Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 3, August 2003, pp. 223–241.

On the absolute N_{q_α}-summability of rth derived conjugate series

A K SAHOO

Department of Mathematics, Government Kolasib College, P. B. No. 23, Kolasib 796 081, India

MS received 26 March 2002

Abstract. The object of the present paper is to study the absolute N_{q_α}-summability of rth derived conjugate series generalizing a known result.

Keywords. Fourier series; conjugate series; derived conjugate series; Nevanlinna summability; kernel.

1. Introduction

1.1.

In the year 1921, Nevanlinna [7] suggested and discussed an interesting method of summation called N_q-method. Moursund [5] applied this method for summation of Fourier series and its conjugate series. Later, Moursund [6] developed N_{q_p}-method (where p is a positive integer) and applied it to pth derived Fourier series. Samal [9] discussed N_{q_α}-method ($0 \le \alpha < 1$) and studied absolute N_{q_α}-summability of some series associated with Fourier series. In his Ph.D. thesis [10] he extended N_{q_p}-method of summation to N_{q_α}-method for any $\alpha \ge 0$ and studied absolute N_{q_α}-summability of Fourier series. Earlier we [8] have studied absolute N_{q_α}-summability of a series conjugate to a Fourier series. In the present paper we shall study the absolute N_{q_α}-summability of rth ($r < \alpha$) derived series of a conjugate series.

1.2.

DEFINITION 1 [6,10]

Let $F(w)$ be a function of a continuous parameter w defined for all $w > 0$. The N_{q_α}-method consists in forming the N_{q_α}-transform or mean

$$N_{q_\alpha} F(w) = \int_0^1 q_\alpha(t) F(wt) \mathrm{d}t$$

and then considering the limit

$$\lim_{w\to\infty} N_{q_\alpha} F(w),$$

where the class of functions $q_\alpha(t)$ is such that

(1) $q_\alpha(t) \geq 0$ for $0 \leq t \leq 1$,

(2) $\displaystyle\int_0^1 q_\alpha(t)\mathrm{d}t = 1$,

(3) $\dfrac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t)$ exists and is absolutely continuous for $0 \leq t \leq 1, \beta = 1, 2, \ldots, k-1$, where $[\alpha] = k$,

(4) $\dfrac{\mathrm{d}^\beta}{\mathrm{d}t^\beta} q_\alpha(t) = 0$ for $t = 1, \beta = 0, 1, 2, \ldots, k-1$,

(5) $\dfrac{\mathrm{d}^k}{\mathrm{d}t^k} q_\alpha(t)$ exists for $0 < t < 1$,

(6) $(-1)^k \dfrac{\mathrm{d}^k}{\mathrm{d}t^k} q_\alpha(t) \geq 0$ and monotonic increasing for $0 < t < 1$,

(7) $\displaystyle\int_0^t \frac{Q_k(u)}{u^{1+\alpha-k}}\mathrm{d}u = O\left(\frac{Q_k(t)}{t^{\alpha-k}}\right)$,

where

$$Q_k(t) = \int_{1-t}^1 (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k} q_\alpha(u)\mathrm{d}u.$$

Also we set

$$Q(t) = \int_{1-t}^1 q_\alpha(u)\mathrm{d}u.$$

If $\lim_{w\to\infty} N_{q_\alpha} F(w)$ exists, we say that N_{q_α}-limit of $F(w)$ exists.

DEFINITION 2 [9,10]

Let $\sum_{n=0}^\infty u_n$ be an infinite series with $S(w) = \sum_{n\leq w} u_n$. If $\lim_{w\to\infty}\{\sum_{n\leq w} u_n Q(1 - (n/w))\} = 1$, we say that $\sum u_n$ is summable by N_{q_α}-method to the sum 1. In short we write that $\sum u_n = 1(N_{q_\alpha})$. Further the series $\sum u_n$ is said to be $|N_{q_\alpha}|$-summable (absolute N_{q_α}-summable) if

$$\int_A^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\leq w} n u_n q_\alpha\left(\frac{n}{w}\right)\right| < \infty$$

for some positive constant A.

For $\alpha = 0$, the method reduces to the original N_q-method [7] and if α is any positive integer p, then the method reduces to N_{q_p}-method of Moursund [6].

1.3.

Let $f(t)$ be a periodic function with period 2π and Lebesque integrable over $(-\pi, \pi)$.

