 0, \|x'(t)\|\to 0 \text{ as } |t|\to\infty,\ 1<p<\infty & \end{array}\right\}. \tag{6}$$

So far the 'homoclinic problem' for second order systems has been studied only in the context of semilinear equations, primarily with smooth potential. We refer to the works of Grossinho *et al* [10], Korman and Lazer [16], Rabinowitz [29],Yanheng [34] and the references therein. Non-smooth semilinear systems were studied only recently by Adly and Goeleven [2] and Hu [13], using different methods. To our knowledge our result is the first one (even in the context of smooth systems) on the existence of homoclinic (to 0) orbits for quasilinear systems. Our approach is based on that of Rabinowitz [29] (see also [10]).

Our hypotheses on the non-smooth potential are the following:

H(j)$_3$: $j\colon \mathbb{R}\times\mathbb{R}^N\longmapsto\mathbb{R}$ is a functional such that $j(t,0)=0$ a.e. on $\mathbb{R}$ and

(i) for all $x\in\mathbb{R}^N$, $t\longmapsto j(t,x)$ is measurable and $2b$-periodic;
(ii) for almost all $t\in\mathbb{R}$, the function $x\longmapsto j(t,x)$ is locally Lipschitz;
(iii) for almost all $t\in\mathbb{R}$, all $x\in\mathbb{R}^N$ and all $u\in\partial j(t,x)$, we have

$$\|u\|\le a_1(t)(1+\|x\|^{p-1}),$$

with $a_1\in L^\infty(\mathbb{R})$;
(iv) there exists $M>0$ such that for almost all $t\in\mathbb{R}$ and all $x\in\mathbb{R}^N$ with $\|x\|\ge M$, we have

$$\mu j(t,x)\le -j^0(t,x;-x) \text{ with } \mu>p;$$

(v) $\lim\limits_{\|x\|\to 0}\frac{pj(t,x)}{\|x\|^p}\le 0$ uniformly for almost all $t\in\mathbb{R}$;
(vi) there exists $x_0\in\mathbb{R}^N$ such that $\int_{-b}^{b} j(t,x_0)\mathrm{d}t>0$.

Remark. Hypothesis H(j)$_3$(v) is equivalent to the following one:

(v)$'$ $\lim\limits_{\|x\|\to 0}\frac{(u,x)_{\mathbb{R}^N}}{\|x\|^p}\le 0$ uniformly for almost all $t\in\mathbb{R}$ and all $u\in\partial j(t,x)$.

First we show that (v) $\Rightarrow$ (v)$'$. From the Lebourg mean value theorem, we know that for almost all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned} j(t,x) - j\left(t, \frac{x}{2}\right) &= \left(u, \frac{x}{2}\right)_{\mathbb{R}^N} \quad \text{with } u \in \partial j\left(t, \lambda\frac{x}{2}\right), \\ &\qquad \lambda \in (1,2) \text{ (depending on } t), \\ \Rightarrow \frac{j(t,x)}{\|x\|^p} &= \frac{j(t,\frac{x}{2})}{\|x\|^p} + \frac{(u,\frac{x}{2})_{\mathbb{R}^N}}{\|x\|^p} \\ &= \frac{j(t,\frac{x}{2})}{2^p\|\frac{x}{2}\|^p} + \frac{\lambda^{p-1}}{2^p}\frac{(u,\frac{\lambda x}{2})_{\mathbb{R}^N}}{\|\frac{\lambda x}{2}\|^p}, \\ \Rightarrow 0 \geq \left(1-\frac{1}{2^p}\right)\lim_{\|x\|\to 0}\frac{j(t,x)}{\|x\|^p} &\geq \lim_{\|x\|\to 0}\frac{\lambda^{p-1}}{2^p}\frac{(u,\frac{\lambda x}{2})_{\mathbb{R}^N}}{\|\frac{\lambda x}{2}\|^p}. \end{aligned}$$

As $\|x\| \to 0$, we have $\lambda \downarrow 1$ and so we conclude that $\lim_{\|x\|\to 0}\frac{(u,x)_{\mathbb{R}^N}}{\|x\|^p} \leq 0$ uniformly for almost all $t \in \mathbb{R}$, i.e. (v)$'$ holds.

Next we show that (v)$'$ $\Rightarrow$ (v). From the previous argument for almost all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^N \setminus \{0\}$ we have

$$\begin{aligned} \frac{j(t,x)}{\|x\|^p} &= \frac{j(t,\frac{x}{2})}{2^p\|\frac{x}{2}\|^p} + \frac{\lambda^{p-1}}{2^p}\frac{(u,\frac{\lambda x}{2})_{\mathbb{R}^N}}{\|\frac{\lambda x}{2}\|^p} \\ \Rightarrow \lim_{\|x\|\to\infty}\frac{j(t,x)}{\|x\|^p} &\leq 0 \quad \left(\text{since } 1-\frac{1}{2^p} > 0 \text{ and } \lambda \downarrow 1 \text{ as } \|x\| \to 0\right), \end{aligned}$$

and the convergence is uniform for almost all $t \in \mathbb{R}$. So (v) holds. Thus we have proved that (v) $\Leftrightarrow$ (v)$'$.

Also the hypothesis on the coefficient function g takes the following form:

H(g)$_1$: $g \in C(\mathbb{R})$, g is $2b$-periodic and for all $t \in [-b, b]$, $g(t) \geq c > 0$.

Theorem 3. *If hypotheses* H(g)$_1$ *and* H(j)$_3$ *hold, then there exists a non-trivial homoclinic solution* $x \in C(\mathbb{R}, \mathbb{R}^N) \cap W^{1,p}(\mathbb{R}, \mathbb{R}^N)$ *for problem* (6).

Proof. We consider the following auxiliary periodic problem:

$$\left\{ \begin{array}{l} -(||x'(t)||^{p-2}x'(t))' + g(t)\|x(t)\|^{p-2}x(t) \in \partial j(t, x(t)) \\ \qquad\qquad \text{a.e on } T[-nb, nb], \\ x(-nb) = x(nb), x'(-nb) = x'(nb),\ 1 < p < \infty \end{array} \right\}. \tag{7}$$

From Theorem 2, we know that problem (7) has a non-trivial solution $x_n \in W^{1,p}_{\text{per}}(T_n, \mathbb{R}^N)$. Let $\varphi_n\colon W^{1,p}_{\text{per}}(T_n, \mathbb{R}^N) \to \mathbb{R}$ be the locally Lipschitz energy functional corresponding to problem (7), i.e.

$$\varphi_n(x) = \frac{1}{p}\|x'\|_p^p + \frac{1}{p}\int_{-nb}^{nb} g(t)\|x(t)\|^p \mathrm{d}t - \int_{-nb}^{nb} j(t, x(t))\mathrm{d}t.$$

Hereafter by L^p_n we shall denote the Lebesgue space $L^p(T_n, \mathbb{R}^N)$ and by $W^{1,p}_n$ the Sobolev space $W^{1,p}(T_n, \mathbb{R}^N)$.

Consider the integral functional $\psi\colon L_1^p \to \mathbb{R}$ defined by $\psi(x) = \int_{-b}^{b} j(t, x(t))\mathrm{d}t$. By virtue of hypothesis H(j)$_3$(vi) we have that $\psi(x_0) > 0$. Because ψ is continuous and $W_0^{1,p}(T_1, \mathbb{R}^N)$ is dense in L_1^p, we can find $\overline{x} \in W_0^{1,p}(T_1, \mathbb{R}^N)$ such that $\psi(\overline{x}) = \int_{-b}^{b} j(t, \overline{x}(t))\mathrm{d}t > 0$. Then recalling that for almost all $t \in T$, all $\|x\| \geq 1$ and all $\lambda \geq 1$, we have $\lambda^{\mu} j(t, x) \leq j(t, \lambda x)$ we can easily see that

$$\begin{aligned}\varphi_1(\lambda\bar{x}) = \frac{\lambda^p}{p}\|\bar{x}'\|_p^p + \frac{\lambda^p}{p}\int_{-b}^{b} g(t)\|\bar{x}(t)\|^p\mathrm{d}t \\ - \int_{-b}^{b} j(t, \lambda\bar{x}(t))\mathrm{d}t \to -\infty \quad \text{as } \lambda \to +\infty.\end{aligned}$$

(Recall $\mu > p$.) So we can find $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0$ we have $\varphi_1(\lambda\bar{x}) < 0$. Define $\hat{x} \in W_0^{1,p}(T_n, \mathbb{R}^N)$ as follows:

$$\hat{x}(t) = \begin{cases} \bar{x}(t), & \text{if } t \in T_1 \\ 0, & \text{if } t \in T_n \setminus T_1 \end{cases}.$$

Then we have $\varphi_n(\lambda\hat{x}) = \varphi_1(\lambda\bar{x}) < 0$ for all $\lambda \geq \lambda_0$ (recall that $j(t, 0) = 0$ a.e. on $\mathbb{R}$).

