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On the fundamental group of real toric varieties

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Abstract. Let $X(\Delta)$ be the real toric variety associated to a smooth fan Δ. The main purpose of this article is: (i) to determine the fundamental group and the universal cover of $X(\Delta)$, (ii) to give necessary and sufficient conditions on Δ under which $\pi_1(X(\Delta))$ is abelian, (iii) to give necessary and sufficient conditions on Δ under which $X(\Delta)$ is aspherical, and when Δ is complete, (iv) to give necessary and sufficient conditions for $\mathcal{C}_\Delta$ to be a $K(\pi, 1)$ space where $\mathcal{C}_\Delta$ is the complement of a real subspace arrangement associated to Δ.

Keywords. Real toric varieties; fundamental group; asphericity; $K(\pi, 1)$ subspace arrangements.

0. Notations

$N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$ and $\langle , \rangle$ = the dual pairing.
$N_{\mathbb{R}} = N \bigotimes_{\mathbb{Z}} \mathbb{R}$. Δ = smooth fan in $N_{\mathbb{R}}$; σ and τ denote cones in Δ.
Let σ be a cone in Δ. $S_\sigma = \sigma^\vee \cap M = \{u \in M : \langle u, v\rangle \geq 0 \; \forall \; v \in \sigma\}$.
$\Delta(k)$ = cones of dimension k. $\Delta(1)$ are the edges and $\#\Delta(1) = d$.
$\Delta(1) = \{\rho_1, \rho_2, \ldots, \rho_d\}$. Let v_j be the primitive vector along ρ_j then, $\langle v_{i_1}, \ldots, v_{i_k}\rangle$ denotes the cone spanned by $\{v_{i_1}, \ldots, v_{i_k}\}$.
$(U_\sigma)_{\mathbb{C}} = \mathrm{Hom}_{sg}(S_\sigma, \mathbb{C})$, $U_\sigma = \mathrm{Hom}_{sg}(S_\sigma, \mathbb{R})$ and $(U_\sigma)_+ = \mathrm{Hom}_{sg}(S_\sigma, \mathbb{R}_+) \forall\; \sigma \in \Delta$ where $\mathbb{R}_+ = \mathbb{R}^+ \cup \{0\}$. Here, Hom_{sg} denotes the semigroup homomorphisms which sends 0 to 1.
X = smooth real toric variety of dimension n associated to Δ.
$X_{\mathbb{C}}$ = the complex toric variety whose real part is X.
X_+ = the non-negative part of X.
$T_2 := \mathrm{Hom}(M, \mathbb{Z}_2) \hookrightarrow T_{\mathbb{R}} := U_{\{0\}} = \mathrm{Hom}(M, \mathbb{R}^*)$; $T_{\mathbb{C}} := \mathrm{Hom}(M, \mathbb{C}^*)$; $T_+ := \mathrm{Hom}(M, \mathbb{R}^+)$.
For every $\sigma \in \Delta$, $x_\sigma \in U_\sigma$ is a *distinguished point* defined as:

$$x_\sigma(u) = \begin{cases} 1 & \forall\; u \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}.$$

O_τ = orbit of x_τ under the action of $T' \simeq (\mathbb{R}^*)^n$ and $V(\tau) = \overline{O_\tau}$.
$\mathrm{Stab}(x_\tau)$ = stabilizer of x_τ under the action of T_2.
$(O_\tau)_+$ = orbit of x_τ under the action of $(\mathbb{R}^+)^n$ and $V(\tau)_+ = \overline{(O_\tau)}_+$.

$W(\Delta) = \langle\, s_j : j = 1, 2, \ldots, d \;\mid\; s_j^2 : 1 \le j \le d,\ (s_i s_j)^2$ whenever $\langle v_i, v_j\rangle \in \Delta\,\rangle$. Then, $W(\Delta)$ is a *right-angled Coxeter group* associated to Δ. In many places when the context is clear, we shall denote $W(\Delta)$ simply by W.
$S_N := (N_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$ be the sphere in $N_{\mathbb{R}}$ and let $\pi : N_{\mathbb{R}} - \{0\} \longrightarrow S_N$ be the projection.

1. Introduction

Let Δ be a smooth fan in the lattice $N \cong \mathbb{Z}^n$. Let $X(\Delta)_{\mathbb{C}}$ be the complex toric variety associated to Δ. Let $X(\Delta)_{\mathbb{R}}$ be the real part of $X(\Delta)_{\mathbb{C}}$ which we call the real toric variety associated to Δ. We shall denote $X(\Delta)_{\mathbb{R}}$ by $X(\Delta)$ for convenience, as it is going to be our main object of study. For the definition and basic facts on real toric varieties (cf. ch. 4 of [10] and §2 of [11]). We mainly follow [10] for notations and background material on toric varieties.

In this paper we describe the fundamental group and the universal cover of $X(\Delta)$. We were motivated by the paper [8] of Davis and Januszkiewicz (cf. Cor. 4.5, p. 415 of [8]), where they prove the corresponding results for the real part of a toric manifold (now also known as a quasitoric manifold). We show that the same results can be obtained for a real toric variety $X(\Delta)$ associated to a smooth fan Δ not necessarily complete, the basic tool being the theory of *developments of complexes of groups* in chapter II.12 of [2]. We further give necessary and sufficient conditions on Δ for $X(\Delta)$ to be aspherical, motivated by the recent papers of Davis, Januszkiewicz and Scott (cf. Theorem 2.2.5, p. 27 of [9]), where they again prove similar results for a small-cover. For this purpose, we too rely primarily on the results of Davis [7], however in many places we give different proofs using the technique of *development* which is consistent with the theme of this paper (cf. Theorem 6.1 in §6).

Besides generalizing the previous results to the setting of a smooth real toric variety $X(\Delta)$, we give a presentation for the fundamental group $\pi_1(X(\Delta))$ completely in terms of the fan; furthermore, we give necessary and sufficient conditions on Δ under which $\pi_1(X(\Delta))$ is abelian, and we also show that the torsion elements are always of order 2.

Finally in §7, when Δ is complete, we relate $X(\Delta)$ to $\mathcal{C}_\Delta$ which is the complement of a real coordinate subspace arrangement in $\mathbb{R}^d$ where d is the number of edges in Δ. We could call $\mathcal{C}_\Delta$ *the real toric subspace arrangement associated to* Δ. It is nothing but the real analogue of the complement of the complex subspace arrangement in $\mathbb{C}^d$, for which $X(\Delta)_{\mathbb{C}}$ is realized as the geometric quotient under the action of $(\mathbb{C}^*)^{d-n}$ (cf. [1,6]). Moreover, since $\mathcal{C}_\Delta$ is homotopically equivalent to a covering space of $X(\Delta)$, we can describe its fundamental group and give conditions on Δ under which it is a $K(\pi, 1)$ space.

Finding $K(\pi, 1)$ arrangements seems to be an interesting problem in topology (cf. [15] and [12]) and we get many such examples. Similar to the results of [12], in our case too it turns out that $\mathcal{C}_\Delta$ is $K(\pi, 1)$ if and only if it is the complement of a union of precisely codimension 2 subspaces (cf. [4,9] for other results related to real subspace arrangements).

Before we state the main theorems let us fix the following terminology:

> Let $\Delta(1)$ denote the edges of Δ, $d = \#\Delta(1)$, and let $\{v_1, v_2, \ldots, v_d\}$ denote the primitive vectors along the edges. We assume that $\{v_1, v_2, \ldots, v_n\}$ form a basis for the lattice N and let $\{u_1, \ldots, u_n\}$ be the dual basis in M.
> Let $W(\Delta) = \langle s_{j_1}, \ldots, s_{j_d} \mid s_j^2 : 1 \le j \le d, (s_i s_j)^2$ whenever $\{v_i, v_j\}$ spans a cone in $\Delta\rangle$ be the right-angled Coxeter group associated to Δ.

We call the fan Δ *flag-like* if and only if the following condition holds for every collection of primitive edge vectors $\{v_{i_1}, \ldots, v_{i_r}\}$ in Δ: if for every $1 \leq k, l \leq r$, $\{v_{i_k}, v_{i_l}\}$ spans a cone in Δ, then $\{v_{i_1}, \ldots, v_{i_r}\}$ together spans a cone in Δ.
Let $X(\Delta)$ be a smooth and connected real toric variety.

We now state the main results in the paper.

Theorem 1.1. *The fundamental group $\pi_1(X(\Delta))$ is abelian if and only if one of the following holds in Δ.*

(i) *For every $1 \leq i, j \leq d$, $\{v_i, v_j\}$ spans a cone in Δ. In this case, $\pi_1(X)$ is isomorphic to $\mathbb{Z}_2^{d-n}$.*
(ii) *For each $1 \leq j \leq d$ there exists at most one $i = i_j$ with $1 \leq i_j \leq n$ such that, $\{v_{i_j}, v_j\}$ does not span a cone in Δ and $\langle u_{i_j}, v_j \rangle = 1 \bmod 2$. Further, for each $n+1 \leq k \leq d$ such that $k \neq j$ we have, $\langle u_{i_j}, v_k \rangle = 0 \bmod 2$.*

Theorem 1.2. *The real toric variety $X(\Delta)$ is aspherical if and only if Δ is flag-like.*

Theorem 1.3. *Let $\mathcal{C}_\Delta$ be the complement of the subspace arrangement related to Δ as above. Then, $\pi_1(\mathcal{C}_\Delta)$ is isomorphic to the commutator subgroup of $W(\Delta)$. Further, $\mathcal{C}_\Delta$ is aspherical if and only if it is the complement of a union of precisely codimension 2 subspaces.*

We prove Theorem 1.1 in §5, Theorem 1.2 in §6 and Theorem 1.3 in §7.

