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On finite groups whose every proper normal subgroup is a union of a given number of conjugacy classes

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Abstract. Let G be a finite group and A be a normal subgroup of G. We denote by $ncc(A)$ the number of G-conjugacy classes of A and A is called n-decomposable, if $ncc(A) = n$. Set $\mathcal{K}_G = \{ncc(A) | A \lhd G\}$. Let X be a non-empty subset of positive integers. A group G is called X-decomposable, if $\mathcal{K}_G = X$.

Ashrafi and his co-authors [1–5] have characterized the X-decomposable non-perfect finite groups for $X = \{1, n\}$ and $n \leq 10$. In this paper, we continue this problem and investigate the structure of X-decomposable non-perfect finite groups, for $X = \{1, 2, 3\}$. We prove that such a group is isomorphic to Z_6, D_8, Q_8, S_4, SmallGroup(20, 3), SmallGroup(24, 3), where SmallGroup(m, n) denotes the mth group of order n in the small group library of GAP [11].

Keywords. Finite group; n-decomposable subgroup; conjugacy class; X-decomposable group.

1. Introduction and preliminaries

Let G be a finite group and let $\mathcal{N}_G$ be the set of proper normal subgroups of G. An element K of $\mathcal{N}_G$ is said to be n-decomposable if K is a union of n distinct conjugacy classes of G. In this case we denote n by $ncc(K)$. Suppose $\mathcal{K}_G = \{ncc(A) | A \in \mathcal{N}_G\}$ and X is a non-empty subset of positive integers. A group G is called X-decomposable, if $\mathcal{K}_G = X$. For simplicity, if $X = \{1, n\}$ and G is X-decomposable, then we say that G is n-decomposable.

In [14], Wujie Shi defined the notion of a complete normal subgroup of a finite group, which we call 2-decomposable. He proved that if G is a group and N a complete normal subgroup of G, then N is a minimal normal subgroup of G and it is an elementary abelian p-group. Moreover, $N \subseteq Z(O_p(G))$, where $O_p(G)$ is a maximal normal p-subgroup of G, and $|N|(|N| - 1)$ divides $|G|$ and in particular, $|G|$ is even.

Shi [14] proved some deep results about finite group G of order $p^a q^b$ containing a 2-decomposable normal subgroup N. He proved that for such a group $|N| = 2, 3, 2^{b_1}$ or $2^{a_1}+1$, where $2^{b_1}-1$ is a Mersenne prime and $2^{a_1}+1$ is a Fermat prime. Moreover, we have (i) if $|N| = 2$, then $N \subseteq Z(G)$, (ii) if $|N| = 3$, then G has order $2^a 3^b$, (iii) if $|N| = 2^{b_1}$, then G has order $(2^{b_1} - 1)2^b$ and (iv) if $|N| = 2^{a_1} + 1$, then G has order $2^a(2^{a_1} + 1)^b$.

Next, Wang Jing [15], continued Wujie Shi's work and defined the notion of a sub-complete normal subgroup of a group G, which we call 3-decomposable. She proved that if N is a sub-complete normal subgroup of a finite group G, then N is a group in which every element has prime power order. Moreover, if N is a minimal normal subgroup of G, then $N \subseteq Z(O_p(G))$, where p is a prime factor of $|G|$. If N is not a minimal normal subgroup of G, then N contains a complete normal subgroup N_1, where N_1 is an elementary abelian group with order p^a and we have:

(a) $N = N_1 Q$ has order $p^a q$ and every element of N has prime power order, $|Q| = q$, $q \neq p$, q is a prime and $G = MN_1$, $M \cap N_1 = 1$, where $M = N_G(Q)$,
(b) N is an abelian p-group with exponent $\leq p^2$ or a special group; if N is not elementary abelian, then $N_1 \leq \Phi(G)$, where $\Phi(G)$ denotes the Frattini subgroup of G.