Let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \tag{1.3.1}$$

The series conjugate to (1.3.1) at $t = x$ is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{1.3.2}$$

$$P(u) = \sum_{i=0}^{r-1} \frac{\theta_i}{i!}u^i \quad \text{for} \quad -\pi \le u \le \pi,$$

where θ_is for $i = 0, 1, 2, \ldots, r-1$ are arbitrary constants.

$$h(u) = \frac{\{f(x+u) - P(u)\} - (-1)^r\{f(x-u) - P(-u)\}}{2u^r},$$

$$H_0(t) = h(t),$$

$$H_\beta(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}h(u)\mathrm{d}u, \quad (\beta > 0),$$

$$h_\beta(t) = \Gamma(1+\beta)t^{-\beta}H_\beta(t), \quad (\beta \ge 0).$$

2. Purpose of the present paper

In the present paper we shall prove the following theorems:

Theorem 1. *Let $\beta = \alpha - r$. If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|\mathrm{d}H_\beta(t)| < \infty$, where $\beta > 0$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$-summable.*

Theorem 2. *Let $\rho = \alpha - r - 1$. If $\rho \ge 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|\mathrm{d}t < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$-summable.*

By taking $\beta = \rho + 1$, $\rho \ge 0$ in Theorem 1, we can obtain Theorem 2 at once as it is known [4] that

$$H_{\rho+1}(+0) = 0 \quad \text{and} \quad \int_0^\pi t^{-\rho-1}|\mathrm{d}H_{\rho+1}(t)| < \infty$$

$$\Longleftrightarrow h_{\rho+1}(t) \in BV(0,\pi) \quad \text{and} \quad \frac{h_{\rho+1}(t)}{t} \in L(0,\pi)$$

$$\Longleftrightarrow \frac{h_\rho(t)}{t} \in L(0,\pi).$$

By taking $q_\alpha(t) = (\alpha+\delta)(1-t)^{\alpha+\delta-1}$, where $\delta > 0$ and $\alpha + \delta < k+1$ ($[\alpha] = k$) in Theorems 1 and 2, we obtain the following corollaries respectively.

COROLLARY 1 [3]

If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|\mathrm{d}H_\beta(t)| < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \beta + r + \delta|$, where $\beta > 0$ and $\delta > 0$.

COROLLARY 2 [3]

If $\rho \ge 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|\mathrm{d}t < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \rho + r + 1 + \delta|$.

3. Notations and lemmas

3.1. *Notations*

For our purpose we use the following notations throughout this paper.

$$[\alpha] = k,$$
$$m = \min\,(k - r, r),$$
$$q^k(u) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k}\, q_\alpha(u),$$
$$(\cos\, nu)_j = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^j \cos\, nu,$$
$$S^{i,j}(x,u) = \sum_{n \le x} (x-n)^i (\cos\, nu)_j,$$
$$G_i(w,u) = \sum_{n \le w} q_\alpha\left(\frac{n}{w}\right) \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i} \cos\, nu, \quad \text{for} \quad i = 0, 1, 2, \ldots, m,$$
$$g_i(x,w,u) = \frac{1}{k!}(-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) \frac{\mathrm{d}}{\mathrm{d}x}\, S^{k,k+1-i}(x,u)$$
$$\text{for} \quad i = 0, 1, 2, \ldots, m.$$

3.2. *Lemmas*

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. *If $\beta > \alpha > 0$, $H_\alpha(t)$ is of $BV(0,\pi)$ and $H_\alpha(+0) = 0$, then $H_\beta(t)$ is an integral in $(0,\pi)$ and for almost all values of t,*

$$H'_\beta(t) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^t (t-u)^{\beta-\alpha-1} \mathrm{d}H_\alpha(u).$$

Lemma 2 [6]. *If $\alpha \ge 1$, the kernel $q_\alpha(t)$ is monotonic decreasing, its derivatives of odd orders less than k are negative and monotonic increasing, its derivatives of even orders less than k are positive and monotonic decreasing and there exists a constant A_k such that*

$$\left| \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\, q_\alpha(t) \right| < A_k \quad (\beta = 0, 1, 2, \ldots, k-1)$$

and

$$\int_0^1 \left| \frac{\mathrm{d}^k}{\mathrm{d}t^k}\, q_\alpha(t) \right| \mathrm{d}t < A_k.$$

Lemma 3 [10]. *$Q_k(t)$ is continuous and monotonic increasing function of t, $Q_k(t) \ge 0$, $Q(0) = 0$ and $Q(1) = 1$.*

This follows directly from the definition of $Q(t)$ and $Q_k(t)$.