In this context, we also mention that the cohomology ring with $\mathbb{Z}_2$ coefficients of smooth, complete real toric varieties and $H_1(X(\Delta), \mathbb{Z})$ has been studied by Jurkiewicz (cf. [11]).

2. The universal cover of $X(\Delta)$

In this section we shall determine the universal cover and the fundamental group of X. For this purpose, we primarily apply the contents of pp. 367–386 of ch. II.12 of [2].

We begin with the elementary topological description of a real toric variety in the following proposition. The proof essentially follows from the proposition on p. 79, ch. 4 of [10] by replacing $X_{\mathbb{C}}$ by X and $\mathbb{S}^1$ by $\mathbb{S}^1 \cap \mathbb{R} \simeq \mathbb{Z}_2$. For details, also see p. 36, §3 of [11].

PROPOSITION 2.1 [10,11]

There is a retraction, $X_+ \overset{i}{\hookrightarrow} X \overset{r}{\to} X_+$ given by the absolute value map, $x \mapsto |x|$ from $\mathbb{R}_+ \subset \mathbb{R} \to \mathbb{R}_+$ which identifies X_+ with the quotient space of X by the action of the compact real 2-torus, $T_2 = \mathrm{Hom}(M, \mathbb{Z}_2)$. Further, there is a canonical mapping $T_2 \times X_+ \to X$ which realizes X as a quotient space, $T_2 \times X_+ / \sim$ where, $(t, x) \sim (t', x')$ if and only if $x = x'$ and $t \cdot (t')^{-1} \in \mathrm{Stab}(x_\tau)$ where, $x \in (O_\tau)_+$. The retraction $X \to X_+$ maps O_τ to $(O_\tau)_+$ and $V(\tau)$ to $V(\tau)_+$ and the fiber over $(O_\tau)_+$ is $T_\tau := \mathrm{Hom}(\tau^\perp \cap M, \mathbb{Z}_2)$ which is a compact real 2-torus of dimension $n - \dim(\tau)$.

We now observe the following property of X_+.

Lemma 2.2. *X_+ is contractible.*

Proof. Recall that $x_{\{0\}}$ is the distinguished point of $(U_{\{0\}})_+ \simeq (\mathbb{R}^+)^n$. We first show that for every $\sigma \in \Delta$, $(U_\sigma)_+ = \mathrm{Hom}_{sg}(S_\sigma, \mathbb{R}_+)$ is contractible to the point $x_{\{0\}} \in (U_{\{0\}})_+ \subseteq (U_\sigma)_+$. This is because, $(1-t)\cdot x + t \cdot x_{\{0\}} \in \mathrm{Hom}_{sg}(S_\sigma, \mathbb{R}_+)$ for every $t \in I = [0,1]$. The only thing we need to check here is that, if both $u, -u \in S_\sigma$ then $((1-t)\cdot x + t\cdot x_{\{0\}})(u) = (1-t)\cdot x(u) + t \cdot x_{\{0\}}(u) > 0$. This clearly holds since, $x(u) > 0$ and $\mathbb{R}^+$ is convex. Thus the map $H_\sigma : (U_\sigma)_+ \times I \to (U_\sigma)_+$ defined as $H_\sigma(x,t) = (1-t)\cdot x + t\cdot x_{\{0\}}$ is a strong deformation retraction of $(U_\sigma)_+$ to the point $x_{\{0\}}$.

Furthermore, by definition, the H_σ's for $\sigma \in \Delta$ are compatible with the inclusions $(U_\tau)_+ \subseteq (U_\sigma)_+$ whenever $\tau < \sigma$ in Δ. Therefore, since X_+ is the union of $(U_\sigma)_+$'s for $\sigma \in \Delta$, we can glue together the maps $\{H_\sigma\}_{\sigma\in\Delta}$ to get a strong deformation retraction H of X_+ to $x_{\{0\}}$. Hence the lemma. □

PROPOSITION 2.3

Let Δ be a smooth fan. We then have the following:

1. *$(X_+, (V(\tau)_+)_{\tau\in\Delta})$ is a stratified space with strata $\{V(\tau)_+\}_{\tau\in\Delta}$ indexed by the poset Δ.*
2. *Associated to this stratified space we have a simple complex of groups $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$ where the local group at the stratum $V(\tau)_+$ is $G_\tau = \mathrm{Stab}(x_\tau)$ under the action of $T_2 = \mathrm{Hom}(M, \mathbb{Z}_2)$ and $\psi_{\sigma\tau} : G_\tau \to G_\sigma$ (for $\tau < \sigma$ in Δ) are canonical inclusions and we have a simple morphism $\varphi = (\varphi_\tau) : G(\Delta) \to T_2 \simeq \mathbb{Z}_2^n$ injective at the local groups.*
3. *For the above simple complex of groups $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$, the direct limit $\widehat{G(\Delta)}$ is isomorphic to $W(\Delta)$. We therefore have a canonical simple morphism $\iota = (\iota_\tau) : G(\Delta) \to W(\Delta)$.*

Proof.

Proof of (1). Since the orbit space decomposition of X_+ under the action of T_+ is obtained by restriction of scalars from that of $X_\mathbb{C}$ under the action of $T_\mathbb{C}$, it follows that $(X_+, V(\tau)_+)$ is a stratified space with strata $V(\tau)_+$ indexed by Δ.

Proof of (2). Let $G_\tau = \mathrm{Stab}(x_\tau) \subseteq T_2$. We then have canonical inclusions, $\psi_{\sigma\tau} : G_\tau \hookrightarrow G_\sigma$ whenever $\tau < \sigma$ in Δ and, $\varphi_\tau : G_\tau \hookrightarrow T_2$ for every $\tau \in \Delta$. Then $G(\Delta) = (G_\tau, \psi_{\sigma\tau})$ is a simple complex of groups over $(X_+, V(\tau)_+)$ where G_τ is the local group along the stratum $V(\tau)_+$. Further, $\varphi = (\varphi_\tau)_{\tau\in\Delta} : G(\Delta) \to T_2$ is a simple morphism injective at the local groups.

Proof of (3). $\widehat{G(\Delta)}$ is by definition the free product of G_τ with the relations $\psi_{\sigma\tau}(h) = h \;\forall\, h \in G_\tau$ whenever $\tau < \sigma$ in Δ. Thus, $\widehat{G(\Delta)}$ is simply the graph product of the vertex groups $G_{\rho_j} \simeq \mathbb{Z}_2$ over the graph $S_N \cap \Delta(2)$ where the vertices of the graph correspond to $\Delta(1)$ and the edges correspond to $\Delta(2)$. Therefore, $\widehat{G(\Delta)} \simeq W(\Delta)$ and (3) follows. □

Let G be a group for which there exists a simple morphism $\varphi : G(\Delta) \to G$, injective at the local groups. Then, $G \times X_+/\sim := \{(g,x) : g \in G, x \in X_+ : (g,x) \sim (g',x') \Leftrightarrow x = x'; g\cdot(g')^{-1} \in G_\tau\}$, where τ is the unique cone such that $x \in O_\tau$. Let $D(\Delta,\varphi) = \sqcup_{\tau\in\mathcal{Q}} G/G_\tau$. Then, $D(\Delta,\varphi)$ is a poset consisting of pairs $(g\cdot G_\tau, \tau)$ where $\tau \in \Delta$ and $g\cdot G_\tau$ is a coset of G_τ in G and, $D(\Delta,\varphi)$ has the partial order, $(g\cdot G_\sigma, \sigma) < (g'\cdot G_\tau, \tau)$ if and only if $\sigma < \tau$ in Δ and $(g')^{-1}\cdot g \in G_\sigma$.

Lemma 2.4. *X is a stratified space over $D(\Delta, \varphi)$. Furthermore, the T_2 action on X is strata-preserving, with X_+ as the strict fundamental domain.*

Proof. By definition, $(T_2 \times X_+/\sim)$ is a stratified space over $D(\Delta, \varphi)$ such that, the action of T_2 on $T_2 \times X_+/\sim$ is strata-preserving where, $t \in T_2$ takes the stratum $(t', V(\tau)_+)$ to the stratum $(tt', V(\tau)_+)$. A strict fundamental domain for this action is the copy $1 \times X_+$ corresponding to the identity element $1 \in T_2$. However, by Proposition 2.1, there is a canonical T_2-equivariant isomorphism from $(T_2 \times X_+)/\sim$ to X. Thus X gets a structure of a stratified space over $D(\Delta, \varphi)$ in such a way that, the action of T_2 on X is strata-preserving and the strict fundamental domain for this action is $X_+ \subseteq X$. □

Theorem 2.5

1. *Let $D(X_+, \varphi)$ and $D(X_+, \iota)$ denote the developments of X_+ with respect to φ and ι respectively. Then, $D(X_+, \varphi) \simeq (T_2 \times X_+/\sim) \simeq X$ and $D(X_+, \iota) \simeq (W \times X_+/\sim) =: \widetilde{X}$. There are strata-preserving actions of W on $D(X_+, \iota)$ and of T_2 on $D(X_+, \varphi)$ with strict fundamental domain X_+.*
2. *X is connected if and only if the primitive vectors along the edges of Δ span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$. In particular, X is connected whenever the primitive vectors contain a $\mathbb{Z}$ basis for N.*
3. *$\widetilde{X} = W \times X_+/\sim$ is the universal cover of X and $\pi_1(X) \simeq \ker(\widehat{\varphi})$, where, $\widehat{\varphi} : W \to T_2$ is the canonical homomorphism induced by φ.*
4. *Let $h : W \to W_{ab} \simeq \mathbb{Z}_2^d$ be the surjective group homomorphism obtained by abelianisation. Associated to the map h, we have a simple morphism $\alpha : G(\Delta) \to \mathbb{Z}_2^d$ such that $\widehat{\alpha} = h$. Then, $D(X_+, \alpha) \simeq \mathbb{Z}_2^d \times X_+/\sim$ is a covering space over X with deck transformation group $\mathbb{Z}_2^{d-n}$, it is a covering space of X and $\pi_1(D(X_+, \alpha)) = [W, W]$.*

Proof. To prove this theorem we use Prop. 12.20 of [2].