Shahryari and Shahabi [12,13] investigated the structure of finite groups which contain a 2- or 3-decomposable subgroup. Riese and Shahabi [8] continued this theme by investigating the structure of finite groups with a 4-decomposable subgroup. Using these works in [1] and [2], Ashrafi and Sahraei characterized the finite non-perfect X-groups, for $X = \{1, n\}$, $n \leq 4$. They also obtained the structure of solvable n-decomposable non-perfect finite groups. Finally, Ashrafi and Zhao [3] and Ashrafi and Shi [4,5] characterized the finite non-perfect X-groups, for $X = \{1, n\}$, where $5 \leq n \leq 10$.

In this paper we continue this problem and characterize the non-perfect X-decomposable finite groups, for $X = \{1, 2, 3\}$. We prove that such a group is solvable and determine the structure of these groups. In fact, we prove the following theorem:

Theorem. *Let G be a non-perfect $\{1, 2, 3\}$-decomposable finite group. Then G is isomorphic to Z_6, D_8, Q_8, S_4,* SmallGroup(20, 3) *or* SmallGroup(24, 3).

Throughout this paper, as usual, G' denotes the derived subgroup of G, Z_n denotes the cyclic group of order n, $E(p^n)$ denotes an elementary abelian p-group of order p^n, for a prime p and $Z(G)$ is the center of G. We denote by $\pi(G)$, the set of all prime divisors of $|G|$ and $\pi_e(G)$ is the set of all orders of elements of G. A group G is called non-perfect, if $G' \neq G$. Also, $d(n)$ denotes the set of positive divisors of n and SmallGroup(n, i) is the ith group of order n in the small group library of GAP [11]. All groups considered are assumed to be finite. Our notation is standard and is taken mainly from [6,7,9,10].

2. Examples

In this section we present some examples of X-decomposable finite groups and consider some open questions. We begin with the finite abelian groups.

Lemma 1. *Let G be an abelian finite group. Set $X = d(n) - \{n\}$, where $n = |G|$. Then G is X-decomposable.*

Proof. The proof is straightforward. □

By the previous lemma a cyclic group of order n is $(d(n) - \{n\})$-decomposable. In the following examples we investigate the normal subgroups of some non-abelian finite groups.

Example 1. Suppose that G is a non-abelian group of order pq, in which p and q are primes and $p > q$. It is a well-known fact that $q|p-1$ and G has exactly one normal subgroup. Suppose that H is the normal subgroup of G. Then H is $(1 + \frac{p-1}{q})$-decomposable. Set $X = \{1, 1 + \frac{p-1}{q}\}$. Then G is X-decomposable.

Example 2. Let D_{2n} be the dihedral group of order $2n$, $n \geq 3$. This group can be presented by

$$D_{2n} = \langle a, b | a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

We first assume that n is odd and $X = \{\frac{d+1}{2} | d | n\}$. In this case every proper normal subgroup of D_{2n} is contained in $\langle a \rangle$ and so D_{2n} is X-decomposable. Next we assume that n is even and $Y = \{\frac{d+1}{2} | d | n : 2 \not| d\} \cup \{\frac{d+2}{2} | d | n; 2 | d\}$. In this case, we can see that D_{2n} has exactly two other normal subgroups $H = \langle a^2, b \rangle$ and $K = \langle a^2, ab \rangle$. To complete the example, we must compute $ncc(H)$ and $ncc(K)$. Obviously, $ncc(H) = ncc(K)$. If $4|n$, then $ncc(H) = \frac{n}{4} + 2$ and if $4 \not| n$, then $ncc(H) = \frac{n+6}{4}$. Set $A = Y \cup \{\frac{n}{4} + 2\}$ and $B = Y \cup \{\frac{n+6}{4}\}$. Our calculations show that if $4|n$, then D_{2n} is A-decomposable and if $4 \not| n$, then dihedral group D_{2n} is B-decomposable.