From the proof of Theorem 1 we know that the solution $x_n \in W_{\mathrm{per}}^{1,p}(T_n, \mathbb{R}^N)$ of problem (7) is obtained via the non-smooth mountain pass theorem and so it satisfies (see [17])

$$\begin{aligned}c_n = \inf_{\gamma\in\Gamma_n}\sup_{t\in[0,1]} \varphi_n(\gamma(t)) = \varphi_n(x_n) \geq \inf[\varphi_n(x)\colon \|x\| = \rho_n] = \xi_n > 0 \\ \text{and } \ 0 \in \partial\varphi_n(x_n),\end{aligned}$$

where $\Gamma_n = \{\gamma \in C([0, 1], W_n^{1,p})\colon \gamma(0) = 0, \gamma(1) = \lambda\hat{x}\}$ with $\lambda \geq \lambda_0$. By continuous extension by constant, we see that for $n_1 \leq n_2$, we have

$$W_{n_1}^{1,p} \subseteq W_{n_2}^{1,p}, \quad \Gamma_{n_1} \subseteq \Gamma_{n_2} \quad \text{and so } \ c_{n_2} \leq c_{n_1}.$$

Therefore we have produced a decreasing sequence $\{c_n\}_{n\geq 1}$ of critical values. For every $n \geq 1$ we have

$$\begin{aligned}c_n = \varphi_n(x_n) = \frac{1}{p}\|x_n'\|_{L_n^p}^p + \frac{1}{p}\int_{-nb}^{nb} g(t)\|x_n(t)\|^p\mathrm{d}t \\ - \int_{-nb}^{nb} j(t, x_n(t))\mathrm{d}t \leq c_1.\end{aligned} \tag{8}$$

Since $0 \in \partial\varphi_n(x_n)$, we can find $x_n^* \in \partial\varphi_n(x_n)$ such that $x_n^* = 0$. So we have

$$\begin{aligned}A(x_n) + g\|x_n\|^{p-2}x_n = u_n, \quad \text{with } \ u_n \in L_n^{\infty}, \\ u_n(t) \in \partial j(t, x_n(t)) \quad \text{a.e. on } \ T_n.\end{aligned} \tag{9}$$

We take the duality brackets (for the pair $(W_n^{1,p}, (W_n^{1,p})^*)$) of (9) with $-x_n$. We obtain

$$\begin{aligned}-\|x_n'\|_{L_n^p}^p - \frac{1}{p}\int_{-nb}^{nb} g(t)\|x_n(t)\|^p\mathrm{d}t = \int_{-nb}^{nb} (u_n(t), -x_n(t))_{\mathbb{R}^N}\mathrm{d}t \\ \leq \int_{-nb}^{nb} j^0(t, x_n(t); -x_n(t))\mathrm{d}t.\end{aligned} \tag{10}$$

Multiply (8) with $\mu > p$ and then add to (10). We obtain

$$\left(\frac{\mu}{p}-1\right)\|x_n'\|^p_{L^p_n}+\left(\frac{\mu}{p}-1\right)\int_{-nb}^{nb} g(t)\|x_n(t)\|^p \mathrm{d}t \\ +\int_{-nb}^{nb}(-j^0(t,x_n(t);-x_n(t))-\mu j(t,x_n(t))\mathrm{d}t \le \mu c_1. \tag{11}$$

Using hypothesis H(j)$_3$(iv), for every $n \ge 1$ we have

$$\int_{-nb}^{nb}(-j^0(t,x_n(t);-x_n(t))-\mu j(t,x_n(t))\mathrm{d}t \\ =\int_{T_n\cap\{\|x_n(t)\|<M\}}(-j^0(t,x_n(t);-x_n(t))-\mu j(t,x_n(t))\mathrm{d}t \\ +\int_{T_n\cap\{\|x_n(t)\|\ge M\}}(-j^0(t,x_n(t);-x_n(t))-\mu j(t,x_n(t))\mathrm{d}t \ge -\xi_1,$$

for some $\xi_1 > 0$ independent of $n \ge 1$. Using this lower bound in (11), we obtain

$$\left(\frac{\mu}{p}-1\right)\|x_n\|^p_{L^p_n}+\left(\frac{\mu}{p}-1\right)\int_{-nb}^{nb} g(t)\|x_n(t)\|^p \mathrm{d}t \le c_1+\xi_1=\xi_2$$

with $\xi_2 > 0$ independent of $n \ge 1$. So it follows that

$$\|x_n\|_{W^{1,p}_n} \le \xi_3, \tag{12}$$

with $\xi_3 > 0$ independent of $n \ge 1$. Moreover, as in ([29], p. 36), we can have that

$$\|x_n\|_{L^\infty_n} \le \xi_4, \tag{13}$$

with $\xi_4 > 0$ independent of $n \ge 1$. We extend by periodicity x_n and u_n to all of $\mathbb{R}$. From (12) and since $W^{1,p}_n$ is embedded compactly in $C_n = C(T_n, \mathbb{R}^N)$, by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, hence $x \in C(\mathbb{R}, \mathbb{R}^N)$. Also because of hypothesis H(j)$_3$(iii), we have

$$\|u_n(t)\| \le \|a_1\|_\infty(1+\|x_n(t)\|^{r-1}) \le \|a_1\|_\infty(1+\xi_4^{r-1})=\xi_5 \\ \text{a.e. on } \mathbb{R} \text{ for all } n\ge 1 \text{ (see eq. (13))},$$

with $\xi_5 > 0$ independent of $n \ge 1$. So we may assume that

$$u_n \xrightarrow{w^*} u \quad \text{in } L^\infty(\mathbb{R},\mathbb{R}^N) \text{ and } u_n \xrightarrow{w} u \quad \text{in } L^q(T_m,\mathbb{R}^N) \\ \text{for all } m\ge 1 \left(\frac{1}{p}+\frac{1}{q}=1\right).$$

Evidently $u \in L^\infty(\mathbb{R}, \mathbb{R}^N) \cap L^q_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ and using Proposition VII.3.13, p. 694, of [14], we have $u(t) \in \partial j(t, x(t))$ a.e. on T_n for all $n \ge 1$, hence $u(t) \in \partial j(t, x(t))$ a.e. on $\mathbb{R}$ (recall that the multifunction $x \to \partial j(t, x(t))$ is upper semicontinuous, see [5], p. 29). For every $\tau > 0$ we have that

$$\int_{-\tau}^{\tau}\|x_n(t)-x(t)\|^p\mathrm{d}t \to 0 \quad \text{as } n\to\infty, \\ \Rightarrow \lim_{n\to\infty}\int_{-\tau}^{\tau}\|x_n(t)\|^p\mathrm{d}t=\int_{-\tau}^{\tau}\|x(t)\|^p\mathrm{d}t.$$

We can find $n_0 \geq 1$ such that for all $n \geq n_0$ we have $[-\tau, \tau] \subseteq T_{n_0}$ and then using (13) we have

$$\int_{-\tau}^{\tau} \|x_n(t)\|^p \mathrm{d}t \leq \int_{-n_0 b}^{n_0 b} \|x_n(t)\|^p \mathrm{d}t \leq \xi_3^p,$$

$$\Rightarrow \int_{-\tau}^{\tau} \|x(t)\|^p \mathrm{d}t \leq \xi_3^p.$$

Because $\tau > 0$ was arbitrary it follows that $x \in L^p(\mathbb{R}, \mathbb{R}^N)$.

Next let $\theta \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$. Then supp $\theta \subseteq T_n = [-nb, nb]$ for some $n \geq 1$. Integrating by parts we have $|\int_{\mathbb{R}}(x_n(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t| = |\int_{\mathbb{R}}(x_n'(t), \theta(t))_{\mathbb{R}^N} \mathrm{d}t|$, hence we have

$$\left|\int_{\mathbb{R}}(x_n(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t\right| = \left|\int_{-nb}^{nb}(x_n'(t), \theta(t))_{\mathbb{R}^N} \mathrm{d}t\right| \leq \|x_n'\|_{L_n^p} \|\theta\|_{L_n^q} \leq \xi_3 \|\theta\|_{L^q(\mathbb{R}, \mathbb{R}^N)}$$

(see (12) and recall that $\theta \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$). Note that $(x_n(t), \theta'(t))_{\mathbb{R}^N} \to (x(t), \theta'(t))_{\mathbb{R}^N}$ uniformly on compact sets (i.e. the convergence is in $C_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$) and $|(x_n(t), \theta'(t))_{\mathbb{R}^N}| \leq \|x_n\|_{L_n^\infty} \|\theta'(t)\| \leq \xi_4 \|\theta'(t)\|$ a.e. on T_n (see (13)).