Proof of (1). By Prop. 2.3, the development $D(X_+, \varphi)$ of X_+, with respect to the simple morphism φ from the simple complex of groups $G(\Delta)$ over X_+ to T_2, is a stratified space over $D(\Delta, \varphi)$ and is isomorphic to $T_2 \times X_+/\sim$ in such a way that, the induced action of T_2 on $T_2 \times X_+/\sim$ is identical to that in Lemma 2.4. Hence by Lemma 2.4, $D(X_+, \varphi)$ is isomorphic to X as a stratified space and further, the isomorphism is equivariant under the strata-preserving action of T_2. Similarly, the development $D(X_+, \iota)$ of X_+, with respect to the canonical simple morphism ι from $G(\Delta)$ to W is isomorphic to $(W \times X_+/\sim)$ which is a stratified space over the poset $D(\Delta, \iota)$ and further, there is a strata-preserving action of W on $D(X_+, \iota)$ with strict fundamental domain, X_+.

Proof of (2). From Lemma 2.2, X_+ is contractible, in particular it is connected. Hence, $D(X_+, \varphi)$ is connected if and only if $\widehat{\varphi}$ is surjective which is equivalent to the assumption that the image of the primitive edge vectors span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$. This certainly happens if a part of the primitive vectors along $\Delta(1)$ form a $\mathbb{Z}$-basis for N.

Proof of (3). From Lemma 2.2 it follows that, X_+ is simply connected and the strata of X_+ are arcwise connected. Further, from Prop. 2.3 we know that $\widehat{G(\Delta)} \simeq W$. Therefore, $(W \times X_+/\sim) \simeq D(X_+, \iota)$ is the universal cover of $X \simeq D(X_+, \varphi)$ and $\ker(\widehat{\varphi}) \simeq \pi_1(X)$ where, $\widehat{\varphi}$ is the canonical surjective homomorphism induced by φ.

Proof of (4). Since $\alpha = h \circ \iota$ and T_2 being abelian $\widehat{\varphi}$ factors through W_{ab}. Therefore, the simple morphism α is injective at the local groups and the development $D(X_+, \alpha) \simeq \mathbb{Z}_2^d \times X_+/\sim$. The remaining claims of (4) follow simply by the direct application of Prop. 12.20 of [2]. □

Remark 2.6 (*Connectedness of X*). If the primitive edge vectors of $\Delta(1)$ do not span $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$, then X is not connected and the number of connected components of X is equal to $[N \otimes_{\mathbb{Z}} \mathbb{Z}_2 : \varphi(W)]$. In fact, Δ is supported on a smaller dimensional lattice and therefore, X is isomorphic to $X' \times (\mathbb{R}^*)^{(k/2)}$, where $k = [N \otimes_{\mathbb{Z}} \mathbb{Z}_2 : \varphi(W)]$ and X' is a connected toric variety of dimension $n - k$. For example, the real toric variety associated to the fan $\Delta = \{e_1, -e_1, \{0\}\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^*$ and has two connected components $\mathbb{S}^1 \times \mathbb{R}^+$ and $\mathbb{S}^1 \times \mathbb{R}^-$. Indeed, for X to be connected it is not necessary that the primitive edge vectors should span N for example the real toric variety associated to the fan $\Delta = \{\langle 2e_1 + 3e_2\rangle, \langle e_1\rangle, \{0\}\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, is smooth and connected but the edge vectors $\{2e_1 + 3e_2, e_1\}$ do not form a $\mathbb{Z}$-basis for N.

3. A presentation for $\pi_1(X)$

Let X be smooth and connected. In this section we shall give a presentation for $\pi_1(X)$ with generators and relations defined purely from the combinatorial structure of Δ.

Let $\{v_1, \ldots, v_n\}$ be primitive vectors along $\Delta(1)$ which form a basis for $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and let $\{u_1, \ldots, u_n\}$ be the dual basis. Let $a_{j,i} = \langle u_i, v_j\rangle \bmod \mathbb{Z}_2$ for $1 \le j \le d$ and $1 \le i \le n$. Then, $A = (a_{j,i})$ is the *characteristic matrix of* Δ *with respect to* $\{v_1, \ldots, v_n\}$.

For $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{Z}_2^n$, let $b_i^j = t_i + a_{j,i}$ for $1 \le i \le n, 1 \le j \le d$ and let $c_i^{p,q} = t_i + a_{p,i} + a_{q,i}$ for $1 \le i \le n$; $1 \le p, q \le d$. We shall denote the vector $(b_i^j)_{i=1,\ldots,n}$ by $\underline{b}^j$ and the vector $(c_i^{p,q})_{i=1,\ldots,n}$ by $\underline{c}^{p,q}$.

In the following proposition we will give a presentation for $\pi_1(X)$ using the above data.

PROPOSITION 3.1

The fundamental group $\pi_1(X)$ *has a presentation with generators*

$$\{y_{j,\underline{t}} : 1 \le j \le d \mid \underline{t} = (t_1, \ldots, t_n) \in \mathbb{Z}_2^n\}$$

and relations

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{1,(0,\ldots,0)}^{t_1} \cdot y_{2,(t_1,0,\ldots,0)}^{t_2} \cdots y_{n,(t_1,\ldots,t_{n-1},0)}^{t_n}\}$$

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{1,\underline{t}} \cdot y_{j,\underline{b}^j} \mid 1 \le j \le d\}$$

$$\bigcup_{\underline{t} \in \mathbb{Z}_2^n} \{y_{p,\underline{t}} \cdot y_{q,\underline{b}^p} \cdot y_{p,\underline{c}^{p,q}} \cdot y_{q,\underline{b}^q} \mid \langle v_p, v_q\rangle \in \Delta\}.$$

Proof. We know from Theorem 2.5 that $\pi_1(X)$ is isomorphic to the kernel of the surjective homomorphism $\widehat{\varphi} : W \to T_2 \simeq \mathbb{Z}_2^n$, where W has the presentation $\langle S \mid R\rangle$ for $S = \{s_1, s_2, \ldots, s_d\}$ and $R = \{s_1^2, s_2^2, \ldots, s_d^2;\ (s_i s_j)^2$ whenever $\{v_i, v_j\}$ spans a cone in $\Delta\}$.

We further have the following commuting diagram:

$$\begin{array}{ccccccccc} 1 & \to & F' & \to & F(S) & \to & \mathbb{Z}_2^n & \to & 1 \\ & & \downarrow & & \downarrow \psi & & \| & & \\ 1 & \to & H & \to & W & \to & \mathbb{Z}_2^n & \to & 1 \end{array},$$

where $F(S)$ denotes the free group on S, ψ denotes the canonical surjection from $F(S)$ to W, H denotes $\pi_1(X)$ and $F' = \psi^{-1}(H)$.

Since $\mathcal{T} = \{s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \mid (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n\}$ is a *Schreier transversal* for F' in $F(S)$, we can apply the Reidemeister–Schreier theorem (cf. [5,13]) to obtain a presentation for $\pi_1(X)$ from that of W. Let

$$S_H = \{y_{j,\underline{t}} : 1 \le j \le d; \underline{t} \in \mathbb{Z}_2^n\}, \tag{3.1}$$

$$R_H^1 = \{\alpha_0(u) \; \forall \, u \in \mathcal{T}\}, \tag{3.2}$$

$$R_{\underline{t}} = \{\alpha_{\underline{t}}(r) \; \forall \, r \in R \; ; \; \underline{t} \in \mathbb{Z}_2^n\}, \tag{3.3}$$

where $0 := (0, 0, \ldots, 0) \in \mathbb{Z}_2^n$, $\alpha_{\underline{t}} : F(S) \to F(S_H)$ is defined recursively as follows: $\alpha_{\underline{t}}(1) := 1$; $\alpha_{\underline{t}}(s_j) = y_{j,\underline{t}}$. Suppose that by induction we have defined $\alpha_{\underline{t}}(w)$ for $w \in F(S)$ then, $\alpha_{\underline{t}}(w \cdot s_j) := \alpha_{\underline{t}}(w) \cdot \alpha_{\underline{t \cdot s_j}}(s_j)$ where, $\underline{t \cdot s_j} \in \mathbb{Z}_2^n$ corresponds to the coset representative $\varphi(w') \in \mathcal{T}$ of $F' \cdot w'$, where $w' = s_1^{t_1} \cdots s_n^{t_n} \cdot s_j$.