Example 3. Let Q_{4n} be the generalized quaternion group of order $4n$, $n \geq 2$. This group can be presented by

$$Q_{4n} = \langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.$$

Set $X = \{\frac{d+1}{2} | d | n \text{ and } 2 \not| d\} \bigcup \{\frac{d+2}{2} | d | 2n \text{ and } 2 | d\}$ and $Y = X \cup \{\frac{n+4}{2}\}$. It is a well-known fact that Q_{4n} has $n + 3$ conjugacy classes, as follows:

$$\{1\};\ \{a^n\};\ \{a^r, a^{-r}\}(1 \leq r \leq n-1);$$
$$\{a^{2j}b | 0 \leq j \leq n-1\};\ \{a^{2j+1}b | 0 \leq j \leq n-1\}.$$

We consider two separate cases that n is odd or even. If n is odd then every normal subgroup of Q_{4n} is contained in the cyclic subgroup $\langle a \rangle$. Thus, in this case Q_{4n} is X-decomposable. If n is even, we have two other normal subgroups $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$ which are both $\frac{n+4}{2}$-decomposable. Therefore, Q_{4n} is Y-decomposable.

Now it is natural to generally ask about the set $\mathcal{K}_G = \{ncc(A) | A \lhd G\}$. We end this section with the following question:

Question 1. Suppose X is a finite subset of positive integers containing 1. Is there a finite group G which is X-decomposable?

3. Main theorem

Throughout this section $X = \{1, 2, 3\}$. The aim of this section is to prove the main theorem of the paper. We will consider two separate cases in which G' is 2- or 3-decomposable. In the following simple lemma, we classify the X-decomposable finite abelian groups.

Lemma 2. *Let G be an abelian X-decomposable finite group. Then $G \cong Z_6$, the cyclic group of order* 6.

Proof. Apply Lemma 1. $\square$

For the sake of completeness, we now define two groups U and V which we will use later. These groups can be presented by

$$U = \langle x, y, z | x^3 = y^4 = 1, y^2 = z^2, z^{-1}yz = y^{-1},$$
$$x^{-1}yx = y^{-1}z^{-1}, x^{-1}zx = y^{-1}\rangle,$$
$$V = \langle x, y | x^4 = y^5 = 1, x^{-1}yx = y^2\rangle.$$

We can see that U and V are groups of orders 24 and 20 which are isomorphic to SmallGroup$(24, 3)$ and SmallGroup$(20, 3)$, respectively. Also, these groups are X-decomposable.

To prove the main result of the paper, we need to determine all of X-decomposable groups of order 8, 12, 18, 20, 24, 36 and 42. The following GAP program determines all the X-decomposable groups of the mentioned orders.

```
AppendTo("x.txt","Begining the Program","\n");
E:=[8,12,18,20,24,36,42];
for m in E do
   n:=NrSmallGroups(m);
   F:=Set([1,2,3]);
         for i in [1,2..n] do
            G1:=[];
            G:=[];
            g:=SmallGroup(m,i);
            h:=NormalSubgroups(g);
            d1:=Size(h);d:=d1-1;
              for j in [1,2..d] do
                 s:=FusionConjugacyClasses(h[j],g);
                 s1:=Set(s);
                 Add(G,s1);
              od;
              for k in G do
                 a:=Size(k);
                 Add(G1,a);
              od;
           G2:=Set(G1);
           if G2=F then AppendTo("x.txt","S(",m,",",i, ")",
           " ");fi;
        od;
od;
```

PROPOSITION 1

Let G be a non-perfect and non-abelian X-decomposable finite group such that G' is 2-decomposable. Then G is isomorphic to D_8, Q_8 or $SmallGroup(20, 3)$.

Proof. Set $G' = 1 \cup Cl_G(a)$. Then it is an easy fact that G' is an elementary abelian r-subgroup of G, for a prime r. First of all, we assume that $|G'| = 2$. Then one can see that $G' = Z(G)$. If G is not a 2-group then there exists an element $x \in G$ of an odd prime order q. Suppose $H = G'\langle x\rangle$. Since H is a cyclic group of order $2q$, $ncc(H) \geq 4$ which is impossible. Hence G is a 2-group. We show that $|G| = 8$. Suppose $|G| \geq 16$. Since $|G'| = 2$ and every subgroup containing G' is normal, we can find a chain $G' < H <$

$K < G$ of normal subgroups of G, a contradiction. So $G \cong D_8$ or Q_8 and by Examples 2 and 3, these groups are X-decomposable.