Set

$$\eta(t) = \begin{cases} \xi_4 \|\theta'(t)\|, & \text{if } t \in \operatorname{supp} \theta \\ 0, & \text{otherwise} \end{cases}.$$

Then $\eta \in L^1(\mathbb{R})$ and we have $|(x_n(t), \theta'(t))_{\mathbb{R}^N}| \leq \eta(t)$ a.e. on $\mathbb{R}$. By the dominated convergence theorem we have

$$\int_{\mathbb{R}}(x_n(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t \to \int_{\mathbb{R}}(x(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t$$

$$\Rightarrow \left|\int_{\mathbb{R}}(x(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t\right| \leq \xi_3 \|\theta\|_{L^q(\mathbb{R}, \mathbb{R}^N)}.$$

From Proposition IX.3, p. 153 of [3], we obtain that $x \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$. Also since $u_n \xrightarrow{w} u$ in $L^q_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, we have that $\int_{\mathbb{R}}(u_n(t), \theta(t))_{\mathbb{R}^N} \to \int_{\mathbb{R}}(u(t), \theta(t))_{\mathbb{R}^N}$, while from the fact that $x_n \to x$ in $C_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ it follows that

$$\int_{\mathbb{R}} g(t)\|x_n(t)\|^{p-2}(x_n(t), \theta(t))_{\mathbb{R}^N} \mathrm{d}t \to \int_{\mathbb{R}} g(t)\|x(t)\|^{p-2}(x(t), \theta(t))_{\mathbb{R}^N} \mathrm{d}t.$$

Also from integration by parts we have

$$\int_{\mathbb{R}}(\|x_n'(t)\|^{p-2}x_n'(t), \theta'(t))_{\mathbb{R}^N} \mathrm{d}t = -\int_{\mathbb{R}}((\|x_n'(t)\|^{p-2}x_n'(t))', \theta(t))_{\mathbb{R}^N} \mathrm{d}t.$$

Because x_n is a solution of (7), we see that $(\|x_n'(\cdot)\|^{p-2}x_n'(\cdot))' \in L_n^q$ for all $n \geq 1$. From this we obtain that $\|x_n'(\cdot)\|^{p-2}x_n'(\cdot) \in W_n^{1,q}$ for all $n \geq 1$. Also by passing to a subsequence if necessary, we may assume that $\|x_n'(\cdot)\|^{p-2}x_n'(\cdot) \xrightarrow{w} v$ in $W^{1,q}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, hence $\|x_n'(\cdot)\|^{p-2}x_n'(\cdot) \to v$ in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ (and in $C_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ too). If $\sigma\colon \mathbb{R}^N \to \mathbb{R}^N$ is defined by $\sigma(x) = \|x\|^{p-2}x$, we have $\sigma^{-1}(\|x_n'(\cdot)\|^{p-2}x_n'(\cdot)) = x_n'(\cdot) \to \sigma^{-1}(v)$ in $L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ and so $\sigma^{-1}(v) = x$, hence $v(\cdot) = \|x'(\cdot)\|^{p-2}x'(\cdot)$. Therefore we have

$$\int_{\mathbb{R}} (\|x_n'(t)\|^{p-2}x_n'(t), \theta'(t))_{\mathbb{R}^N}\,\mathrm{d}t \to \int_{\mathbb{R}} (\|x'(t)\|^{p-2}x'(t), \theta'(t))_{\mathbb{R}^N}\,\mathrm{d}t$$
$$= -\int_{\mathbb{R}} ((\|x'(t)\|^{p-2}x'(t))', \theta'(t))_{\mathbb{R}^N}\,\mathrm{d}t \quad \text{(by integration by parts)}.$$

Since for all $n \geq 1$ large we have

$$\int_{\mathbb{R}} (\|x_n'(t)\|^{p-2}x_n'(t), \theta'(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_{\mathbb{R}} g(t)\|x_n(t)\|^{p-2}(x_n(t), \theta(t))_{\mathbb{R}^N}\,\mathrm{d}t = \int_{\mathbb{R}} (u(t), \theta(t))_{\mathbb{R}^N}\,\mathrm{d}t$$

by passing to the limit as $n \to \infty$ and using the convergences established above, we obtain

$$-\int_{\mathbb{R}} ((\|x'(t)\|^{p-2}x'(t))', \theta(t))_{\mathbb{R}^N}\,\mathrm{d}t + \int_{\mathbb{R}} g(t)\|x(t)\|^{p-2}(x(t), \theta(t))_{\mathbb{R}^N}\,\mathrm{d}t = \int_{\mathbb{R}} (u(t), \theta(t))_{\mathbb{R}^N}\,\mathrm{d}t.$$

Because $\theta \in C_0^\infty(\mathbb{R},\mathbb{R}^N)$ is arbitrary, it follows that

$$-(\|x'(t)\|^{p-2}x'(t))' + g(t)\|x(t)\|^{p-2}x(t) = u(t) \quad \text{a.e. on } \mathbb{R}$$

and $u \in L^q_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$, $u(t) \in \partial j(t, x(t))$ a.e. on $\mathbb{R}$.

Next we show that $x(\pm\infty) = x'(\pm\infty) = 0$. Recall that from previous arguments we have $x \in W^{1,p}(\mathbb{R},\mathbb{R}^N)$. So from Corollary VII.8, p. 130 of [3], we have $x(t) \to 0$ as $|t| \to \infty$. Hence we have $x(\pm\infty) = 0$.

Since $u(t) \in \partial j(t, x(t))$ a.e. on $\mathbb{R}$, from hypothesis H(j)$_3$(iii) we have $\|u(t)\| \leq a_1(t)(1 + \|x(t)\|^{p-1})$ a.e. on $\mathbb{R}$. Because $x \in W^{1,p}(\mathbb{R},\mathbb{R}^N)$, we have $\|x(\cdot)\|^{p-2}x(\cdot) \in L^q(\mathbb{R},\mathbb{R}^N)$ and so $u \in L^q(\mathbb{R},\mathbb{R}^N)$. Therefore, $\|x'(\cdot)\|^{p-2}x'(\cdot) \in W^{1,q}(\mathbb{R},\mathbb{R}^N)$ and once again from p. 130 of [3], we have that $\|x'(t)\|^{p-1} \to 0$ as $|t| \to \infty$, hence $x'(t) \to 0$ as $|t| \to \infty$. Therefore $x'(\pm\infty) = 0$ and we have proved that x is a homoclinic (to 0) solution.

It remains to show that x is non-trivial. For every $n \geq 1$, we have

$$A(x_n) + g\|x_n\|^{p-2}x_n = u_n$$
$$\Rightarrow c\|x_n\|^p_{L^p_n} \leq \int_{-nb}^{nb} (u_n(t), x_n(t))_{\mathbb{R}^N}\,\mathrm{d}t.$$

Set

$$h_n(t) = \begin{cases} \dfrac{(u_n(t), x_n(t))_{\mathbb{R}^N}}{\|x_n(t)\|^p}, & \text{if } x_n(t) \neq 0 \\ 0, & \text{if } x_n(t) = 0 \end{cases}.$$

We have

$$c\|x_n\|^p_{L^p_n} \le \int_{-nb}^{nb} (u_n(t), x_n(t))_{\mathbb{R}^N}\,\mathrm{d}t = \int_{-nb}^{nb} h_n(t)\|x_n(t)\|^p\mathrm{d}t \le \operatorname*{ess\,sup}_{T_n} h_n\|x_n\|^p_{L^p_n}.$$

By virtue of hypothesis H(j)$_3$(v) (see the remark following H(j)$_3$), given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that for almost all $t \in \mathbb{R}$, all $\|x\| \le \delta$ and all $u \in \partial j(t, x)$ we have

$$\frac{(u, x)_{\mathbb{R}^N}}{\|x\|^p} \le \varepsilon. \tag{14}$$

If $x = 0$, then $x_n \to 0$ in $C_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ and so we can find $n_0 \ge 1$ such that for all $n \ge n_0$ and all $t \in T_n$, we have $\|x_n(t)\| \le \delta$. Therefore for all $n \ge n_0$ and almost all $t \in T_n$, we have $h_n(t) \le \varepsilon$ (see (14)) and so $c \le \operatorname{ess\,sup}_{T_n} h_n = \operatorname{ess\,sup}_{\mathbb{R}} h_n \le \varepsilon$ for all $n \ge n_0$ (recall that x_n, u_n were extended by periodicity to all of $\mathbb{R}$). Let $\varepsilon \downarrow 0$ to obtain $0 < c \le 0$, a contradiction. This proves that $x \neq 0$.

Therefore $x \in C(\mathbb{R}, \mathbb{R}^N) \cap W^{1,p}(\mathbb{R}, \mathbb{R}^N)$ is the desired non-trivial, homoclinic (to 0) solution of the non-smooth non-linear periodic system. QED

6. Scalar equations

In this last part of the paper we study the scalar (i.e. $N = 1$) problem. We approach the problem using a generalized Landesmann–Lazer type condition, which is more general than the one used by Tang [33] in the context of smooth semilinear periodic equations. So our work is a two-fold generalization of the work of Tang.

First we examine the following non-linear scalar periodic problem:

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t, x(t)) \quad \text{a.e on } T = [0, b] \\ x(0) = x(b), \quad x'(0) = x'(b), \quad 1 < p < \infty \end{array} \right\}. \tag{15}$$

The conditions on the non-smooth potential j are the following:

H(j)$_4$: j: $T \times \mathbb{R} \longmapsto \mathbb{R}$ is a functional such that $j(\cdot, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \longmapsto j(t, x)$ is measurable;
(ii) for almost all $t \in T$, the function $x \longmapsto j(t, x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t, x)$, we have

$$|u| \le a_1(t) + c_1(t)|x|^{r-1},$$

$1 \le r < +\infty$ with $a_1, c_1 \in L^{r'}(T)$, $\frac{1}{r} + \frac{1}{r'} = 1$;
(iv) $\lim\limits_{|x|\to\infty} \frac{u}{x} = 0$ uniformly for almost all $t \in T$ and all $u \in \partial j(t, x)$;
(v) there exist functions $j_+, j_- \in L^1(T)$ such that $j_+(t) = \liminf\limits_{x\to+\infty} \frac{j(t,x)}{x}$ and $j_-(t) = \limsup\limits_{x\to-\infty} \frac{j(t,x)}{x}$ a.e. on T and $\int_0^b j_-(t)\mathrm{d}t < 0 < \int_0^b j_+(t)\mathrm{d}t$.