Note that, $\forall \, \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$ we have

(i) $\alpha_0(s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}) = (\alpha_0(s_1))^{t_1} \cdot (\alpha_{(t_1,0,0,\ldots,0)}(s_2))^{t_2} \cdots (\alpha_{(t_1,t_2,\ldots,t_{n-1},0)}(s_n))^{t_n}$,

(ii) $\alpha_{\underline{t}}(s_j^2) = \alpha_{\underline{t}}(s_j) \cdot \alpha_{\underline{b}^j}(s_j)$,

(iii) $\alpha_{\underline{t}}((s_p \cdot s_q)^2) = \alpha_{\underline{t}}(s_p) \cdot \alpha_{\underline{b}^p}(s_q) \cdot \alpha_{\underline{c}^{p,q}}(s_p) \cdot \alpha_{\underline{b}^q}(s_q) \; \forall \, 1 \le j \le d$.

It follows from definition (3.2) and from the identity (i) above that

$$\begin{aligned} R_H^1 &= \left\{\alpha_0(s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}) \quad \middle| \; \underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n\right\} \\ &= \left\{\begin{array}{l} (y_{1,(0,0,\ldots,0)}^{t_1} \cdot y_{2,(t_1,0,\ldots,0)}^{t_2} \cdots y_{n,(t_1,t_2,\ldots,t_{n-1},0)}^{t_n}) \\ |\underline{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n \end{array}\right\}. \end{aligned}$$

Also the definition (3.3) and the identities (ii) and (iii) above, imply that

$$\begin{aligned} R_{\underline{t}} &= \left\{\begin{array}{l} \alpha_{\underline{t}}(s_1^2), \ldots, \alpha_{\underline{t}}(s_d^2) \; ; \\ (\alpha_{\underline{t}}(s_p s_q)^2) \text{ whenever } \{v_p, v_q\} \text{ spans a cone in } \Delta \end{array}\right\} \\ &= \left\{\begin{array}{l} y_{1,\underline{t}} \cdot y_{1,\underline{b}^1}, \ldots, y_{d,\underline{t}} \cdot y_{d,\underline{b}^d} \; ; \\ y_{p,\underline{t}} \cdot y_{q,\underline{b}^p)} \cdot y_{p,\underline{c}^{p,q}} \cdot y_{q,\underline{b}^q} \\ \text{whenever } \{v_p, v_q\} \text{ spans a cone in } \Delta \end{array}\right\}. \end{aligned}$$

Here, $b_i^j = t_i + a_{ji} \; \forall \; 1 \le j \le d$; $1 \le i \le n$ and $c_i^{p,q} = t_i + a_{pi} + a_{qi} \; \forall \; 1 \le i \le n$; $1 \le p, q \le d$.

Let $R_H^2 := \cup_{\underline{t} \in \mathbb{Z}_2^n} R_{\underline{t}}$. Then, from the Reidemeister–Schreier theorem it follows that, $\pi_1(X)$ has the presentation $\langle S_H \mid R_H^1; \; R_H^2 \rangle$. □

Lemma 3.2. *The presentation can be simplified with lesser number of generators and relations by expressing the* $y_{j,\underline{t}} \in S_H$ *as words in* S *and using the relations in* W *if we make further assumption on* Δ *that,* $\langle v_p, v_q\rangle \in \Delta$ *for all* $1 \le p, q \le n$.

Proof. $\alpha_{\underline{t}}(s_j) = s_1^{t_1} \cdots s_n^{t_n} \cdot s_j \cdot \varphi(s_1^{t_1} \cdots s_n^{t_n} \cdot s_j)$ where $\varphi(w) \in \mathcal{T}$ is the coset representative of $F' \cdot w$ for $w \in W$. Notice further that in W, the $\{s_1, \ldots, s_n\}$ commute among themselves by our assumption on $\{v_1, v_2, \ldots, v_n\}$. Hence we have, $\varphi(s_1^{t_1} \cdots s_n^{t_n} \cdot s_j) = s_1^{t_1+a_{j,1}} \cdot s_2^{t_2+a_{j,2}} \cdots s_n^{t_n+a_{j,n}}$, $\forall\ (t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$. This implies that we have the following identities:

(i) $y_{j,\underline{t}} = s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot (s_j) \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_j^{t_j+1} \cdots s_n^{t_n} = 1$ for all $1 \le j \le n$; $\underline{t} \in \mathbb{Z}_2^n$ (since, $a_{j,i} = \delta_{j,i}$ for all $1 \le j \le n$),

(ii) $y_{j,0} = \alpha_0(s_j) = s_j \cdot s_1^{a_{j,1}} \cdot s_2^{a_{j,2}} \cdots s_n^{a_{j,n}}$ for all $n+1 \le j \le d$,

(iii) $y_{j,\underline{t}} = \alpha_{\underline{t}}(s_j) = s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot (s_j \cdot s_1^{a_{j,1}} \cdot s_2^{a_{j,2}} \cdots s_n^{a_{j,n}}) \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}$ for all $n+1 \le j \le d$,

(iv) $y_{j,\underline{b}^j} = \alpha_{\underline{b}^j}(s_j) = s_1^{t_1+a_{j,1}} \cdots s_n^{t_n+a_{j,n}} \cdot (s_j) \cdot s_1^{t_1} \cdots s_n^{t_n} = (s_1^{t_1} \cdots s_n^{t_n} \cdot (s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}) \cdot s_1^{t_1} \cdots s_n^{t_n})^{-1} = (y_{j,\underline{t}})^{-1}$.

Let $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ then, from (ii) and (iii) above, we see that the generators of $\pi_1(X)$ are, $\{s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n} \cdot S_j \cdot s_1^{t_1} \cdot s_2^{t_2} \cdots s_n^{t_n}\ \forall\, n+1 \le j \le d;\ \forall\, \underline{t} \in \mathbb{Z}_2^n\}$. Further, since R_H^1 consists of words in $\{y_{j,\underline{t}}$ for $1 \le j \le n$ and $\underline{t} \in \mathbb{Z}_2^n\}$, from (i) we see that $R_H^1 = \{1\}$. Furthermore, (iv) implies that the first set of relations in $R_{\underline{t}}$ are of the form $\{y_{j,k} \cdot (y_{j,k})^{-1}\}$, therefore they trivially hold in the group $\pi_1(X)$. Thus, finally, the number of generators reduce to $(d-n) \cdot 2^n$ and they are: $\{y_{j,\underline{t}}$ for $n+1 \le j \le d;\ \underline{t} \in \mathbb{Z}_2^n\}$ and the non-trivial relations are of the form: $R_{\underline{t}} = \{y_{p\varphi(\underline{t})} \cdot y_{q\varphi(\underline{b}^p)} \cdot y_{p\varphi(\underline{c}^{p,q})} \cdot y_{q\varphi(\underline{b}^q)}$ whenever $\{v_p, v_q\}$ spans a cone in $\Delta\}\ \forall\, \underline{t} \in \mathbb{Z}_2^n$. Therefore, the final presentation for $\pi_1(X)$ is $\langle S_H, R_H\rangle$, where

$$
\begin{aligned}
S_H &= \{y_{j,\underline{t}} \text{ where } n+1 \le j \le d \text{ and } \underline{t} \in \mathbb{Z}_2^n\},\\
R_{\underline{t}} &= \{y_{p,\varphi(\underline{t})} \cdot y_{q,\varphi(\underline{b}^p)} \cdot y_{p,\varphi(\underline{c}^{p,q})} \cdot y_{q,\varphi(\underline{b}^q)}\\
&\qquad \text{whenever } \{v_p, v_q\} \text{ spans a cone in } \Delta\}\ \forall\, \underline{t} \in \mathbb{Z}_2^n,\\
R_H &= \cup_{\underline{t} \in \mathbb{Z}_2^n} R_{\underline{t}}.
\end{aligned}
$$

Further, $\pi_1(X)$ is generated as a subgroup of $W(\Delta)$ by $s_1^{t_1} \cdots s_n^{t_n} \cdot S_j \cdot s_1^{t_1} \cdots s_n^{t_n}$, where $\underline{t} \in \mathbb{Z}_2^n$ and $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \le j \le d$. □

Remark 3.3. By the classification of two-dimensional smooth complete fans (cf. p. 42 of [10] and p. 57 of [11]) we observe that, except the torus $\mathbb{S}^1 \times \mathbb{S}^1$, all other smooth complete real toric surfaces correspond bijectively to the two-dimensional compact non-orientable manifolds. This is because, they are obtained by successively blowing up $\mathbb{P}^2_{\mathbb{R}}$ at T-fixed points and are therefore homeomorphic to $\mathbb{P}^2_{\mathbb{R}} \# \cdots \# \mathbb{P}^2_{\mathbb{R}}$ ($d-2$ copies). However, the classical presentation for the fundamental group is apparently different from the presentation we have obtained, especially because it has only one relation. In the cases when $d = 3$ and 4, where the spaces are $\mathbb{P}^2_{\mathbb{R}}$, $\mathbb{S}^1 \times \mathbb{S}^1$ or the Klein-bottle $\simeq \mathbb{P}^2_{\mathbb{R}} \# \mathbb{P}^2_{\mathbb{R}}$, the presentations we give agrees with some of the classical ones. We hope to simplify the presentation given above to reduce the number of generators and relations so that in general it agrees with the classical cases.