We next assume that $|G'| \geq 3$. If $Z(G) \neq 1$ then it is easy to see that $|Z(G)| = 2$ or 3. Suppose $|Z(G)| = 3$. Then $G'Z(G) = G$ or G'. If $G'Z(G) = G$, then $G \cong G' \times Z(G)$. This implies that G is abelian, a contradiction. Thus $Z(G) \leq G'$. This leads to a contradiction, since G is non-perfect and G' is 2-decomposable. Thus $|Z(G)| = 2$. But in this case $H = Z(G)G'$ is 3-decomposable, which is impossible. Therefore, $Z(G) = 1$ and by Theorem 2.1 of [12], we have $|G| = |G'|(|G'|-1)$ and G is a Frobenius group with kernel G' and its complement is abelian. Suppose $|G'| = r^n$, then $|G| = r^n(r^n - 1)$. Take K to be any proper non-trivial subgroup of T, where T is a Frobenius complement of G'. Then KG' is 3-decomposable and for any $x \neq 1$ in K we have $|Cl_G(x)| = |G|/|T| = |G'|$. So $|KG'| = 2|G'|$ and therefore we get $|K| = 2$. Since T is abelian, this forces 2 to be the only proper divisor of $|T|$ and hence $|T| = 4$. So $|G| = 20$ and clearly G is a semidirect product of Z_5 by T. Further, $Z(G) = 1$ forces $T \leq \mathrm{Aut}(Z_5) \cong Z_4$. Hence $T = \mathrm{Aut}(Z_5)$. Therefore $G \cong \mathrm{Aut}(Z_5) \propto Z_5$. To complete the proof, we show that $G \cong$ SmallGroup(20, 3) and it is X-decomposable. Let x and y be elements of G with $o(x) = 4$ and $o(y) = 5$. Since G is a centerless group containing five involutions, it has exactly two non-trivial, proper normal subgroups $A = \langle y \rangle$ and $B = A\langle x^2 \rangle$ of orders 5 and 10, respectively. Clearly B is non-abelian and so it is isomorphic to the dihedral group of order 10. This shows that the elements of $B - A$ are conjugate in B. But $x^{-1}yx = y^i$, $i = 2, 3, 4$. Suppose $i = 4$. Since $B \cong D_{10}$, we have that $x^2yx^{-2} = y^{-1}$. Thus we get that $xyx^{-1} = x^{-1}y^{-1}x$. Consequently if $i = 4$ then we have that $x^{-1}y^{-1}x = y^{-1}$ and then G would be abelian, a contradiction. Also the two groups constructed by $i = 2$ and $i = 3$ will be isomorphic. Hence without loss of generality, we can assume that $i = 2$ and so $G \cong V \cong$ SmallGroup(20, 3). This shows that non-identity elements of A will be conjugate in G and so A is 2-decomposable and B is 3-decomposable. This completes the proof. □

PROPOSITION 2

Let G be a non-perfect and non-abelian X-decomposable finite group such that G' is 3-*decomposable. Then G is isomorphic to S_4 or* SmallGroup(24, 3).

Proof. Set $G' = 1 \cup Cl_G(a) \cup Cl_G(b)$. Our main proof will consider three separate cases.

Case 1. $a^{-1} \notin Cl_G(a)$. In this case $Cl_G(b) = Cl_G(a^{-1})$ and by Proposition 1 of [13], G' is an elementary abelian p-subgroup of G, for some odd prime p. Suppose H is a 2-decomposable subgroup of G. Then by Corollary 1.7 of [12], we can see that $H = Z(G)$ has order two. Thus $G \cong Z(G) \times G'$, which is a contradiction.