We consider the locally Lipschitz functional φ: $W^{1,p}_{\mathrm{per}}(T) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\|x'\|^p_p - \int_0^b j(t, x(t))\mathrm{d}t.$$

PROPOSITION 4

If hypotheses H(j)$_4$ *hold, then* φ *satisfies the non-smooth PS-condition.*

Proof. We consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$ such that

$$|\varphi(x_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ and all } n \geq 1 \text{ and } m(x_n) \to 0.$$

As before we choose $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We have

$$x_n^* = A(x_n) - u_n \quad \text{with } u_n \in L^{r'}(T), u_n(t) \in \partial j(t, x_n(t)) \text{ a.e. on } T.$$

We claim that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W^{1,p}_{\text{per}}(T) \quad \text{and} \quad y_n \to y \text{ in } C_{\text{per}}(T).$$

(Recall that $W^{1,p}_{\text{per}}(T)$ is embedded compactly in $C_{\text{per}}(T)$.) From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$, we have

$$\frac{|\varphi(x_n)|}{\|x_n\|^p} = \left| \frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t, x_n(t))}{\|x_n\|^p}\mathrm{d}t \right| \leq \frac{M_1}{\|x_n\|^p}. \tag{16}$$

By virtue of hypothesis H(j)$_4$(v) we have $\int_0^b \frac{j(t,x_n(t))}{\|x_n\|^p}\mathrm{d}t \to 0$. So from (16) and the weak lower semicontinuity of the norm in a Banach space, we obtain $\frac{1}{p}\|y'\|_p^p = 0$, hence $y = c \in \mathbb{R}$. If $c = 0$, then we have $\|y'\|_p \to 0$, hence $y_n \to 0$ in $W^{1,p}_{\text{per}}(T)$. But for every $n \geq 1$, $\|y_n\| = 1$ and so we have a contradiction. Therefore $y = c \neq 0$ and without any loss of generality we may assume that $y = c > 0$ (the analysis is the same if instead we assume that $y = c < 0$). Recall that $W^{1,p}_{\text{per}}(T) = \mathbb{R} \oplus V$ with $V = \{v \in W^{1,p}_{\text{per}}(T)\colon \int_0^b v(t)\mathrm{d}t = 0\}$. We have $x_n = \bar{x}_n + \hat{x}_n$ with $\bar{x}_n \in \mathbb{R}$, $\hat{x}_n \in V$, $n \geq 1$. Then $y_n = \bar{y}_n + \hat{y}_n$ with $\bar{y}_n = \frac{\bar{x}_n}{\|x_n\|}$, $\hat{y}_n = \frac{\hat{x}_n}{\|x_n\|}$, $n \geq 1$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{\text{per}}(T)$, we have $-\varepsilon_n \leq \langle x_n^*, y_n\rangle \leq \varepsilon_n$ with $\varepsilon_n \downarrow 0$, hence

$$-\varepsilon_n \leq \frac{1}{\|x_n\|}\left[\|\hat{x}_n'\|_p^p - \int_0^b u_n(t)\hat{x}_n(t)\mathrm{d}t\right] \leq \varepsilon_n. \tag{17}$$

Since we have assumed that $y = c > 0$, we have that for all $t \in T$, $x_n(t) \to +\infty$ as $n \to \infty$. We claim that this convergence is uniform in $t \in T$. Indeed let $\varepsilon > 0$ be such that $0 < \varepsilon < c$ (recall that we have assumed $c > 0$). Since $y_n \to c$ in $C(T)$, we can find $n_0 \geq 1$ such that for all $n \geq n_0$ and all $t \in T$, $|y_n(t) - c| < \varepsilon$, hence $0 < c - \varepsilon < |y_n(t)|$. Because $\|x_n\| \to \infty$ given $\beta > 0$ we can find $n_1 \geq 1$ such that for all $n \geq n_1$ we have $\|x_n\| \geq \beta > 0$. So for all $n \geq n_2 = \max\{n_0, n_1\}$ and all $t \in T$ we have $\frac{|x_n(t)|}{\beta} \geq \frac{|x_n(t)|}{\|x_n\|} = |y_n(t)| > c - \varepsilon = \theta > 0 \Rightarrow |x_n(t)| \geq \beta\theta > 0$. Because $\beta > 0$ is arbitrary and $\theta > 0$, we infer that $\min_T |x_n(t)| \to +\infty$. Then we have

$$\int_0^b u_n(t)\hat{x}_n(t)\mathrm{d}t = \int\limits_{\{x_n(t)\neq 0\}} \frac{u_n(t)}{x_n(t)} x_n(t)\hat{x}_n(t)\mathrm{d}t.$$

Evidently $|\{x_n = 0\}| \to 0$, while by virtue of hypothesis H(j)$_4$(iv), given $\varepsilon > 0$ we can find $n_3 \geq 1$ such that for all $n \geq n_3$ and almost all $t \in T$ we have $|u_n(t)/x_n(t)| < \varepsilon$. So for $n \geq n_3$ we have

$$\left| \int\limits_{\{x_n(t)\neq 0\}} \frac{u_n(t)}{x_n(t)} x_n(t)\hat{x}_n(t)\mathrm{d}t \right| \leq \varepsilon \|\hat{x}_n\|_2^2 \leq \varepsilon c_2 \|\hat{x}_n\|^p \quad \text{for some } \ c_2 > 0.$$

From (17) and the Poincaré–Wirtinger inequality, we have

$$\frac{1}{\|x_n\|}[c_3\|\hat{x}_n\|^p - \varepsilon c_2\|\hat{x}_n\|^p] = (c_3 - \varepsilon c_2)\frac{\|\hat{x}_n\|^p}{\|x_n\|} \leq \varepsilon_n \quad \text{for some } \ c_3 > 0.$$

Choose $\varepsilon > 0$ such that $\varepsilon c_2 < c_3$. Since $\varepsilon_n \downarrow 0$ it follows that $\frac{\|\hat{x}_n\|^p}{\|x_n\|} \to 0$, hence once again by the Poincaré–Wirtinger inequality, we have

$$\frac{\|x_n'\|_p^p}{\|x_n\|} \to 0 \quad \text{as } \ n \to \infty. \tag{18}$$

Recall that for $n \geq 1$, we have

$$-\frac{M_1}{\|x_n\|} \leq \frac{1}{p}\frac{\|x_n'\|_p^p}{\|x_n\|} - \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t \leq \frac{M_1}{\|x_n\|}. \tag{19}$$

(We assume without any loss of generality that $\|x_n\| \geq \xi > 0$ for all $n \geq 1$; recall that $\|x_n\| \to \infty$.) We can write that

$$\int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t + \int\limits_{\{x_n(t)\neq 0\}} \frac{j(t, x_n(t))}{x_n(t)} y_n(t)\mathrm{d}t + \int\limits_{\{x_n(t)=0\}} \frac{j(t,0)}{\|x_n\|}\mathrm{d}t.$$

By virtue of hypothesis H(j)$_4$(v), given $\varepsilon > 0$ we can find $M = M(\varepsilon) > 0$ such that for almost all $t \in T$ and all $x \geq M$, we have $j(t,x)/x \geq j_+(t) - \varepsilon$. Recall that $x_n(t) \to +\infty$ uniformly in $t \in T$ (i.e. $\min_T x_n \to +\infty$). So we can find $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$\begin{aligned}
& j_+(t) - \varepsilon \leq \frac{j(t, x_n(t))}{x_n(t)} \quad \text{a.e. on } \ T \\
\Rightarrow & \int\limits_{\{x_n(t)\neq 0\}} (j_+(t) - \varepsilon) y_n(t) \leq \int\limits_{\{x_n(t)\neq 0\}} \frac{j(t, x_n(t))}{x_n(t)} y_n(t)\mathrm{d}t \\
\Rightarrow & \int_0^b j_+(t) c\mathrm{d}t \leq \liminf_{n\to\infty} \int\limits_{\{x_n(t)\neq 0\}} \frac{j(t, x_n(t))}{x_n(t)} y_n(t)\mathrm{d}t \\
& \qquad\qquad (\text{since } \varepsilon > 0 \text{ was arbitrary}).
\end{aligned}$$

Also we have $\int_{\{x_n(t)=0\}} (j(t,0)/\|x_n\|)\mathrm{d}t \to 0$. So finally we have

$$c\int_0^b j_+(t)\mathrm{d}t \leq \liminf_{n\to\infty} \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t.$$

Using this and (18) in (19) we obtain $c\int_0^b j_+(t)\mathrm{d}t \le 0$, hence $\int_0^b j_+(t)\mathrm{d}t \le 0$ (recall that we have assumed that $c > 0$). This contradicts hypothesis H(j)$_4$(v). Therefore $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{\mathrm{per}}(T)$ is bounded and then arguing as in the proof of Theorem 1 we conclude that φ satisfies the non-smooth PS-condition. QED

PROPOSITION 5

If hypotheses H(j)$_4$ *hold, then* $\lim_{\substack{|\xi|\to\infty\\ \xi\in\mathbb{R}}} \varphi(\xi) = -\infty$ (*i.e.* $\varphi|_{\mathbb{R}}$ *is anticoercive*).