Lemma 4 [10,8]. *$\int_0^1 q^k(t)/((1-t)^{\alpha-k})\mathrm{d}t$ exists.*

Lemma 5 [8]. *Let $x > 0$.*

(i) *If $1/x < u \leq \pi$, then*

$$S^{i,j}(x,u) = \begin{cases} O(x^i u^{-j-1}) & \text{for} \quad 0 \leq j \leq i, \\ O(x^j u^{-i-1}) & \text{for} \quad j > j \geq 0. \end{cases}$$

(ii) *If $1/x \geq u > 0$, then*

$$S^{i,j}(x,u) = O(x^{i+j+1}).$$

Lemma 6 [2]. *Let $\lambda = \{\lambda_n\}$ be a positive monotonic increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$. Then*

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \leq x} a_n$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n \leq x} (x - \lambda_n)^r a_n (r > 0).$$

If k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k A_\lambda^k(x).$$

Lemma 7 [8,10]. *For $\alpha \geq 1$,*

$$\sum_{n \leq w} (-1)^n n^k q_\alpha\left(\frac{n}{w}\right) = O\left\{q^k\left(1 - \frac{1}{w}\right)\right\} + O\left\{w Q_k\left(\frac{1}{w}\right)\right\}.$$

Lemma 8 [8,10]. *For $\alpha \geq 1$ and $r = 0, 1, 2, \ldots, k-1$,*

$$\sum_{n \leq w} (-1)^n n^r q_\alpha\left(\frac{n}{w}\right) = O(1).$$

Lemma 9. *For $i = 0, 1, 2, \ldots, k-1$,*

$$\sum_{n \leq w} (-1)^n n^i \in |N q_\alpha|.$$

Proof. For $i = 0, 1, 2, \ldots, k-2$,

$$\begin{aligned} \int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n \leq w} n(-1)^n n^i q_\alpha\left(\frac{n}{w}\right)\right| &= \int_1^\infty O(1)\frac{\mathrm{d}w}{w^2} \quad \text{by Lemma 8} \\ &= O(1) \end{aligned}$$

and

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w}(-1)^n n^k q_\alpha\left(\frac{n}{w}\right)\right|$$

$$= \int_1^\infty O\left\{q^k\left(1-\frac{1}{w}\right)\right\}\frac{\mathrm{d}w}{w^2}$$

$$+ \int_1^\infty O\left\{wQ_k\left(\frac{1}{w}\right)\right\}\frac{\mathrm{d}w}{w^2} \quad \text{by Lemma 7}$$

$$= O\left(\int_0^1 q^k(u)\mathrm{d}u\right) + O\left(\int_0^1 \frac{Q_k(u)}{u}\,\mathrm{d}u\right)$$

$$= O(1)$$

by Lemma 2 and the definitions of $q^k(u)$ and $Q_k(u)$. This completes the proof of Lemma 9.

Lemma 10. *For* $i = 0, 1, 2, \ldots, m$,

$$G_i(w,u) = \int_1^w g_i(x,w,u)\mathrm{d}x.$$

Proof.

$$G_i(w,u) = \sum_{n\le w} q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu$$

$$= q_\alpha(1)\sum_{n\le w}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu$$

$$- \int_1^w \frac{\mathrm{d}}{\mathrm{d}x}\, q_\alpha\left(\frac{x}{w}\right)\left\{\sum_{n\le x}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}x$$

$$= -\int_1^w \frac{\mathrm{d}}{\mathrm{d}x}\, q_\alpha\left(\frac{x}{w}\right)\frac{1}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\left\{\sum_{n\le x}(x-n)^k\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}x$$

by Lemma 6

$$= \left[\frac{1}{k!}\sum_{\rho=1}^{k-1}(-1)^\rho\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\rho q_\alpha\left(\frac{x}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k-\rho} S^{k,k+1-i}\,(x,u)\right]_{x=1}^w$$

$$+ \int_1^w \frac{(-1)^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}\, S^{k,k+1-i}\,(x,u)\mathrm{d}x$$

(integrating by parts for $(k-1)$ times)

$$= \int_1^w g_i(x,w,u)\mathrm{d}x$$

as the integrated part vanishes for $x = w$ and $x = 1$.