Case 2. $a^{-1} \in Cl_G(a), b^{-1} \in Cl_G(b)$ and $(o(a), o(b)) = 1$. In this case, by ([13], Lemma 6), $|G'| = pq^n$, for some distinct primes p, q, and by Lemma 4 of [13], $Z(G') = 1$. Also, by Lemma 5 of [13], $G'' = 1 \cup Cl_G(a)$ has order q^n. Since G' is 3-decomposable, $|G : G'| = r$, r is prime. Thus $|G| = prq^n$ and $|\pi(G)| = 2$ or 3. Suppose $|\pi(G)| = 2$. Then by Shi's result [14], mentioned in the introduction, $|G''| = 2, 3, 2^{b_1}$ or $2^{a_1} + 1$, where $2^{b_1} - 1$ is a Mersenne prime and $2^{a_1} + 1$ is a Fermat prime. If $|G''| = 2$, then G' is a cyclic group of order $2p$, a contradiction. If $|G''| = 3$, then G' is a cyclic group of order $3p$ or isomorphic to the symmetric group on three symbols. Since G' is centerless, $G' \cong S_3$.

This shows that $|G| = 12$ or 18 and by our program in GAP language, there is no X-decomposable group of order 12 or 18. We now assume that $|G''| = 2^{b_1}$. Hence by Shi's result, mentioned before, $|G| = 2^b(2^{b_1}-1)$ and $|G'| = 2^{b_1}(2^{b_1}-1)$. Suppose $x \in G' - G''$ and $y \in G''$. Then we can see that $|Cl_G(x)| = 2^{b_1}(2^{b_1} - 2)$ and $|Cl_G(y)| = 2^{b_1} - 1$. Since $|Cl_G(x)|$ is a divisor of $|G|$, $2^{b_1-1} - 1|2^{b_1} - 1$, which implies that $b_1 = 2$. Therefore $|G'| = 12$ and $|G| = 24$ or 36. Again using our GAP program, we can see that $|G| = 24$ and since G' is centerless, $G \cong S_4$. Next we suppose that $|G''| = 2^{a_1} + 1$. Apply Shi's result again to obtain $|G| = 2^a(2^{a_1}+1)^b$. So G' is a dihedral group of order $2(2^{a_1}+1)$ and $|G| = 2(2^{a_1}+1)^2$ or $4(2^{a_1}+1)$. If $|G| = 2(2^{a_1}+1)^2$ then G has a 3-decomposable subgroup of order $(2^{a_1}+1)^2$. This subgroup has a G-conjugacy class of length $q^2-q = 2^{a_1}(2^{a_1}+1)$ and so $a_1 = 1$ and $|G| = 18$, a contradiction. If $|G| = 4(2^{a_1}+1)$, then $q(q-1)|4(2^{a_1}+1)$ and so $a_1 = 1, 2$. This implies that $|G| = 12, 20$, which is impossible.

Therefore it is enough to assume that $|G| = prq^n$, for distinct primes p, q and r. Since G'' is a 2-decomposable subgroup of order q^n, $q^n(q^n-1)||G|$. Thus $q^n - 1|pr$. Suppose $n = 1$. Then $|G| = pqr$, $|G'| = pq$ and G' has two G-conjugacy classes of lengths $q-1$ and $q(p-1)$. Hence $p - 1|r$ and $q - 1|pr$. If $p = 2$, then G has a 3-decomposable subgroup H of order qr. Since H has a G-conjugacy class of length $q(r-1)$, $r = 3$. Thus $q - 1|6$. This shows that $q = 7$ and $|G| = 42$. But by our GAP program, there is no X-decomposable group of order 42, a contradiction. If $p \neq 2$, then $p = 3, r = 2$ and a similar argument shows that $|G| = 42$, which is impossible. Thus $n \neq 1$. We now assume that $q \neq 2$. Since $q^n - 1|pr$, $q - 1 = p$ or r. This shows that $q = 3$ and one of p or r is equal to 2. If p or r take the value 2, using arguments similar to the case $n = 1$, we get r or p equals 3 respectively. This is a contradiction as $q = 3$. Finally, we assume that $q = 2$. Then $|G'| = 2^n p$ and $|G| = 2^n pr$. Since G' has a G-conjugacy class of length $2^n(p-1)$, $p - 1|r$. Therefore, $p = 2$ or $r = 2$ which is our final contradiction.