Proof. Suppose that the result of the proposition is not true. Then we can find $\{\xi_n\}_{n\ge1} \subseteq \mathbb{R}$ such that $|\xi_n| \to \infty$ and $\gamma \in \mathbb{R}$ such that $\gamma \le \varphi(\xi_n)$ for all $n \ge 1$. So $\gamma \le \liminf_{n\to\infty} \varphi(\xi_n) = \liminf_{n\to\infty}(-\int_0^b j(t,\xi_n)\mathrm{d}t)$. First suppose that $\xi_n \to +\infty$. We may assume that $\xi_n > 0$ for all $n \ge 1$. We have

$$\begin{aligned} &\frac{\gamma}{\xi_n} \le -\int_0^b \frac{j(t,\xi_n)}{\xi_n}\mathrm{d}t,\\ \Rightarrow &\limsup_{n\to\infty} \int_0^b \frac{j(t,\xi_n)}{\xi_n}\mathrm{d}t \le 0. \end{aligned}$$

On the other hand as in the proof of Proposition 4, we obtain that

$$\int_0^b j_+(t)\mathrm{d}t \le \liminf_{n\to\infty} \int_0^b \frac{j(t,\xi_n)}{\xi_n}\mathrm{d}t \le 0,$$

which is a contradiction. Similarly if $\xi_n \to -\infty$, we obtain $\int_0^b j_-(t)\mathrm{d}t \ge 0$, again a contradiction. QED

Recall the direct sum decomposition $W^{1,p}_{\mathrm{per}}(T) = \mathbb{R} \oplus V$ with $V = \{v \in W^{1,p}_{\mathrm{per}}(T)\colon \int_0^b v(t)\mathrm{d}t = 0\}$.

PROPOSITION 6

If hypotheses H(j)$_4$ *hold, then* $\varphi|_V$ *is coercive* (*i.e.* $\varphi(v) \to +\infty$ *as* $\|v\| \to \infty$, $v \in V$).

Proof. For every $v \in V$ we have $\varphi(v) = \frac{1}{p}\|v'\|_p^p - \int_0^b j(t,v(t))\mathrm{d}t$. From the Poincaré–Wirtinger inequality we know that on V, $\|v'\|_p$ is an equivalent norm. So

$$\frac{\varphi(v)}{\|v\|^p} \ge c_4 - \int_0^b \frac{j(t,v(t))}{\|v\|^p}\mathrm{d}t \quad \text{for some } c_4 > 0.$$

Recall that $\int_0^b (j(t,v(t))/\|v\|^p)\mathrm{d}t \to 0$. So $\liminf_{\substack{\|v\|\to\infty\\ v\in V}} \frac{\varphi(v)}{\|v\|^p} \ge c_4 > 0$, hence $\varphi(v) \to +\infty$ as $\|v\| \to \infty$, $v \in V$. QED

These auxiliary results lead to the following existence theorem.

Theorem 7. *If hypotheses* H(j)$_4$ *hold, then problem* (15) *has at least one solution* $x \in C^1(T)$ *with* $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,r'}(T)$.

Proof. Propositions 4–6 permit the application of the non-smooth saddle point theorem. So we obtain $x \in W^{1,p}_{\text{per}}(T)$ such that $0 \in \partial\varphi(x)$, hence $A(x) = u$ with $u \in L^{r'}(T)$, $u(t) \in \partial j(t, x(t))$ a.e. on T. As in the proof of Theorem 1 we show that $x \in C^1(T)$, $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,r'}(T)$ and it solves problem (15). QED

As we have already mentioned in the beginning of this section, our generalized Landesmann–Lazer type condition (see hypothesis H(j)$_4$(v)) generalizes the one used by Tang [33] (for smooth potentials). In the next proposition we are going to show this. For this purpose we introduce the following functions:

$$g_1(t, x) = \min\{u: u \in \partial j(t, x)\} \quad \text{and} \quad g_2(t, x) = \max\{u: u \in \partial j(t, x)\},$$

$$G_1(t, x) = \begin{cases} \frac{2j(t,x)}{x} - g_1(t, x), & \text{if } |x| \neq 0 \\ 0, & \text{if } |x| = 0 \end{cases}$$

and

$$G_2(t, x) = \begin{cases} \frac{2j(t,x)}{x} - g_2(t, x), & \text{if } |x| \neq 0 \\ 0, & \text{if } |x| = 0 \end{cases},$$

$$G_1^-(t) = \limsup_{x\to-\infty} G_1(t, x) \quad \text{and} \quad G_2^+(t) = \liminf_{x\to+\infty} G_2(t, x).$$

The functions G_1^-, G_2^+ are essentially the ones used by Tang [33] in the context of smooth, semilinear (i.e. $p = 2$) periodic problems. In that case, since $j(t, \cdot) \in C^1(\mathbb{R})$, we have $g_1 = g_2$, and hence $G_1 = G_2$.

PROPOSITION 8

For all $t \in T \setminus D$ with $|D| = 0$, $G_2^+(t) \le j_+(t)$ and $j_-(t) \le G_1^-(t)$.

Proof. Let $D \subseteq T$ the Lebesgue-null set outside of which hypotheses H(j)$_4$(ii) $\to$ (v) hold. Let $t \in T \setminus D$ and set $k_\varepsilon^+(t) = G_2^+(t) - \varepsilon$. We can find $M_1 > 0$ such that for all $x \ge M_1$ we have

$$\begin{aligned} &G_2^+(t) - \varepsilon = k_\varepsilon^+(t) \le G_2(t, x) \\ \Rightarrow &\frac{k_\varepsilon^+(t)}{x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{k_\varepsilon^+(t)}{x}\right) \le \frac{G_2(t, x)}{x^2}. \end{aligned}$$

For all $u \in \partial j(t, x)$, we have

$$\frac{G_2(t, x)}{x^2} = \frac{2j(t, x)}{x^3} - \frac{g_2(t, x)}{x^2} \le \frac{2j(t, x)}{x^3} - \frac{u}{x^3}.$$

From p. 48 of [5], we know that $x \to 2j(t, x)/x^2$ is locally Lipschitz on $[M_1, +\infty)$ and

$$\partial\left(\frac{j(t, x)}{x^2}\right) \subseteq \frac{\partial j(t, x)}{x^2} - \frac{2j(t, x)}{x^3}.$$

Therefore for all $t \in T$, all $x \ge M_1$ and all $u \in \partial j(t, x)$, we have

$$\begin{aligned} &u \le \frac{g_2(t, x)}{x^2} - \frac{2j(t, x)}{x^3} = -\frac{1}{x^2} G_2(t, x) \\ \Rightarrow &u \le \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{k_\varepsilon^+(t)}{x}\right). \end{aligned}$$

Since for $t \in T \setminus D$, the function $t \to j(t,x)/x^2$ is locally Lipschitz on $[M_1, +\infty)$, it is differentiable at every $x \in [M_1, +\infty) \setminus D_1(t)$, $|D_1(t)| = 0$. We set

$$u_0(t,x) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}(\frac{j(t,x)}{x^2}), & \text{if } x \in [M_1, +\infty) \setminus D_1(t), \\ 0, & \text{if } x \in D_1(t). \end{cases}$$

We fix $t \in T \setminus D$ and choose $x \in [M_2, +\infty) \setminus D_1(t)$. Then $u_0(t,x) \in \partial(j(t,x)/x^p)$ and so

$$u_0(t,x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{j(t,x)}{x^2}\right) \le \frac{\mathrm{d}}{\mathrm{d}x}\frac{k_\varepsilon^+(t)}{x}. \tag{20}$$

Let $y < x$ and $y \in [M_1, +\infty) \setminus D_1(t)$. We integrate (20) over the interval $[y, x]$ and obtain

$$\frac{j(t,x)}{x^2} - \frac{j(t,y)}{y^2} \le k_\varepsilon^+(t)\left(\frac{1}{x} - \frac{1}{y}\right). \tag{21}$$

By virtue of hypotheses H(j)$_4$(iii), (iv) and the Lebourg mean value theorem, given $\varepsilon > 0$ for all $t \in T \setminus D$ and all $x \ge 0$, we have

$$\begin{aligned} & -\varepsilon x^2 - c_1 x + j(t,0) \le j(t,x) \quad \text{for some } c_1 > 0 \\ \Rightarrow & -\varepsilon \le \liminf_{x \to +\infty} \frac{j(t,x)}{x^2}, \\ \Rightarrow & \ 0 \le \liminf_{x \to +\infty} \frac{j(t,x)}{x^2}. \end{aligned}$$

So if we go to (21) and pass to the limit as $x \to +\infty$, we obtain

$$\begin{aligned} & k_\varepsilon^+(t) \le \frac{j(t,y)}{y}, \\ \Rightarrow & \ G_2^+(t) \le \liminf_{x \to +\infty} \frac{j(t,y)}{y} = j_+(t). \end{aligned}$$

Similarly we obtain that for all $t \in T \setminus D$, $|D| = 0$, we have $j_-(t) \le G_1^-(t)$. QED

Remark. This proposition shows that our generalized Landesman–Lazer type condition (hypothesis H(j)$_4$(v)) is more general than the one used by Tang [33]. Here is an example of a non-smooth locally Lipschitz potential which satisfies H(j)$_4$(v) but does not satisfy the condition of Tang. Again for simplicity we drop the t-dependence

$$j(x) = \max\{x^{1/3}, x^{1/2}\} + \ln(1 + |x|) + \cos x + x.$$

A simple calculation shows that $j_+ = 1$, $j_- = -1$ but $G_1^- = G_2^+ = 0$.

When dealing with the semilinear (i.e $p = 2$) case, we can consider problems at resonance in an eigenvalue of any order. Similar problems (but with smooth potential) were studied by Mawhin and Ward [25], p. 67 of Mawhin and Willem [26], Mawhin and Schmitt [24] (problems near resonance) and Tang [33] (who employed his more restrictive version of the generalized Landesman–Lazer condition (see Proposition 8).