Lemma 11. *For $wt \leq \pi$ and $i = 0, 1, 2, \ldots, m$,*

$$\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u = O(w^{\alpha-r+1}).$$

Proof. For $i = 0, 1, 2, \ldots, m$,

$$\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \left(\sum_{n\leq w} q_\alpha\left(\frac{n}{w}\right)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i} \cos nu\right)\mathrm{d}u \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} O(w^{k+2-i})\mathrm{d}u \\
&\quad = O\left\{\left(t+\frac{1}{w}\right)^{r-i} w^{k+2-i}\int_t^{t+(1/w)}(u-t)^{k-\alpha}\mathrm{d}u\right\} \\
&\quad = O\left\{\left(\frac{wt+1}{w}\right)^{r-i} w^{k+2-i}\cdot\frac{1}{w^{k-\alpha+1}}\right\} \\
&\quad = O(w^{\alpha-r+1}) \quad \text{as} \quad wt \leq \pi.
\end{aligned}$$

Lemma 12. *For $i = 0, 1, 2, \ldots, m$ and $wt \leq \pi$,*

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u = O(w^{\alpha-r+1}).$$

Proof. By the use of Lemma 10,

$$\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^w g_i(x,w,u)\mathrm{d}x \\
&\quad = \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u\,\frac{1}{k!} \\
&\qquad \times \int_1^w (-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \frac{1}{(k-1)!}\int_1^w (-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\mathrm{d}x \\
&\qquad \times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\, S^{k-1,k+1-i}(x,u)\mathrm{d}u \\
&\quad = \frac{1}{(k-1)!}\int_1^w (-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\mathrm{d}x\; w^{\alpha-k} \\
&\qquad \times \int_{t+(1/w)}^{\xi} u^{r-i}\; S^{k-1,k+1-i}\;(x,u)\mathrm{d}u, \qquad (3.2.1)
\end{aligned}$$

for some $t + (1/w) < \xi < \pi$, by an application of the mean value theorem. For $i \geq 2$, using Lemma 5(i) in (3.2.1), we get

$$
\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&= \frac{1}{(k-1)!}\int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha-k}\mathrm{d}x \\
&\quad \times \int_{t+(1/w)}^{\xi} u^{r-i} O(x^{k-1}u^{-k-2+i})\mathrm{d}u \\
&= \frac{1}{(k-1)!}\int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha-k} O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\}\mathrm{d}x \\
&= O\left(w^{\alpha-r+1}\int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) x^{k-1}\mathrm{d}x\right) \\
&= O\left(w^{\alpha-r+1}\int_0^1 q^k(\theta)\mathrm{d}\theta\right) \\
&= O\left(w^{\alpha-r+1}\right) \quad \text{by Lemma 2.}
\end{aligned}
$$

For $i = 1$,

$$
\begin{aligned}
&\int_{t+(1/w)}^{\xi} u^{r-i}\, S^{k-1,k+1-i}\,(x,u)\mathrm{d}u \\
&= \int_{t+(1/w)}^{\xi} u^{r-1}\, S^{k-1,k}\,(x,u)\mathrm{d}u \\
&= \left[u^{r-1}\, S^{k-1,k-1}\,(x,u)\right]_{t+(1/w)}^{\xi} \\
&\quad -(r-1)\int_{t+(1/w)}^{\xi} u^{r-2}\, S^{k-1,k-1}\,(x,u)\mathrm{d}u \\
&= O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\} + \int_{t+(1/w)}^{\xi} u^{r-2} O(x^{k-1}u^{-k})\mathrm{d}u \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 5(i)} \\
&= O(w^{2k-r}).
\end{aligned}
$$

Similarly, for $i = 0$, integrating by parts twice and using Lemma 5(i), it follows that

$$
\int_{t+(1/w)}^{\xi} u^{r-i} S^{k-1,k+1-i}(x,u)\mathrm{d}u = O(w^{2k-r}).
$$

Hence, for $i \leq 1$, using the above estimation in (3.2.1)

$$
\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\mathrm{d}u \\
&\quad = O\left(\int_1^w (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha+k-r}\mathrm{d}x\right) \\
&\quad = O\left(w^{\alpha-r+1}\int_0^1 q^k(\theta)\mathrm{d}\theta\right) \\
&\quad = O(w^{\alpha-r+1}) \quad \text{by Lemma 2.}
\end{aligned}
$$

This completes the proof of Lemma 12.