Case 3. $a^{-1} \in Cl_G(a), b^{-1} \in Cl_G(b)$ and $(o(a), o(b)) \neq 1$. In this case by Proposition 2 of [13], we have that G' is a metabelian p-group. Since G' is a maximal subgroup of G, we have that $|G : G'| = q$, where q is prime. If $q = p$ then G is p-group and so $G' \leq \Phi(G)$. This shows that G is cyclic, a contradiction. Thus $|G| = p^n q$, for distinct primes p and q. Suppose H is a 2-decomposable subgroup of G. If H is central then $H = Z(G)$. We first assume that $Z(G) \not\leq G'$. Then $G \cong G' \times Z(G)$ which implies that $Z(G') = 1$, a contradiction. Next, suppose that $Z(G) \leq G'$. Then G' has a G-conjugacy class of length $2^n - 2$ and so $p = 2$ and $q = 2^{n-1} - 1$. Without loss of generality we can assume that $|Cl_G(a)| = 1$ and $|Cl_G(b)| = 2^n - 2$. This shows that $|C_G(b)| = |G|/|Cl_G(b)| = 2^{n-1}$ and G' cannot be abelian because if G' is abelian we would get $G' \leq C_G(b)$. Since G' is non-abelian, $o(b) = 4$ and G' has a unique subgroup of order 2. Thus $G' \cong Q_8$ and G is a semidirect product of Q_8 by Z_3. Assume that $H = \langle x \rangle$ is the cyclic group of order 3 and $N = Q_8$. Then $\mathrm{Aut}(N)$ has a unique conjugacy type of automorphism of order 3. Therefore $G = H \propto_\theta N$, where $\theta(x)$ is an automorphism of order 3 of the group Q_8. We now show that G is X-decomposable and it is isomorphic to SmallGroup(24, 3). The possible orders for a non-trivial proper normal subgroups of G are 2, 4, 6, 8, 12. Clearly $Z(G) = Z(N)$ and $G/Z(G) \cong A_4$. But A_4 does not have normal subgroups of order 2, 3 and 6, so every normal subgroup of G has order 2 or 8 and these are unique. On the other hand, using Example 3, we can assume that $N = \langle y, z | y^4 = 1, y^2 = z^2, z^{-1}yz = y^{-1}\rangle$. Thus G is isomorphic to a group which has the same presentation as the group V, which we defined before. This shows that $G \cong$ SmallGroup(24, 3) and also N is 3-decomposable.

Now it remains to investigate the case that H is not central. Thus $H \leq G'$ and G' has a G-conjugacy class of length $p^i - 1$, for some $1 \leq i \leq n-1$. Thus $p^i - 1 = q$. If $p = 3$, then $i = 1$, $q = 2$ and we can show that $|G| = 18$. Thus $|G'| = 9$ and so G' is abelian. Further $|H| = 3$, so assuming without loss of generality that $b \in G' \backslash H$, we get $|Cl_G(b)| = 6$. Hence $|C_G(b)| = 3$ which is a contradiction as it must be at least 9. Hence $p = 2$, i is prime and $|G| = 2^{2i}(2^i - 1)$ or $2^{1+i}(2^i - 1)$. Suppose $|G| = 2^{1+i}(2^i - 1)$, Q is a Sylow q-subgroup of G and $N = HQ$. Since G/N is abelian, $G' \leq N$, which is impossible as $|G'|$ does not divide $|N|$. Finally, we assume that $|G| = 2^{2i}(2^i - 1)$. We may assume without loss of generality that $H = 1 \cup Cl_G(a)$. Then we get that $|Cl_G(a)| = 2^i - 1$ and so $|C_G(a)| = 2^{2i}$. Thus $C_G(a) = G'$. Hence we get $H \leq Z(G')$ and so $H \leq C_G(b)$. But $|C_G(b)| = 2^i = |H|$ and so $H = C_G(b)$. Thus $b \in H$, which is our final contradiction. This completes the proof. □

Now we are ready to prove our main result.

Theorem. *Let G be a non-perfect X-decomposable finite group. Then G is isomorphic to* Z_6, D_8, Q_8, S_4, SmallGroup(20, 3) *or* SmallGroup(24, 3).

Proof. The proof is straightforward and follows from Lemma 2, Proposition 1 and Proposition 2. □

We end this paper with the following question:

Question 2. Is there any classification of perfect X-decomposable finite groups?

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