The problem under consideration is the following:

$$\left\{\begin{array}{ll} -x''(t) - m^2\omega^2 x(t) \in \partial j(t, x(t)) - h(t) & \text{a.e on } T = [0, b] \\ x(0) = x(b), x'(0) = x'(b),\ h \in L^1(T) & \end{array}\right\}. \tag{22}$$

Here $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and $\omega = 2\pi/b$ (see §2). Our hypotheses on $j(t, x)$ are the following:

H(j)$_5$: j: $T \times \mathbb{R} \longmapsto \mathbb{R}$ is a functional such that $j(\cdot, 0) \in L^1(T)$ and

(i) for all $x \in \mathbb{R}$, $t \longmapsto j(t, x)$ is measurable;
(ii) for almost all $t \in T$, the function $x \longmapsto j(t, x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t, x)$, we have

$$|u| \leq a_1(t)(1 + |x|^{r-1}),$$

$1 \leq r < +\infty$ with $a_1 \in L^{r'}(T)$, $\frac{1}{r} + \frac{1}{r'} = 1$;
(iv) $\lim\limits_{|x|\to\infty} \frac{u}{x} = 0$ uniformly for almost all $t \in T$ and all $u \in \partial j(t, x)$;
(v) there exist functions $j_\pm \in L^1(T)$ such that $j_+(t) = \liminf_{x\to+\infty} \frac{j(t,x)}{x}$ and $j_-(t) = \limsup_{x\to-\infty} \frac{j(t,x)}{x}$ a.e. on T and $\int_0^b h(t)\sin(m\omega t + \theta)\mathrm{d}t < \int_0^b j_+(t)\sin(m\omega t + \theta)^+ - j_-(t)\sin(m\omega t + \theta)^- \mathrm{d}t$ for all $\theta \in \mathbb{R}$.

In our analysis of problem (22) we shall use the following subspaces of $W^{1,2}_{\mathrm{per}}(T)$:

$$\begin{aligned} \bar{H} &= \operatorname{span}\{\sin k\omega t, \cos k\omega t : k = 0, 1, ..., m-1\}, \\ N_m &= \operatorname{span}\{\sin m\omega t, \cos m\omega t\}, \\ \hat{H} &= (\bar{H} + N_m)^\perp = \operatorname{span}\{\sin k\omega t, \cos k\omega t : k \geq m+1\}. \end{aligned}$$

We have $W^{1,2}_{\mathrm{per}}(T) = \bar{H} \oplus N_m \oplus \hat{H}$ and so if $x \in W^{1,2}_{\mathrm{per}}(T)$, we have $x = \bar{x} + x^0 + \hat{x}$ with $\bar{x} \in \bar{H}$, $x^0 \in N_m$ and $\hat{x} \in \hat{H}$.

We start with an auxiliary result concerning the subspace $\hat{H}$.

Lemma 9. *There exists* $c > 0$ *such that for all* $x \in \hat{H}$ *we have* $c\|x\|^2 \leq \|x'\|_2^2 - \lambda_m\|x\|_2^2$.

Proof. Let $\psi(x) = \|x'\|_2^2 - \lambda_m\|x\|_2^2$ and suppose that the result is not true. We can find $\{x_n\}_{n\geq 1} \subseteq \hat{H}$ such that $\psi(x_n) \downarrow 0$. Set $y_n = x_n/\|x_n\|$, $n \geq 1$. We may assume that $y_n \xrightarrow{w} y$ in $W^{1,2}_{\mathrm{per}}(T)$ and $y_n \to y$ in $L^2(T)$. Thus in the limit as $n \to \infty$, we obtain $\|y'\|_2^2 \leq \lambda_m\|y\|_2^2$ and so $y = 0$. Hence $\|y_n'\|_2 \to 0$ and so $y_n \to 0$ in $W^{1,2}_{\mathrm{per}}(T)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. QED

We introduce the locally Lipschitz functional φ: $W^{1,2}_{\mathrm{per}}(T) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{m^2\omega^2}{2}\|x\|_2^2 - \int_0^b j(t, x(t))\mathrm{d}t + \int_0^b h(t)x(t)\mathrm{d}t.$$

PROPOSITION 10

If hypotheses H(j)$_5$ *hold, then* φ *satisfies the non-smooth PS-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(T)$ be a sequence such that

$$|\varphi(x_n)| \leq M_1 \text{ for all } n \geq 1 \text{ and some } M_1 > 0$$

and

$$m(x_n) \to 0 \text{ as } n \to \infty.$$

Again we find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$ and $x_n^* = A(x_n) - m^2\omega^2 x_n - u_n + h$, with $u_n \in L^{r'}(T)$, $u_n(t) \in \partial j(t, x_n(t))$ a.e. on T. Here $A \in \mathcal{L}(W^{1,2}_{\text{per}}(T), W^{1,2}_{\text{per}}(T)^*)$ is defined $\langle A(x), y\rangle = \int_0^b x'(t)y'(t)\mathrm{d}t$ for all $x, y \in W^{1,2}_{\text{per}}(T)$. Of course A is a maximal monotone, bounded linear operator. We claim that $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(T)$ is bounded. If this is not the case, we may assume that $\|x_n\| \to \infty$. We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$ and we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,2}_{\text{per}}(T) \text{ and } y_n \to y \text{ in } C_{\text{per}}(T) \text{ as } n \to \infty.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(T)$ we have

$$\begin{aligned} &|\langle x_n^*, v\rangle| \leq \varepsilon_n \|v\| \text{ for all } v \in W^{1,2}_{\text{per}}(T) \text{ with } \varepsilon_n \downarrow 0, \\ \Rightarrow &\left| \int_0^b y_n' v' \mathrm{d}t - \lambda_m \int_0^b y_n v \mathrm{d}t - \int_0^b \frac{u_n}{\|x_n\|} v\, \mathrm{d}t + \int_0^b \frac{h}{\|x_n\|} v\, \mathrm{d}t \right| \leq \varepsilon_n \frac{\|v\|}{\|x_n\|} \\ &\text{with } \lambda_m = m^2\omega^2. \end{aligned} \tag{23}$$

By virtue of hypothesis H(j)$_5$(iv), given $\varepsilon > 0$ we can find $M > 0$ such that for almost all $t \in T$, all $|x| \geq M$ and all $u \in \partial j(t, x)$, we have $|u/x| \leq \varepsilon$. So we can write that

$$\begin{aligned} \left| \int_0^b \frac{u_n}{\|x_n\|} v \mathrm{d}t \right| &= \left| \int_{\{|x_n(t)|\geq M\}} \frac{u_n}{x_n} y_n v\, \mathrm{d}t + \int_{\{|x_n(t)|<M\}} \frac{u_n}{\|x_n\|} v\, \mathrm{d}t \right| \\ &\leq \varepsilon \|y_n\|_\infty \|v\|_1 + \int_{\{|x_n(t)<M|\}} \frac{a_1(t)(1+M^{r-1})}{\|x_n\|} |v| \mathrm{d}t \\ &\qquad \text{(see hypothesis H(j)}_5\text{(iii))} \\ \Rightarrow \limsup_{n\to\infty} \left| \int_0^b \frac{u_n}{\|x_n\|} v\, \mathrm{d}t \right| &\leq \varepsilon \|y\|_\infty \|v\|_1. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we infer that $\lim_{n\to\infty} \int_0^b \frac{u_n}{\|x_n\|} v \mathrm{d}t = 0$. Also we have $\lim_{n\to\infty} \int_0^b \frac{h}{\|x_n\|} v \mathrm{d}t = 0$. So from (23) and if $v = y_n - y$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \left[\int_0^b y_n'(y_n - y)' \mathrm{d}t - \lambda_m \int_0^b y_n(y_n - y)\mathrm{d}t \right] = 0, \\ \Rightarrow &\lim_{n\to\infty} \int_0^b y_n'(y_n' - y')\mathrm{d}t \quad (\text{recall that } y_n \to y \text{ in } C_{\text{per}}(T)), \\ \Rightarrow &\|y_n'\|_2 \to \|y'\|_2. \end{aligned}$$

Because $y_n' \xrightarrow{w} y'$ in $L^2(T)$, it follows that $y_n' \to y'$ in $L^2(T)$ and so $y_n \to y$ in $W^{1,2}_{\text{per}}(T)$.

Since $|\varphi(x_n)|/\|x_n\|^2 \leq M_1/\|x_n\|^2$ and $\int_0^b (j(t, x_n(t))/\|x_n\|^2)\mathrm{d}t \to 0$ (see hypothesis H(j)$_5$(v)), if in (23) $v = y_n$ and we pass to the limit as $n \to \infty$, we obtain that $\|y'\|_2^2 = \lambda_m \|y\|_2^2$, hence $y(t) = \xi_1 \sin m\omega t + \xi_2 \cos m\omega t$ with $\xi_1, \xi_2 \in \mathbb{R}$ and so $y(t) = r \sin(m\omega t + \theta)$ with $r = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}$, $\tan\theta = \xi_1/\xi_2$.