Remark 3.4. Note that the fundamental group $\pi_1(X)$ and hence its presentation depends only on the 2-skeleton $\Delta(2)$ of Δ.

4. The coxeter group $W(\Delta)$

In this section we prove some general results on right-angled Coxeter groups and in particular for $W(\Delta)$. Let $M = (m_{ij})$ denote the Coxeter matrix corresponding to W.

Lemma 4.1. *$[W, W]$ is abelian if and only if for all $1 \leq j \leq d$ there exists at most one i such that $\langle v_i, v_j \rangle \notin \Delta$.*

Proof. If there exist $i \neq k$ such that $\{v_i, v_j\}$ and $\{v_k, v_j\}$ does not span a cone in Δ then, $[s_i, s_j] \cdot [s_k, s_j] \neq [s_k, s_j] \cdot [s_i, s_j]$ in $[W, W]$.

Conversely, if for each $1 \leq j \leq d$, there exists at most one i such that $\langle v_i, v_j \rangle \notin \Delta$ then, using the relations in W, it is easy to see that for any word $w \in W$, $w \cdot [s_i, s_j] \cdot w^{-1} = [s_i, s_j]$ or $[s_j, s_i]$. (It is $[s_j, s_i]$ iff either one of s_i or s_j but not both occurs in the reduced expression of w.) Now, $[W, W]$ is the normal subgroup of W generated by the commutators $\{[s_i, s_j] \mid \langle v_i, v_j \rangle \notin \Delta\}$. Therefore, under the above assumption, $\{[s_i, s_j] \mid \langle v_i, v_j \rangle \notin \Delta\}$ in fact generate $[W, W]$ as a subgroup of W. Further, since they commute among themselves, $[W, W]$ is abelian. □

Lemma 4.2. *A word $w \in W$ is of finite order if and only if it is of order* 2. *Moreover, in this case, w is a conjugate in W to a word w' which is of the form, $w' = s_{j_1} \cdots s_{j_l}$ with $s_{j_p} \cdot s_{j_q} = s_{j_q} \cdot s_{j_p}$ $\forall\, 1 \leq p, q \leq l$.*

Proof. Suppose $w = v \cdot w' \cdot v^{-1}$ where, w' is as above and $v \in W$. Then w is clearly of order 2. On the other hand, if w is not of the above form then the reduced expression for w is of the form $w = s_{i_1} \cdots s_{i_k}$ where s_{i_p} and s_{i_q} do not commute for some $1 \leq p, q \leq k$. Indeed, by repeatedly using the relation $s_i \cdot s_j = s_j \cdot s_i$ whenever $m_{ij} = 2$, we can assume without loss of generality that, up to conjugation w is of the form $s_{i_1} \cdot s_{i_2} \cdots s_{i_k}$, where s_{i_1} and s_{i_k} do not commute. Then it follows that, for any positive integer r, $w^r = (s_{i_1} \cdots s_{i_k}) \cdot (s_{i_1} \cdots s_{i_k}) \cdots (s_{i_1} \cdots s_{i_k})$ is in fact a reduced expression in W. Hence, w is of infinite order. □

Lemma 4.3. *Let $w = s_{i_1} \cdots s_{i_k} \in W$, where $\langle v_{i_1}, \ldots, v_{i_k} \rangle \in \Delta$ and let $w' = s_{j_1} \cdots s_{j_l}$ where, $\langle v_{j_p}, v_{j_q} \rangle \in \Delta$ for all $1 \leq p, q \leq l$ but $\langle v_{j_1}, \ldots, v_{j_l} \rangle \notin \Delta$. Then, $w \notin N(w')$, where $N(w')$ is the normal subgroup generated by w' in W.*

Proof. Suppose on the contrary $w = v_1 \cdot w' \cdot v_1^{-1} \cdot v_2 \cdot w' \cdot v_2^{-1} \cdots v_r \cdot w' \cdot v_r^{-1}$ for some $v_1, v_2, \ldots v_r \in W$. By Lemma 4.2 we know that $(w')^2 = 1$. Hence, the above expression can be rewritten as

1. $w = [v_1, w'] \cdot [w', v_2] \cdot [v_3, w'] \cdots [w', v_r]$ if r is even,
2. $w = [v_1, w'] \cdot [w', v_2] \cdots [v_r, w'] \cdot w'$ if r is odd.

This implies that, $w \in [W, W]$ if r is even and $w \cdot w' = w \cdot (w')^{-1} \in [W, W]$ if r is odd.

Now let $h : W \to \mathbb{Z}_2^d$ be the abelianisation map which takes s_j to the coordinate vector $e_j = (0, 0, \ldots, 1, \ldots, 0)$ (with 1 at the jth position). Also, by our choice of w and w'

we observe that, $\{s_{i_1}, \ldots, s_{i_k}\}$ and $\{s_{j_1}, \ldots, s_{j_l}\}$ pairwise commute in W and the tuples $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_l)$ are distinct.

Therefore, $h(w) = \Sigma_{p=1}^{k} e_{i_p} \neq (0, \ldots, 0)$ when r is even and $h(w \cdot w') = \Sigma_{p=1}^{k} e_{i_p} + \Sigma_{q=1}^{l} e_{j_q} \neq (0, \ldots, 0)$ when r is odd. This is a contradiction since on the other hand, w and $w \cdot (w')^{-1} \in [W, W]$ when r is even and r is odd respectively. This proves the lemma. □

Remark 4.4. Lemma 4.1 if phrased differently will be: $[W, W]$ is abelian if and only if there exists at most one i for every j such that $m_{i,j} \neq 2$, holds not just for right-angled Coxeter groups, but for more general class of Coxeter groups with $m_{i,j} = 2$ or $m_{i,j} \geq 5 \; \forall \; i, j$.

5. Criterion for $\pi_1(X)$ to be abelian

Let X be smooth and connected. In the following theorem we give conditions on Δ under which $\pi_1(X)$ is abelian. We shall follow the notations in §3 and further assume that $\langle v_p, v_q \rangle \in \Delta$ for every $1 \leq p, q \leq n$ as in Lemma 3.2.

Theorem 5.1. *$\pi_1(X)$ is abelian if and only if one of the following holds in Δ.*

1. *For every $1 \leq i, j \leq d$, $\{v_i, v_j\}$ spans a cone in Δ. In this case, $\pi_1(X)$ is isomorphic to $\mathbb{Z}_2^{d-n}$.*
2. *For each $1 \leq j \leq d$ there exists at most one $i = i_j$ with $1 \leq i_j \leq n$ such that, $\{v_{i_j}, v_j\}$ does not span a cone in Δ and $\langle u_{i_j}, v_j \rangle = 1 \bmod 2$. Further, for each $n+1 \leq k \leq d$ such that $k \neq j$ we have, $\langle u_{i_j}, v_k \rangle = 0 \bmod 2$.*

Proof. Recall that we have an exact sequence, $1 \to [W, W] \to \pi_1(X) \to \mathbb{Z}_2^{d-n} \to 1$ and further, $[W, W]$ is generated as a normal subgroup of W by $[s_{i_p}, s_{i_q}]$ whenever $\{v_{i_p}, v_{i_q}\}$ does not span a cone in Δ.

Step 1. Since $[W, W]$ is a subgroup of $\pi_1(X)$, if $\pi_1(X)$ is abelian then $[W, W]$ must be abelian. By Lemma 4.1, $[W, W]$ is abelian if and only if for every v_j there exists at most one v_i such that $\{v_j, v_i\}$ does not span a cone in Δ.

Further, $[W, W] = \{1\}$ if and only if any two $\{v_j, v_i\}$ for $1 \leq i, j \leq d$ spans a cone in Δ which implies that, $W \simeq \mathbb{Z}_2^d$ and $\pi_1(X) \simeq \mathbb{Z}_2^{d-n}$.

Step 2. On the other hand, if $[W, W] \neq \{1\}$ then there exists $\{v_j, v_i\}$ which does not span a cone in Δ. However, since $[W, W]$ is abelian, this $i = i_j$ must be unique for every such j. Thus, in W, s_j and s_{i_j} do not commute but they both commute with s_k for every $1 \leq k \leq n$.

Step 3. Suppose now that, for some $n+1 \leq j \leq d$ we have $n+1 \leq i_j \leq d$, then $\pi_1(X)$ is non-abelian for if S_j denotes the word $s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ in W then,

$$\begin{aligned} S_j \cdot S_{i_j} &= s_j \cdot s_{i_j} \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{i_j,1}} \cdots s_n^{a_{i_j,n}} \\ &\neq s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{i_j,1}} \cdots s_n^{a_{i_j,n}} = S_{i_j} \cdot S_j. \end{aligned}$$

Hence if $\pi_1(X)$ is abelian then, for every $n+1 \leq j \leq d$ there is a unique index i_j such that $\langle v_j, v_{i_j} \rangle \notin \Delta$ and further, $1 \leq i_j \leq n$.