Lemma 13.

$$
\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = O\left(\frac{w^{\alpha-k}}{t^{k+1-r}}\, q^k\left(1-\frac{\pi}{wt}\right)\right).
\end{aligned}
$$

Proof. For some $1<\xi<w-(\pi/t)$, by an application of the mean value theorem,

$$
\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \\
&\qquad \times \int_1^{w-(\pi/t)} \frac{(-1)^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \int_t^{t+(1/w)} \frac{1}{k!}u^{r-i}(u-t)^{k-\alpha}\left[(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-(\pi/t)}\mathrm{d}u \\
&\qquad \times \int_\xi^{w-(\pi/t)} \frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\mathrm{d}x \\
&\quad = \frac{1}{k!}\int_t^{t+(1/w)} \frac{u^{r-i}(u-t)^{k-\alpha}}{w^k}\, q^k\left(1-\frac{\pi}{wt}\right)\left[S^{k,k+1-i}(x,u)\right]_{x=\xi}^{w-(\pi/t)}\mathrm{d}u.
\end{aligned}
\tag{3.2.2}
$$

For $i=0$, using Lemma 5(i) in (3.2.2), we get

$$
\begin{aligned}
&\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x \\
&\quad = \frac{1}{k!}\int_t^{t+(1/w)} \frac{u^{r}(u-t)^{k-\alpha}}{w^k}\, q^k\left(1-\frac{\pi}{wt}\right) O\left\{\frac{\left(w-\frac{\pi}{t}\right)^{k+1}}{u^{k+1}}\right\}\mathrm{d}u
\end{aligned}
$$

$$= O\left(\frac{wq^k\left(1-\frac{\pi}{wt}\right)}{t^{k+1-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}\,\mathrm{d}u\right)$$

$$= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{kt}\right)}{t^{k+1-r}}\right).$$

For $i \geq 1$, using Lemma 5(i) in (3.2.2), we obtain

$$\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\mathrm{d}x$$

$$= \frac{1}{k!}\int_t^{t+(1/w)} \frac{u^{r-i}(u-t)^{k-\alpha}}{w^k} q^k\left(1-\frac{\pi}{wt}\right) O\left(\frac{w^k}{u^{k+2-i}}\right)\mathrm{d}u$$

$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right)}{t^{k+2-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}\mathrm{d}u\right)$$

$$= O\left(\frac{w^{\alpha-k+1}}{t^{k+2-r}} q^k\left(1-\frac{\pi}{wt}\right)\right).$$

$$= O\left(\frac{w^{\alpha-k}}{t^{k+1-r}} q^k\left(1-\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wt > \pi.$$

This completes the proof of Lemma 13.

Lemma 14. *For* $i = 0, 1, 2, \ldots, m$,

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\, S^{k,k+1-i}\left(w-\frac{\pi}{t}, u\right)\mathrm{d}u = O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right).$$

Proof. By an application of the mean value theorem for some $t + (1/w) < \xi < \pi$,

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} S^{k,k+1-i}\left(w-\frac{\pi}{t}, u\right)\mathrm{d}u$$

$$= w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i} S^{k,k+1-i}\left(w-\frac{\pi}{t}, u\right)\mathrm{d}u$$

$$= w^{\alpha-k}\left[u^{r-i}\ S^{k,k-i}\left(w-\frac{\pi}{t}, u\right)\right]_{u=t+(1/w)}^{\xi}$$

$$- (r-i)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i-1} S^{k,k-i}\left(w-\frac{\pi}{t}, u\right)\mathrm{d}u$$

$$= w^{\alpha-k} O\left\{\frac{w^k}{\left(t+\frac{1}{w}\right)^{k+1-r}}\right\} + w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-i-1} O\left(\frac{w^k}{u^{k+1-i}}\right)\mathrm{d}u$$

by Lemma 5(i)

$$= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right) + O\left(w^{\alpha}\int_{t+(1/w)}^{\xi}\frac{1}{u^{k+2-r}}\mathrm{d}u\right)$$

$$= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right).$$

Lemma 15. *For $i = 0, 1, 2, \ldots, m$,*

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{1}^{w-(\pi/t)} g_i(x, w, u)\mathrm{d}x$$
$$= O\left\{\frac{w^{\alpha-k} q^k\left(1-\frac{\pi}{wt}\right)}{t^{k+1-r}}\right\}.$$