We write $y_n = \bar{y}_n + y_n^0 + \hat{y}_n$ with $\bar{y}_n \in \bar{H}$, $y_n^0 \in N_m$, $\hat{y}_n \in \hat{H}$, $n \geq 1$. Using $v = -\bar{y}_n + y_n^0 + \hat{y}_n \in W^{1,2}_{\mathrm{per}}(T)$ as our test function, we obtain

$$\begin{aligned}
&\left| \int_0^b x_n'(-\bar{y}_n + y_n^0 + \hat{y}_n)'\mathrm{d}t - \lambda_m \int_0^b x_n(-\bar{y}_n + y_n^0 + \hat{y}_n)\mathrm{d}t \right.\\
&\left. - \int_0^b u_n(-\bar{y}_n + y_n^0 + \hat{y}_n)\mathrm{d}t + \int_0^b h(-\bar{y}_n + y_n^0 + \hat{y}_n)\mathrm{d}t \right|\\
&\leq \varepsilon_n \| - \bar{y}_n + y_n^0 + \hat{y}_n\| \leq 3\varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0\\
&\Rightarrow \frac{1}{\|x_n\|} \left| \int_0^b x_n'(-\bar{x}_n' + x_n^{0'} + \hat{x}_n')\mathrm{d}t - \lambda_m \int_0^b x_n(-\bar{x}_n + x_n^0 + \hat{x}_n)\mathrm{d}t \right|\\
&- \left| \int_0^b \left(\frac{u_n - h}{\|x_n\|}\right)(-\bar{x}_n + x_n^0 + \hat{x}_n)\mathrm{d}t \right| \leq 3\varepsilon_n
\end{aligned} \tag{24}$$

where $x_n = \bar{x}_n + x_n^0 + \hat{x}_n$ with $\bar{x}_n \in \bar{H}$, $x_n^0 \in N_m$, $\hat{x}_n \in \hat{H}$. Because $x_n^0 \in N_m$, we have $x_n^0 = \xi_n^1 \sin m\omega t + \xi_n^2 \cos m\omega t$ and so for all $n \geq 1$ we have $\|x_n^{0'}\|_2^2 = \lambda_m \|x_n^0\|_2^2$. Since $|\langle x_n^*, v\rangle| \leq \varepsilon_n \|v\|$ for all $v \in W^{1,2}_{\mathrm{per}}(T)$, taking $v = x_n^0$ and exploiting the orthogonality relations, we have

$$\begin{aligned}
&\left| \frac{1}{\|x_n\|}(\|x_n^{0'}\|_2^2 - \lambda_m \|x_n^0\|_2^2) - \int_0^b \frac{u_n}{\|x_n\|} x_n^0 \mathrm{d}t + \int_0^b \frac{h}{\|x_n\|} x_n^0 \mathrm{d}t \right| < \varepsilon_n,\\
&\Rightarrow \int_0^b \frac{u_n - h}{\|x_n\|} x_n^0 \mathrm{d}t \to 0.
\end{aligned} \tag{25}$$

Also from Lemma 3 of [33], we have that $\beta_1 \|\bar{x}_n\|^2 \leq \lambda_m \|x_n\|_2^2 - \|\bar{x}_n'\|_2^2$ for all $n \geq 1$ and some $\beta_1 > 0$ while from Lemma 9, we have $\beta_1 \|\hat{x}_n\|^2 \leq \|\hat{x}_n'\|_2^2 - \lambda_m \|\hat{x}_n'\|_2^2$ for all $n \geq 1$. Using the orthogonality relations among the three subspaces $\bar{H}$, N_m, and $\hat{H}$, we obtain

$$\int_0^b x_n'(-\bar{x}_n' + x_n^{0'} + \hat{x}_n')\mathrm{d}t = -\|\bar{x}_n'\|_2^2 + \|x_n^{0'}\|_2^2 + \|\hat{x}_n'\|_2^2$$

and

$$-\lambda_m \int_0^b x_n(-\bar{x}_n + x_n^0 + \hat{x}_n)\mathrm{d}t = \lambda_m \|\bar{x}_n\|_2^2 - \lambda_m \|x_n^0\|_2^2 - \lambda_m \|\hat{x}_n\|_2^2.$$

Thus finally we can write that

$$\begin{aligned}
&\int_0^b x_n'(-\bar{x}_n' + x_n^{0'} + \hat{x}_n')\mathrm{d}t - \lambda_m \int_0^b x_n(-\bar{x}_n + x_n^0 + \hat{x}_n)\mathrm{d}t\\
&= \lambda_m \|\bar{x}_n\|_2^2 - \|\bar{x}_n'\|_2^2 + \|\hat{x}_n'\|_2^2 - \lambda_m \|\hat{x}_n\|_2^2\\
&\geq \beta_2 \| - \bar{x}_n + \hat{x}_n\|^2 \quad \text{for all } n \geq 1 \text{ and some } \beta_2 > 0.
\end{aligned} \tag{26}$$

Moreover, recalling that for almost all $t \in T$, all $|x| \geq M$ and all $u \in \partial j(t,x)$ we have $|u/x| \leq \varepsilon$ and using also hypothesis H(j)$_5$(iii), we have

$$\begin{aligned}
\int_0^b \frac{u_n}{\|x_n\|}(-\bar{x}_n + \hat{x}_n)\mathrm{d}t &= \frac{1}{\|x_n\|}\int_{\{|x_n(t)|\geq M\}} \frac{u_n}{x_n} x_n(-\bar{x}_n + \hat{x}_n)\mathrm{d}t \\
&\quad + \int_{\{|x_n(t)|<M\}} \frac{u_n}{\|x_n\|}(-\bar{x}_n + \hat{x}_n)\mathrm{d}t \\
&\leq \frac{1}{\|x_n\|}\varepsilon(\|\bar{x}_n\|_2^2 + \|\hat{x}_n\|_2^2) + \frac{1}{\|x_n\|}\beta_3\| - \bar{x}_n + x_n\| \\
&\quad \text{for some } \beta_3 > 0 \\
&\leq \frac{\varepsilon}{\|x_n\|}\|w_n\|^2 + \frac{1}{\|x_n\|}\beta_3\|w_n\| \\
&\quad \text{with } w_n = -\bar{x}_n + \hat{x}_n, n \geq 1. \qquad (27)
\end{aligned}$$

Also for all $n \geq 1$, we have

$$\begin{aligned}
\left|\int_0^b h(-\bar{y}_n + \hat{y}_n)\mathrm{d}t\right| &\leq \|h\|_1 \, \| - \bar{y}_n + \hat{y}_n\|_\infty \leq \beta_4\| - \bar{y}_n + \hat{y}_n\| \\
&= \frac{\beta_4}{\|x_n\|}\|w_n\| \quad \text{for some } \beta_4 > 0. \qquad (28)
\end{aligned}$$

Using (25)$\rightarrow$(28) in (14), we obtain

$$\frac{1}{\|x_n\|}(\beta_2 - \varepsilon)\|w_n\|^2 - \frac{1}{\|x_n\|}\beta_5\|w_n\| \leq \varepsilon_n' \quad \text{for some } \beta_5 > 0 \text{ and with } \varepsilon_n' \downarrow 0.$$

Choosing $\varepsilon < \beta_2$, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\|x_n\|}(\beta_6\|w_n\|^2 - \beta_5\|w_n\|) \leq 0 \quad \text{with } \beta_6 = \beta_2 - \varepsilon > 0 \\
\Rightarrow\ &\limsup_{n\to\infty} \frac{\|w_n\|^2}{\|x_n\|}\left(\beta_6 - \frac{\beta_5}{\|w_n\|}\right) \leq 0, \\
\Rightarrow\ &\frac{\|w_n\|^2}{\|x_n\|} \to 0 \ \text{(by passing to a subsequence if necessary)}.
\end{aligned}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\mathrm{per}}(T)$ we have

$$|\varphi(x_n)| = \left|\frac{1}{2}\|x_n'\|_2^2 - \frac{\lambda_m}{2}\|x_n\|_2^2 - \int_0^b j(t, x_n(t))\mathrm{d}t + \int_0^b h(t)x_n(t)\mathrm{d}t\right| \leq M_1.$$

Divide by $\|x_n\|$ and use the orthogonality of the subspaces and the equality $\|x_n^{0'}\|_2^2 = \lambda_m\|x_n^0\|_2^2$ for all $n \geq 1$. We obtain

$$\begin{aligned}
\frac{|\varphi(x_n)|}{\|x_n\|} = \Bigg|&\frac{1}{2}\frac{\|w_n'\|_2^2}{\|x_n\|} - \frac{\lambda_m}{2}\frac{\|w_n\|_2^2}{\|x_n\|} - \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t \\
&+ \int_0^b h(t)y_n(t)\mathrm{d}t\Bigg| \leq \frac{M_1}{\|x_n\|}. \qquad (29)
\end{aligned}$$

Recall that $W_{\rm per}^{1,2}(T) = \mathbb{R} \oplus V$ with $V = \{v \in W_{\rm per}^{1,p}(T)\colon \int_0^b v(t)\mathrm{d}t = 0\} = \mathbb{R}^{\perp}$. So $w_n = \xi_n + v_n$ with $\xi_n \in \mathbb{R}$, $v_n \in V$, $n \geq 1$. We have $\|w_n\|_2^2 = \|\xi_n\|_2^2 + \|v_n\|_2^2 = b\xi_n^2 + \|v_n\|_2^2$, $n \geq 1$. So

$$\frac{\lambda_m}{2}\|w_n\|_2^2 = \frac{\lambda_m}{2}\xi_n^2 b + \frac{\lambda_m}{2}\|v_n\|_2^2,$$

$$\Rightarrow \frac{\lambda_m}{2}\frac{\|w_n\|_2^2}{\|x_n\|} = \frac{\lambda_m}{2}\frac{\xi_n^2 b}{\|x_n\|} + \frac{\lambda_m}{2}\frac{\|v_n\|_2^2}{\|x_n\|} \leq \lambda_m \frac{\|w_n\|^2}{\|x_n\|} \to 0.$$