Step 4. Now, if for some $n+1 \leq k \leq d$ with $k \neq j$ we have $a_{k,i_j} = \langle u_i, v_{i_j} \rangle \bmod \mathbb{Z}_2 = 1$, then $\pi_1(X)$ is non-abelian. This is because, if $w = [S_k, S_j] \in \pi_1(X)$ then $w \neq 1$ which we can see by the following cases:

If $a_{k,i_k} = 0$ and $a_{j,i_k} = 0$, then $w = [s_{i_j}, s_j] \neq 1$.
If $a_{k,i_k} = 1$ and $a_{j,i_k} = 1$, then $w = [s_{i_j}, s_j] \neq 1$.
If $a_{k,i_k} = 1$ and $a_{j,i_k} = 0$, then $w = [s_{i_j}, s_j] \cdot [s_k, s_{i_k}] \neq 1$.
If $a_{k,i_k} = 0$ and $a_{j,i_k} = 1$, then $w = [s_{i_j}, s_j] \cdot [s_k, s_{i_k}] \neq 1$.

(Here we omit the proofs of the assertion that $w \neq 1$ in each case, as it follows easily from the relations in W).

Step 5. If $a_{j,i_j} = 0$, then again $\pi_1(X)$ is non-abelian since, the elements $s_{i_j} \cdot S_j \cdot s_{i_j}$ and S_j^{-1} do not commute in $\pi_1(X)$. This is because, by Lemma 4.2 $[s_{i_j}, s_j]$ is an element of infinite order in W and hence, $(s_{i_j} \cdot S_j \cdot s_{i_j}) \cdot S_j^{-1} = [s_{i_j}, s_j] \neq [s_j, s_{i_j}] = S_j^{-1} \cdot (s_{i_j} \cdot S_j \cdot s_{i_j})$.

Step 6. Therefore, if $\pi_1(X)$ is abelian and $[W, W] \neq 1$, then it is necessary that the following conditions must hold:

For every $1 \leq j \leq d$, there exists a unique index i_j with $1 \leq i_j \leq n$ such that $\{v_j, v_{i_j}\}$ does not span a cone in Δ and $a_{j,i_j} = 1$. Further, for every $n+1 \leq k \leq d$ such that $k \neq j$, we have $a_{k,i_j} = 0$.

We shall now prove that these conditions are in fact sufficient for $\pi_1(X)$ to be abelian.

Claim.

(i) S_j and S_k commute for $n+1 \leq j, k \leq d$.
(ii) $w \cdot S_j \cdot w^{-1}$ and S_j commute where, $w = s_1^{t_1} \cdots s_n^{t_n}$ for every $\underline{t} = (t_1, \ldots, t_n) \in \mathbb{Z}_2^n$ and $n+1 \leq j \leq d$.

Proof of the Claim.

(i)
$$\begin{aligned} S_j \cdot S_k &= s_j \cdot s_k \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \{\text{since } a_{j,i_k} = 0 \text{ by assumption}\} \\ &= s_k \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \{\text{since } k \neq i_j\} \\ &= s_k \cdot s_1^{a_{k,1}} \cdots s_n^{a_{k,n}} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \{\text{since } a_{k,i_j} = 0 \text{ by assumption}\} \\ &= S_k \cdot S_j. \end{aligned}$$

(ii) Let $w = s_1^{t_1} \cdots s_n^{t_n}$ such that $(t_1, t_2, \ldots, t_n) \in \mathbb{Z}_2^n$.

$$w \cdot S_j \cdot w^{-1} = (s_1^{t_1} \cdots s_n^{t_n}) \cdot (s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}) \cdot (s_1^{t_1} \cdots s_n^{t_n}) = z.$$

(a) If $t_{i_j} = 0$, then $z = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} = S_j$ {since $s_j \cdot w = w \cdot s_j$}. Thus $w \cdot S_j \cdot w^{-1} = S_j$.
(b) If $t_{i_j} = 1$, then $z = s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}} \cdot s_{i_j} = s_{i_j} \cdot S_j \cdot s_{i_j}$. Further,

$$\begin{aligned} (w \cdot S_j \cdot w^{-1}) \cdot (S_j) &= s_{i_j} \cdot S_j \cdot s_{i_j} \cdot S_j \\ &= s_{i_j} \cdot s_j \cdot s_1^{a_{j,1}} \cdots \widehat{s_{i_j}} \cdots s_n^{a_{j,n}} \cdot S_j \text{ {since } } a_{j,i_j} = 1\} \\ &= s_{i_j} \cdot s_j^2 \cdot s_{i_j} = 1. \end{aligned}$$

This implies that, $w \cdot S_j \cdot w^{-1} = S_j^{-1}$.

Hence, $w \cdot S_j \cdot w^{-1}$ commutes with S_k for all $n+1 \leq j,k \leq d$, since we have either $w \cdot S_j \cdot w^{-1} = S_j$ or S_j^{-1} in each of the cases. Therefore, since the generators commute among themselves, we conclude that $\pi_1(X)$ is abelian. □

Remark 5.2. If Δ is complete, then the condition $a_{j,i_j} = 1$ will be forced after Step 4 in which case, we shall skip Step 5. However this is not true in general for example, in the non-complete fan $\Delta = \{\{0\}, \langle e_1, e_2\rangle, \langle -2e_1 + e_2\rangle\}$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$.

Remark 5.3 (*Torsion elements*). By Lemma 4.2, since $\pi_1(X)$ is a subgroup of W, the torsion elements in $\pi_1(X)$ are always of order 2. In particular, when $\pi_1(X)$ is abelian, $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \leq j \leq d$ is of order 2 iff $\langle v_j, v_i\rangle \in \Delta$ for all $1 \leq i \leq n$ and it is of infinite order iff there exists a unique $1 \leq i_j \leq n$ such that $\langle v_j, v_{i_j}\rangle \notin \Delta$ since in this case, $a_{j,i_j} = 1$ and $S_j^2 = [s_j, s_{i_j}] \neq 1$ in $[W,W] \subset W$.

Remark 5.4. If $\pi_1(X)$ is abelian then $\pi_1(X)$ is generated by $S_j = s_j \cdot s_1^{a_{j,1}} \cdots s_n^{a_{j,n}}$ for $n+1 \leq j \leq d$. Let $\{j_1, j_2, \ldots, j_r\} = J = \{j \mid n+1 \leq j \leq d$ and $\langle v_j, v_{i_j}\rangle \notin \Delta$ for some $1 \leq i_j \leq n\}$. Therefore, if $j \notin J$ then, $\langle v_j, v_i\rangle \in \Delta$ for every $1 \leq i \leq n$. Thus, $\pi_1(X) \simeq \mathbb{Z}_2^{d-n-r} \oplus \mathbb{Z}^r$ where, $\mathbb{Z}^r = \langle S_{j_p} = s_{j_p} \cdot s_1^{a_{j_p,1}} \cdots s_n^{a_{j_p,n}}$ for $1 \leq p \leq r\rangle$. Furthermore, $[W,W] = \langle [s_{j_p}, s_{i(j_p)}] = S_{j_p}^2$ for $1 \leq p \leq r\rangle \subset W$ is a free abelian of rank r. We therefore have the following commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [W,W] & \longrightarrow & \pi_1(X) & \longrightarrow & \mathbb{Z}_2^{d-n} & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \mathbb{Z}^r & \xrightarrow{\times 2} & \mathbb{Z}^r \oplus \mathbb{Z}_2^{d-n-r} & \longrightarrow & \mathbb{Z}_2^r \oplus \mathbb{Z}_2^{d-n-r} & \longrightarrow & 1 \end{array}.$$

Remark 5.5. If $\pi_1(X)$ is abelian then necessarily $d \leq 2n$ because, to every $n+1 \leq j \leq d$, we associate a unique i_j with $1 \leq i_j \leq n$. Examples of toric varieties with abelian fundamental group are: (i) products of real projective spaces, (ii) toric bundles with base as aspherical toric variety with abelian fundamental group and fibre $\mathbb{P}^n_{\mathbb{R}}$ for $n \geq 2$. (However, this is not true for a non-trivial bundle with fibre $\mathbb{P}^1_{\mathbb{R}}$ for example, $\pi_1((\mathbb{F}_1)_{\mathbb{R}})$ is non-abelian where $(\mathbb{F}_1)_{\mathbb{R}}$ denotes the real part of the Hirzebruch surface $\mathbb{F}_1$.)

6. Asphericity of X

Let $S_N := (N_{\mathbb{R}} - \{0\})/\mathbb{R}_{>0}$ and $\pi : N_{\mathbb{R}} - \{0\} \longrightarrow S_N$ be the projection. Let $\mathcal{S}_\Delta$ denote the simplicial complex associated to the smooth fan Δ, where each k-dimensional $\sigma \in \Delta$ corresponds to a $(k-1)$-dimensional spherical simplex $\pi(\sigma - \{0\})$. If further we assume Δ to be complete, then it gives rise to a triangulation of S_N (cf. p. 52 of [14]).

Recall that a simplicial complex $\mathcal{S}$ with vertices $\mathcal{V} = \{v_i\}$ is called a *flag complex* if the following condition holds for every finite subset $\{v_1, v_2, \ldots, v_n\}$ of $\mathcal{V}$: If $\{v_i, v_j\}$ span a simplex in $\mathcal{S}$ for all $i, j \in \{1, 2, \ldots, n\}$ then $\{v_1, v_2, \ldots, v_n\}$ span a simplex of $\mathcal{S}$.