Proof. For some $1 < \eta < w - (\pi/t)$, by an application of the mean value theorem

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \int_{t}^{w-(\pi/t)} g_i(x, w, u)\mathrm{d}x$$
$$= \frac{1}{k!}\int_{t-(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\mathrm{d}u \left[(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-(\pi/t)}$$
$$\times \int_{\eta}^{w-(\pi/t)} \frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x, u)\mathrm{d}x$$
$$= \frac{q^k\left(1-\frac{\pi}{wt}\right)}{k!\, w^k}\left\{\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} S^{k,k+1-i}\left(w-\frac{\pi}{t}, u\right)\mathrm{d}u\right.$$
$$\left. - \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\, S^{k,k+1-i}(\eta, u)\mathrm{d}u\right\}$$
$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right) w^\alpha}{w^k\, t^{k+1-r}}\right)$$
$$= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right) w^{\alpha-k}}{t^{k+1-r}}\right),$$

since by Lemma 14, the first integral is $O(w^\alpha/(t^{k+1-r}))$ and the second integral is dominated by the first integral.

Lemma 16. *For $i = 0, 1, 2, \ldots, m$ and $wu > \pi$,*

$$\int_{w-(\pi/t)}^{w} g_i(x, w, u)\mathrm{d}x = O\left(w^2\, u^{-k+i} Q_k\left(\frac{\pi}{wt}\right)\right).$$

Proof. For $0 \leq 1$, by use of Lemma 5(i),

$$\int_{w-(\pi/t)}^{w} g_i(x, w, u)\mathrm{d}x$$
$$= \int_{w-(\pi/t)}^{w} \frac{(-1)^k}{(k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) S^{k-1,k+1-i}(x, u)\mathrm{d}x$$
$$= \frac{1}{(k-1)!}\int_{w-(\pi/t)}^{w} (-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{x^{k+1-i}}{u^k}\right)\mathrm{d}x$$

$$= O\left(\frac{w^{2-i}}{u^k}\int_{1-(\pi/wt)}^{1} q^k(\theta)\mathrm{d}\theta\right)$$
$$= O\left(w^2 u^{-k+i}\ Q_k\left(\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wu > \pi.$$

For $i \geq 2$, by use of Lemma 5(i)

$$\int_{w-(\pi/t)}^{w} g_i(x, w, u)\mathrm{d}x$$
$$= \int_{w-(\pi/t)}^{w} \frac{(-1)^k}{(k-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) S^{k-1,k+1-i}(x, u)\mathrm{d}x$$
$$= \frac{1}{(k-1)!}\int_{w-(\pi/t)}^{w} (-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{x^{k-1}}{u^{k+2-i}}\right)\mathrm{d}x$$
$$= O\left(u^{-k-2+i}\int_{1-(\pi/wt)}^{1} q^k(\theta)\mathrm{d}\theta\right)$$
$$= O\left(w^2 u^{-k+i}\ Q_k\left(\frac{\pi}{wt}\right)\right) \quad \text{as} \quad wu > \pi.$$

Hence

$$\int_{w-(\pi/t)}^{w} g_i(x, w, u)\mathrm{d}x = O\left(w^2\, u^{-k+i}\ Q_k\left(\frac{\pi}{wt}\right)\right).$$

Lemma 17. *For $i = 0, 1, 2, \ldots, m$ and $wt > \pi$,*

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\ S^{k,k+1-i}(x, u)\mathrm{d}u = O\left(\frac{w^\alpha}{t^{k-r}}\right).$$

Proof. Let $i = 0$. By mean value theorem for some $t + (1/w) < \xi < \pi$,

$$\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x} S^{k,k+1-i}(x, u)\,\mathrm{d}u$$
$$= k\int_{t+(1/w)}^{\pi} u^r(u-t)^{k-\alpha} S^{k-1,k+1}(x, u)\,\mathrm{d}u$$
$$= kw^{\alpha-k}\int_{t+(1/w)}^{\xi} u^r S^{k-1,k+1}(x, u)\,\mathrm{d}u$$
$$= kw^{\alpha-k}\left[u^r S^{k-1,k}(x, u)\right]_{u=t+(1/w)}^{\xi}$$
$$- krw^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-1} S^{k-1,k}(x, u)\,\mathrm{d}u$$