Also we have

$$\frac{1}{2}\|w_n'\|_2^2 = \frac{1}{2}\|v_n'\|_2^2 \leq \frac{\beta_7}{2} \quad \text{with } \beta_7 > 0$$

$$\text{(by the Poincaré–Wirtinger inequality)},$$

$$\Rightarrow \frac{1}{2}\frac{\|w_n'\|_2^2}{\|x_n\|} \leq \frac{\beta_7}{2}\frac{\|w_n\|^2}{\|x_n\|} \to 0 \quad \text{as } n \to \infty.$$

Moreover, we have

$$\int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t = \int_{\{x_n(t)\neq 0\}} \frac{j(t, x_n(t))}{x_n} y_n(t)\mathrm{d}t + \int_{\{x_n(t)=0\}} \frac{j(t,0)}{\|x_n\|}\mathrm{d}t.$$

Note that on $\{t \in T\colon y(t) > 0\}$ we have $x_n(t) \to +\infty$ and on $\{t \in T\colon y(t) < 0\}$ we have that $x_n(t) \to -\infty$. In addition $\int_{\{x_n(t)=0\}} \frac{j(t,0)}{\{\|x_n\|\}}\mathrm{d}t \to 0$. So via Fatou's lemma, we have

$$\begin{aligned}\liminf_{n\to\infty} \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t &\geq \liminf_{n\to\infty} \int_{\{x_n(t)\neq 0\}} \frac{j(t, x_n(t))}{x_n(t)} y_n(t)\mathrm{d}t \\ &\geq \int_0^b j_+(t)y^+(t)\mathrm{d}t - \int_0^b j_-(t)y^-(t)\mathrm{d}t.\end{aligned}$$

Because $\frac{\|w_n\|}{\|x_n\|} \to 0$, we have that $y \in N_m$ and so

$$\begin{aligned}\liminf_{n\to\infty} \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t &\geq \int_0^b j_+(t)r\sin(m\omega t + \theta)^+\mathrm{d}t \\ &\quad - \int_0^b j_-(t)r\sin(m\omega t + \theta)^-\mathrm{d}t, \theta \in \mathbb{R}. \end{aligned} \tag{30}$$

From (29) and since $\frac{\|w_n\|^2}{\|x_n\|} \to 0$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t &= \int_0^b h(t)y(t)\mathrm{d}t = \int_0^b h(t)r\sin(m\omega t + \theta)\mathrm{d}t \\ &< r\left(\int_0^b j_+(t)\sin(m\omega t + \theta)^+\mathrm{d}t - \int_0^b j_-(t)\sin(m\omega t + \theta)^-\mathrm{d}t\right) \\ &\quad \text{(by hypothesis } \mathrm{H(j)_5(v)}). \end{aligned} \tag{31}$$

Comparing (30) and (31), we reach a contradiction. This proves that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{\text{per}}(T)$ is bounded, hence we may assume that $x_n \xrightarrow{w} x$ in $W^{1,2}_{\text{per}}(T)$ and $x_n \to x$ in $C_{\text{per}}(T)$. As before (see the proof of Theorem 1) we can finish the proof and conclude that φ satisfies the non-smooth PS-condition. QED

Let $H_1 = \text{span}\{\sin k\omega t, \cos k\omega t\colon k = 0, 1, ..., m\}$ and $H_2 = H_1^{\perp}$.

PROPOSITION 11

If hypotheses H(j)$_5$ *hold, then* $\varphi(x) \to -\infty$ *as* $\|x\| \to \infty$, $x \in H_1$.

Proof. Suppose that the conclusion of the proposition was not true. Then we can find $\beta \in \mathbb{R}$ and a sequence $\{x_n\}_{n\geq 1} \subseteq H_1$ such that $\|x_n\| \to \infty$ and $\varphi(x_n) \geq \beta$ and all $n \geq 1$. We have

$$\frac{1}{2}\|x_n'\|_2^2 - \frac{\lambda_m}{2}\|x_n\|_2^2 - \int_0^b j(t, x_n(t))\mathrm{d}t + \int_0^b h(t)x_n(t)\mathrm{d}t \geq \beta.$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that $y_n \xrightarrow{w} y$ in $W^{1,2}_{\text{per}}(T)$ and $y_n \to y$ in $C_{\text{per}}(T)$. Because H_1 is finite dimensional and $y_n \in H_1$ for all $n \geq 1$, we have $y_n \to y$ in $W^{1,2}_{\text{per}}(T)$. For all $n \geq 1$ we have

$$\frac{1}{2}\|y_n'\|_2^2 - \frac{\lambda_m}{2}\|y_n\|_2^2 - \int_0^b \frac{j(t, x_n(t))}{\|x_n\|^2}\mathrm{d}t + \int_0^b \frac{h(t)}{\|x_n\|}y_n(t)\mathrm{d}t \geq \frac{\beta}{\|x_n\|^2},$$
$$n \geq 1.$$

Clearly $\int_0^b \frac{j(t,x_n(t))}{\|x_n\|^2}\mathrm{d}t \to 0$ and $\int_0^b \frac{h(t)}{\|x_n\|}y_n(t)\mathrm{d}t \to 0$. So in the limit as $n \to \infty$, we obtain

$$\frac{\lambda_m}{2}\|y\|_2^2 \leq \frac{1}{2}\|y'\|_2^2$$
$$\Rightarrow \frac{1}{2}\|y'\|_2^2 = \frac{\lambda_m}{2}\|y_n\|_2^2 \quad (\text{since } y \in H_1)$$
$$\Rightarrow y \in N_m.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq H_1$ we have

$$-\int_0^b j(t, x_n(t))\mathrm{d}t + \int_0^b h(t)x_n(t)\mathrm{d}t \geq \varphi(x_n) \geq \beta$$
$$\Rightarrow \int_0^b \frac{j(t, x_n(t))}{\|x_n\|}\mathrm{d}t \leq -\frac{\beta}{\|x_n\|} + \int_0^b h(t)y_n(t)\mathrm{d}t.$$

Arguing as in the proof of Proposition 10, in the limit as $n \to \infty$, we obtain

$$r\int_0^b j_+(t)\sin(m\omega t + \theta)^+\mathrm{d}t - r\int_0^b j_-(t)\sin(m\omega t + \theta)^-\mathrm{d}t$$
$$\leq r\int_0^b h(t)\sin(m\omega t + \theta)\mathrm{d}t,$$

which contradicts hypothesis H(j)$_5$(v). This proves the proposition. QED

PROPOSITION 12

If hypotheses $\mathrm{H(j)}_5$ *hold, then* $\varphi(x) \to +\infty$ *as* $\|x\| \to \infty$, $x \in H_2$.

Proof. For $x \in H_2$, we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\|x'\|_2^2 - \frac{\lambda_m}{2}\|x\|_2^2 - \int_0^b j(t, x(t))\mathrm{d}t + \int_0^b h(t)x(t)\mathrm{d}t \\ &\geq c\|x\|^2 - \int_0^b j(t, x(t))\mathrm{d}t + \int_0^b h(t)x(t)\mathrm{d}t \quad \text{(see Lemma 9)} \\ \Rightarrow \frac{\varphi(x)}{\|x\|^2} &\geq c - \int_0^b \frac{j(t, x(t))}{\|x\|^2}\mathrm{d}t + \int_0^b \frac{h(t)}{\|x\|}y(t)\mathrm{d}t.\end{aligned}$$

Remark that $\int_0^b \frac{j(t,x(t))}{\|x\|^2}\mathrm{d}t \to 0$ and $\int_0^b \frac{h(t)}{\|x\|}y(t)\mathrm{d}t \to 0$ as $\|x\| \to \infty$, $x \in H_2$. So we have

$$\begin{aligned}&\liminf_{\substack{\|x\|\to\infty \\ x\in H_2}} \frac{\varphi(x)}{\|x\|^2} \geq c > 0 \\ \Rightarrow\ &\varphi(x) \to +\infty \quad \text{as } \|x\| \to \infty, x \in H_2.\end{aligned}$$

QED

Propositions 10–12, permit the use of the non-smooth saddle point theorem which gives $x \in W^{1,2}_{\mathrm{per}}(T)$ such that $0 \in \partial\varphi(x)$. As in the proof of Theorem 1, we can check that $x \in C^1(T)$, $x' \in W^{1,1}(T)$ and also x solves (22). So we can state the following existence theorem.

Theorem 13. *If hypotheses* $\mathrm{H(j)}_5$ *hold, then for every* $h \in L^1(T)$, *problem* (22) *has a solution* $x \in C^1(T)$ *with* $x' \in W^{1,1}(T)$.

Remark. Theorem 13 generalizes Theorems 2 and 3 of Tang [33]. The generalization is two-fold. On the one hand we assume a more general Landesmann–Lazer type condition (see hypothesis $\mathrm{H(j)}_5$(v) and Proposition 8) and on the other hand we have a non-smooth potential function. Moreover, in Tang the potential function is independent of the time-variable $t \in T$.

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