Hence, $\mathcal{S}_\Delta$ is a flag complex if and only if for every collection of primitive edge vectors $\{v_{i_1}, \ldots, v_{i_r}\}$, if $\{\langle v_{i_k}, v_{i_l}\rangle \in \Delta \ \forall\ 1 \leq k,l \leq r\}$ then $\langle v_{i_1}, \ldots, v_{i_r}\rangle \in \Delta$. We shall say that Δ is *flag-like* whenever $\mathcal{S}_\Delta$ is a flag complex.

Theorem 6.1. *X is aspherical if and only if Δ is flag-like.*

Proof. If $\widetilde{X}$ is contractible then we claim that Δ is flag-like.

Suppose on the contrary, Δ is not flag-like. Then, $\exists\ \{v_{j_1}, \ldots, v_{j_l}\}$ such that $\forall\ 1 \leq p, q \leq l$, $\langle v_{j_p}, v_{j_q}\rangle \in \Delta$ but, $\langle v_{j_1}, \ldots, v_{j_l}\rangle \notin \Delta$.

Let $w' = s_{j_1} \ldots s_{j_l} \in W$ and let $N(w')$ be the normal subgroup of W generated by w' as in Lemma 4.3. Also let $\theta : W \to W/N(w')$ be the canonical surjection. Clearly, $\lambda = (\lambda_\tau = \theta \circ \iota_\tau)$ is a simple morphism from $G(\Delta) \to W/N(w')$. Further, Lemma 4.3 implies that, $\lambda_\tau : G_\tau \subseteq T_2 \to W/N(w')$ is injective $\forall\ \tau \in \Delta$. Hence, λ is injective at the local groups. Now, the development $D(X_+, \lambda)$ of X_+ with respect to λ, has $D(X_+, \iota) \simeq \widetilde{X}$ as the universal cover and its fundamental group $\pi_1(D(X_+, \lambda) \simeq N(w')$ has w' as a torsion element. This is a contradiction since $D(X_+, \lambda)$ is a $K(\pi, 1)$ space, because of our assumption that $\widetilde{X}$ is contractible.

For proving the converse, we apply Corollary 10.3 of the main result of [7] to the reflection system $(\Gamma = W, V = S)$ on $M = \widetilde{X}$ with fundamental chamber $Q = X_+$ (which is contractible by Lemma 2.2). Here, for every $T \subseteq S$, $Q_T = \cap_{s_j \in T} V(\rho_j)_+$. Let W_T be the subgroup generated by T in W. Then, the following statements are equivalent:

1. Q_T is acyclic for all $T \subseteq S$ with W_T finite.
2. Δ is flag-like.

Proof of (1) $\Rightarrow$ (2). Let $\rho_{j_1}, \ldots, \rho_{j_l}$ be edges such that $\{\rho_{j_p}, \rho_{j_q}\}$ spans a cone in Δ for all $1 \leq p, q \leq l$. Then, (1) implies that $Q_T = \cap_{r=1}^{l} V(\rho_{j_r})_+ = V(\tau)_+$ is non-empty since, by Lemma 4.2, $W_T = \langle s_{j_i} \ldots s_{j_l}\rangle$ is a finite subgroup of W. This implies that, $\tau = \langle \rho_{j_1}, \ldots, \rho_{j_l}\rangle$ is a non-empty cone in Δ.

Proof of (2) $\Rightarrow$ (1). Let $T = \{s_{j_1}, \ldots, s_{j_l}\} \subseteq S$ be such that W_T is finite. Then, in particular, $w' = s_{j_1} \cdots s_{j_l}$ is an element of finite order in W. By Lemma 4.2, the edge vectors $v_{j_1} \ldots, v_{j_l}$ pairwise span cones in Δ. The assumption (2) further implies that, $v_{j_1}, \ldots, v_{j_l}$ together span a cone τ in Δ. Thus, $Q_T = \cap_{r=1}^{l} V(\rho_{j_r})_+ = V(\tau)_+$ is non-empty. Moreover, $V(\tau)$ being a smooth toric variety, its non-negative part $V(\tau)_+$ is contractible by Lemma 2.2, and is hence acyclic if it is non-empty.

We therefore conclude from Corollary 10.3 of [7] that, if Δ is a flag-like then $M = \widetilde{X}$ is contractible. □

Remark 6.2. In fact, since (1) $\Leftrightarrow$ (2) above, it is clear that Corollary 10.3 of [7] also proves the first implication of the above theorem. However, in our particular case (where W is a right-angled Coxeter group), the argument given above is self-contained and is an application of the 'method of development' which is consistent with the theme of this paper.

The following are some corollaries of the above theorem.

COROLLARY 6.3

If X is aspherical then $V(\tau)$ is aspherical for every cone τ.

Proof. This is immediate because, $V(\tau)$ is the toric variety associated to the fan $\mathrm{Star}(\tau)$ which by definition (cf. p. 52 of [10]) is smooth and flag-like whenever Δ is smooth and flag-like. A proof for this is as follows: Let $\overline{\rho_{i_1}}, \ldots, \overline{\rho_{i_k}}$ be edge vectors which pairwise

span cones in Star(τ). Therefore by the definition of Star(τ), the edges of τ and $\rho_{i_1}, \ldots \rho_{i_k}$ pairwise span cones in Δ. Since Δ is flag-like, this implies that $\gamma = \langle \tau, \rho_{i_1} \ldots, \rho_{i_k} \rangle$ is a cone in Δ and hence, $\overline{\gamma} = \langle \overline{\rho_{i_1}}, \ldots, \overline{\rho_{i_k}} \rangle$ is a cone in Star(τ). Thus, Star(τ) is flag-like. □

COROLLARY 6.4

Let X be smooth and complete. We can blow up X along a number of T-stable subvarieties to get a smooth complete toric variety X' which is aspherical.

Proof. Since Δ is a smooth and complete fan, $\mathcal{S}_\Delta$ is a simplicial decomposition of the sphere S_N. It is known that the barycentric subdivision of any simplicial complex is a flag complex (cf. [2], p. 210). Therefore, if Δ' is the refinement of Δ obtained by taking the cones over the simplices in the barycentric subdivision of $\mathcal{S}_\Delta$, then Δ' is a flag-like fan. It is not difficult to see that Δ' is also smooth and complete. Hence, the smooth complete toric variety $X(\Delta')$ which is obtained by blowing up X along certain T-stable subvarieties is aspherical. □

Remark 6.5. However, in some cases we need lesser number of blow ups to arrive at an aspherical space. For e.g. (i) $\mathbb{P}^2_{\mathbb{R}}$ blown up at a T-fixed point is the Hirzebruch surface $(\mathbb{F}_1)_{\mathbb{R}}$ (the Klein-bottle) and $(\mathbb{F}_1)_{\mathbb{R}}$ is aspherical. (ii) $\mathbb{P}^2_{\mathbb{R}} \times \mathbb{S}^1$ needs to be blown up along a T-stable $\mathbb{P}^1_{\mathbb{R}}$ to get an aspherical space $(\mathbb{F}_1)_{\mathbb{R}} \times \mathbb{S}^1$.

7. Subspace arrangement related to Δ

Throughout this section we assume that Δ is a smooth and complete fan.

In this section we define a real subspace arrangement associated to Δ whose complement in $\mathbb{R}^d$ is denoted by $\mathcal{C}_\Delta$. Recall from [6] that, $X_{\mathbb{C}} \simeq X'_{\mathbb{C}}/(\mathbb{C}^*)^{d-n}$ where, $X'_{\mathbb{C}}$ is the complement of a complex subspace arrangement in $\mathbb{C}^d$. By restricting scalars to $\mathbb{R}$ in the above quotient, we show that $X \simeq \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n}$ where $\mathcal{C}_\Delta \simeq X'_{\mathbb{R}}$. We compute the fundamental group of $\mathcal{C}_\Delta$ and also give necessary and sufficient conditions for it to be a $K(\pi, 1)$ space.

DEFINITION 7.1

A collection $\mathcal{P} = \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$ of edges in Δ is called a primitive collection if $\{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$ together does not span a cone in Δ but every proper subcollection of $\mathcal{P}$ spans a cone in Δ. For the primitive collection $\mathcal{P}$ let $\mathcal{A}(P) = \{(x_1 \ldots, x_d) \in \mathbb{R}^d \mid x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}$.

DEFINITION 7.2

(i) The coordinate subspace arrangement in $\mathbb{R}^d$ corresponding to a fan Δ denoted by $\mathcal{A}_\Delta$ is defined as follows: $\mathcal{A}_\Delta = \cup_{\mathcal{P}} \mathcal{A}(\mathcal{P})$, where the union is taken over all primitive collections $\mathcal{P}$ of edges in Δ.
(ii) Let $\mathcal{C}_\Delta$ denote the complement of $\mathcal{A}_\Delta$ in $\mathbb{R}^d$, i.e. $\mathcal{C}_\Delta := \mathbb{R}^d - \mathcal{A}_\Delta$.

Let $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_r\}$ be the set of all primitive collections in Δ consisting of two edges. Let $\mathcal{P}_i = \{\rho_{i_p}, \rho_{i_q}\}$ where $1 \leq i_p, i_q \leq d \ \forall\, 1 \leq i \leq r$.

The following lemma generalizes the description of a smooth complete complex toric variety as given in [6] and [1] to the corresponding real and non-negative parts. Although this follows almost immediately from the complex case, we give a proof for it since we have not seen the result mentioned anywhere explicitly.