$$
\begin{aligned}
&= kw^{\alpha-k} O\left\{\frac{x^k}{(t+(1/w))^{k-r}}\right\} \\
&\quad - krw^{\alpha-k}\left[u^{r-1}S^{k-1,k-1}(x,u)\right]_{u=t+(1/w)}^{\xi} \\
&\quad + kr(r-1)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-2}S^{k-1,k-1}(x,u)\,\mathrm{d}u \quad \text{by Lemma 5(i)} \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) + krw^{\alpha-k} O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\} \\
&\quad + kr(r-1)w^{\alpha-k}\int_{t+(1/w)}^{\xi} u^{r-2} O\left(\frac{x^{k-1}}{u^k}\right)\mathrm{d}u \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) + O\left(\frac{w^{\alpha-1}}{t^{k-r+1}}\right) \\
&= O\left(\frac{w^\alpha}{t^{k-r}}\right) \quad \text{as} \quad wt > \pi.
\end{aligned}
$$

For $i > 1$, using the technique used in the proof of Lemma 14, it can be proved that

$$
\begin{aligned}
&\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\,S^{k,k+1-i}(x,u)\,\mathrm{d}u \\
&\quad = O\left(\frac{w^{\alpha-1}}{t^{k-r+1}}\right) \\
&\quad = O\left(\frac{w^\alpha}{t^{k-r}}\right) \quad \text{as} \quad wt > \pi.
\end{aligned}
$$

This completes the proof of Lemma 17.

Lemma 18. *For $i = 0, 1, 2, \ldots, m$,*

$$
\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| = O\left(\frac{1}{t^{\alpha-r}}\right).
$$

Proof.

$$
\begin{aligned}
&\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\quad \le \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\qquad + \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| \\
&\quad = \int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}O(w^{\alpha-r+1}), \quad \text{by Lemmas 11 and 12} \\
&\quad = O\left(\frac{1}{t^{\alpha-r}}\right).
\end{aligned}
$$

Lemma 19. *For $i = 0, 1, 2, \ldots, m$ and $wt > \pi$,*

$$\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\,\mathrm{d}u$$
$$= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k \left(\frac{\pi}{wt}\right)}{t^{k-r}}\right).$$

Proof. Using Lemma 10,

$$\begin{aligned}
&\int_t^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u)\,\mathrm{d}u \\
&= \int_t^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x \\
&= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x \\
&\quad + \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w} g_i(x,w,u)\,\mathrm{d}x \\
&= J_1 + J_2, \quad \text{say.}
\end{aligned}$$

Using Lemmas 13 and 16,

$$\begin{aligned}
J_1 &= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\,\mathrm{d}x \\
&\quad + \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{w-(\pi/t)}^{w} g_i(x,w,u)\,\mathrm{d}x \\
&= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) \\
&\quad + O\left(\int_t^{t+(1/w)} u^{r-k}(u-t)^{k-\alpha} w^2 Q_k\left(\frac{\pi}{wt}\right)\mathrm{d}u\right) \\
&= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k \left(\frac{\pi}{wt}\right)}{t^{k-r}}\right) \quad \text{as} \quad k \geq r
\end{aligned}$$

and

$$\begin{aligned}
J_2 &= \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_1^{w-(\pi/t)} g_i(x,w,u)\,\mathrm{d}x \\
&\quad + \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\,\mathrm{d}u \int_{w-(\pi/t)}^{w} g_i(x,w,u)\,\mathrm{d}x \\
&= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \int_{w-(\pi/t)}^{w} \mathrm{d}x \\
&\quad \times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} g_i(x,w,u)\,\mathrm{d}x \quad \text{by Lemma 15,}
\end{aligned}$$

$$
\begin{aligned}
&= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \frac{1}{k!}\int_{w-(\pi/t)}^{w} (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right)\,\mathrm{d}x\\
&\quad\times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}S^{k,k+1-i}(x,u)\,\mathrm{d}x\\
&= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right)\\
&\quad+\frac{1}{k!}\int_{w-(\pi/t)}^{w}(-1)^k\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{w^\alpha}{t^{k-r}}\right)\mathrm{d}x \quad \text{by Lemma 17}\\
&= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}}{t^{k-r}}\int_{1-(\pi/wt)}^{1} q^k(\theta)\,\mathrm{d}\theta\right)\\
&= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right).
\end{aligned}
$$

This completes the proof of Lemma 19.