Lemma 7.3. *The real toric variety X corresponding to a smooth complete fan Δ is the geometric quotient of $\mathcal{C}_\Delta$ by the real algebraic torus $(\mathbb{R}^*)^{d-n}$ and we have a locally trivial principal bundle with total space $\mathcal{C}_\Delta$, base X and structure group $(\mathbb{R}^*)^{d-n}$, i.e., $\mathcal{C}_\Delta \to \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n} \simeq X$. Similarly, $X_+ \simeq (\mathcal{C}_\Delta)_+/(\mathbb{R}^+)^{d-n}$.*

Proof. Let $\sigma = \langle v_1 \dots , v_n\rangle \in \Delta(n)$ be such that $\{v_1, \dots , v_n\}$ form a $\mathbb{Z}$ basis for N. Let $\{u_1, \dots , u_n\}$ be the dual basis. Let $N'' \simeq \mathbb{Z}^{d-n}$; $N' \simeq \mathbb{Z}^d$ and let $\{e'_j : 1 \le j \le d\}$, $\{e''_k : 1 \le k \le d-n\}$ denote the natural bases of N' and N'' respectively. Further, let $g : N' \to N$ map e'_j to v_j for every $1 \le j \le d$ and let $f : N'' \hookrightarrow N'$ be the map which takes e''_{d-j+1} to $e'_j - (\Sigma_{i=1}^n \langle u_i, v_j\rangle \cdot e'_i)$ for every $n+1 \le j \le d$. From the results of [6] we know that there is an exact sequence of fans:

$$0 \longrightarrow (\Delta'', N'') \overset{f}{\hookrightarrow} (\Delta', N') \xrightarrow{g} (\Delta, N) \longrightarrow 0,$$

where $\Delta'' = \{0\}$ and Δ' is the fan consisting of the cones $\tau' = \langle e'_{j_1}, \dots , e'_{j_k}\rangle$ corresponding to every $\tau = \langle v_{j_1}, \dots , v_{j_k}\rangle \in \Delta$. Observe that, the real toric varieties corresponding to Δ'' and Δ' are $X(\Delta'') \simeq (\mathbb{R}^*)^{d-n}$ and $X(\Delta') \simeq \mathbb{R}^d - Z$ respectively, where Z is the zero locus in $\mathbb{R}^d$ of the monomials $x_{\hat{\sigma}} = \prod_{\rho \notin \sigma} x_\rho$ for every $\sigma \in \Delta$. Moreover, it is easy to see that $\mathbb{R}^d - Z$ is also isomorphic to the complement of the subspace arrangement $\mathbb{R}^d - \mathcal{A}_\Delta = \mathcal{C}_\Delta$ defined above (cf. p. 130 of [4]). Hence, from the above exact sequence of fans, we see that the smooth complete real toric variety X is the base space of a principal bundle with total space $(\mathbb{R}^d - Z) \simeq (\mathbb{R}^d - \mathcal{A}_\Delta) \simeq \mathcal{C}_\Delta$ and structure group $(\mathbb{R}^*)^{d-n}$(cf. p. 59 of [14] and p. 27 of [6]). Similarly, by restricting to the non-negative parts we see that X_+ is the base space of a principal fibre bundle with total space $\mathbb{R}^d_+ - Z_+$ and structure group $(\mathbb{R}^+)^{d-n}$. Thus we have the following:

$$X \simeq (\mathbb{R}^d - Z)/(\mathbb{R}^*)^{d-n} \simeq \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n},$$
$$X_+ \simeq \mathbb{R}^d_+ - Z_+/(\mathbb{R}^+)^{d-n} \simeq (\mathcal{C}_\Delta)_+/(\mathbb{R}^+)^{d-n}. \qquad \square$$

Remark 7.4. Note that the only property of a smooth and complete fan which we use in the above proof is that $\{v_1, \dots , v_n\}$ form a $\mathbb{Z}$ basis of N. Thus Lemma 7.3 is true even for a smooth (not necessarily complete) fan Δ, for which the primitive vectors along $\Delta(1)$ contains a $\mathbb{Z}$ basis for N.

Lemma 7.5. *$\pi_1(\mathcal{C}_\Delta)$ is isomorphic to the commutator subgroup $[W, W]$ of the Coxeter group W defined in §2, which is generated as a normal subgroup of W by $[s_{i_p}, s_{i_q}]$ for $1 \le i \le r$ where, $\mathcal{P}_i = \{\rho_{i_p}, \rho_{i_q}\}\ \forall\ 1 \le i \le r$.*

Proof. From Lemma 7.3 we know that $X \simeq \mathcal{C}_\Delta/(\mathbb{R}^*)^{d-n}$. Moreover, since $(\mathbb{R}^*)^{d-n} \simeq (\mathbb{R}^+)^{d-n} \times \mathbb{Z}_2^{d-n}$, $X_1 = \mathcal{C}_\Delta/(\mathbb{R}^+)^{d-n}$ is a regular covering space over X with deck transformation group $\mathbb{Z}_2^{d-n}$. In fact, it is the same covering space of X as in Theorem 2.5(4). Also observe that $\mathcal{C}_\Delta$ and X_1 are of the same homotopy type since $\mathcal{C}_\Delta$ is a fibre bundle over X_1 with contractible fibre $(\mathbb{R}^+)^{d-n}$. Therefore we have, $\pi_1(\mathcal{C}_\Delta) \simeq [W, W]$. $\square$

In the following theorem we shall find the necessary and sufficient conditions on Δ and hence on the arrangement $\mathcal{A}_\Delta$, under which $\mathcal{C}_\Delta$ is aspherical.

Theorem 7.6. *$\mathcal{C}_\Delta$ is aspherical if and only if $\mathcal{A}_\Delta$ is a union of precisely codimension* 2 *subspaces.*

Proof. Since $\mathcal{C}_\Delta$ is of the homotopy type of a finite regular covering space over X, it follows that X is aspherical if and only if $\mathcal{C}_\Delta$ is aspherical. From Theorem 6.1 the necessary and sufficient condition for X to be aspherical is that Δ is flag-like. Therefore, it suffices to show that Δ is flag-like if and only if $\mathcal{A}_\Delta$ is a union of precisely codimension two subspaces.

Now, by Definition 7.1, the condition for Δ to be flag-like is equivalent to the condition that in Δ there are no primitive collections consisting of more than two edges. Also by Definition 7.2, $\mathcal{A}_\Delta = \cup_{\mathcal{P}} \mathcal{A}_{\mathcal{P}}$, where the union is over primitive collections $\mathcal{P}$ in Δ and where $\mathcal{A}_{\mathcal{P}}$ is a subspace in $\mathbb{R}^d$ of codimension precisely equal to the number of edges in $\mathcal{P}$. Thus, Δ is flag-like if and only if $\mathcal{A}_\Delta = \cup_{\mathcal{P}_i} \mathcal{A}(\mathcal{P}_i)$ where the union runs over the primitive collections $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ consisting of two edges or equivalently, $\mathcal{A}_\Delta$ is a union of codimension two subspaces. Hence the theorem. □

Remark 7.7 (*$K(\pi, 1)$-arrangements*). The barycentric subdivision of any simplicial complex is a flag complex. Hence, given a smooth complete fan Δ, we can obtain several smooth complete flag-like fans whose cones are the cones over the simplices of the repeated barycentric subdivisions of $\mathcal{S}_\Delta$. We therefore get several examples of $K(\pi, 1)$ arrangements finding which seems to be of interest in the topology of arrangements (cf. [15] and [12]). However, note that even if we start with a flag-complex, an arbitrary subdivision need not result in a flag-complex. For example, let Δ be the fan consisting of the faces of $\sigma = \langle e_1, e_2, e_3 \rangle$ in $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$. If we refine Δ by adding the edge vector through $v = e_1 + e_2 + e_3$, then the resulting fan Δ' is not flag-like since, e_1, e_2, e_3, v pairwise span cones in Δ' but together do not span any cone.

Remark 7.8. Indeed, both Lemma 7.5 and Theorem 7.6 follow directly from the fact that $\mathcal{C}_\Delta$ is a smooth non-complete toric variety associated to the fan $\Delta' = \{\langle e_{j_1}, \ldots, e_{j_k} \rangle$ for every cone $\tau = \langle v_{j_1}, \ldots, v_{j_k} \rangle \in \Delta\}$ in $N' = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_d$ (cf. Lemma 7.3) and applying Theorems 2.5 and 6.1. However, since $\mathcal{C}_\Delta$ has been defined specifically as the complement of real coordinate subspace arrangement related to a smooth complete fan Δ, we therefore describe both its fundamental group and criterion for asphericity using Δ.

Remark 7.9. Since $\mathcal{C}_\Delta$ is the toric variety associated to the fan Δ', we can apply Theorems 3.1 and 4.1 respectively to give a presentation for $\pi_1(\mathcal{C}_\Delta)$ and give conditions on Δ' for it to be abelian. In particular, it follows from Theorems 7.6 and 4.1 that $\mathcal{C}_\Delta$ is $K(\pi, 1)$ with $\pi_1(\mathcal{C}_\Delta)$ abelian, if and only if it is the complement of subspaces of codimension precisely 2 which pairwise intersect at $\{0\}$. Moreover, it also follows from Lemma 4.2 that $\pi_1(\mathcal{C}_\Delta) = [W, W]$ is always torsion free.

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