Lemma 20. *For $i = 0, 1, 2, \ldots, m$,*

$$
\int_{\pi/t}^{\infty}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right| = O\left(\frac{1}{t^{\alpha-r}}\right).
$$

Proof. By the use of Lemma 19, we get

$$
\begin{aligned}
&\int_{\pi/t}^{\infty}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\\
&\quad= O\left(\int_{\pi/t}^{\infty}\frac{w^{\alpha-k-2}q^k\left(1-\frac{\pi}{wt}\right)}{t^{k-r+1}}\,\mathrm{d}w\right) + O\left(\int_{\pi/t}^{\infty}\frac{w^{\alpha-k-1}Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\,\mathrm{d}w\right)\\
&\quad= O\left(\frac{1}{t^{\alpha-r}}\int_0^1\frac{q^k(\theta)}{(1-\theta)^{\alpha-k}}\,\mathrm{d}\theta\right) + O\left(\frac{1}{t^{\alpha-r}}\int_0^1\frac{Q_k(u)}{u^{\alpha-k+1}}\,\mathrm{d}u\right)\\
&\quad= O\left(\frac{1}{t^{\alpha-r}}\right) \quad \text{by Lemma 4.}
\end{aligned}
$$

4. Proof of the theorem

Proof of Theorem 1. We have for $r \geq 1$,

$$
\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^r B_n(x) &= \frac{2}{\pi}\int_0^\pi \frac{(-1)^r}{2}\left\{f(x+u)-(-1)^r f(x-w)\right\}\\
&\qquad\times\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu\,\mathrm{d}u
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^r \frac{2}{\pi} \int_0^{\pi} h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&\quad + (-1)^r \frac{2}{\pi} \int_0^{\pi} \frac{1}{2}\left\{P(u) - (-1)^r P(-u)\right\} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&= \alpha_n + \beta_n, \quad \text{say.}
\end{aligned}
$$

For the proof of our theorem it is enough to show that

$$\sum \alpha_n \in \left|N_{q_\alpha}\right|$$

and

$$\sum \beta_n \in \left|N_{q_\alpha}\right|.$$

Now

$$
\begin{aligned}
n\alpha_n &= (-1)^r \frac{2}{\pi} \int_0^{\pi} n h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^r \sin nu \,\mathrm{d}u \\
&= (-1)^{r+1} \frac{2}{\pi} \int_0^{\pi} h(u) u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu \,\mathrm{d}u \\
&= (-1)^{r+1} \frac{2}{\pi} \left[\sum_{j=1}^{k-r} (-1)^{j-1} H_j(u) \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{j-1} \right. \\
&\quad \left. \times \left\{u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu\right\}\right]_{u=0}^{\pi} \\
&\quad + (-1)^{k+1} \frac{2}{\pi} \int_0^{\pi} H_{k-r}(u) \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r} \left\{u^r \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1} \cos nu\right\} \mathrm{d}u \\
&= J_1(n) + J_2(n), \quad \text{say.}
\end{aligned} \tag{4.1}
$$

Since for $j = 1, 2, \ldots, k-r$, $H_j(+0) = O$ it is clear that $J_1(n)$ is the sum of the terms containing $(-1)^n n^p$, where p is even and $r + 1 \leq p \leq k$.

By the use of Lemma 9, for $p = 1, 2, \ldots, k$,

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left|\sum_{n \leq w} n^p (-1)^n q_\alpha \left(\frac{n}{w}\right)\right| < \infty.$$

Hence

$$\int_1^{\infty} \frac{\mathrm{d}w}{w^2} \left|\sum_{n \leq w} J_1(n) q_\alpha \left(\frac{n}{w}\right)\right| < \infty. \tag{4.2}$$

Now

$$\begin{aligned}
J_2(n) &= (-1)^{k+1}\frac{2}{\pi}\int_0^\pi H_{k-r}(u)\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r}\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u\\
&= \frac{2(-1)^{k+1}}{\pi\,\Gamma(k-\alpha+1)}\int_0^\pi \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r}\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u\\
&\quad\times\int_0^u (u-t)^{k-\alpha}\mathrm{d}H_\beta(t)\quad \text{by Lemma 1 as } \beta=\alpha-r \text{ and } [\alpha]=k\\
&= \frac{2(-1)^{k+1}}{\pi\,\Gamma(k-\alpha+1)}\int_0^\pi \mathrm{d}H_\beta(t)\int_t^\pi (u-t)^{k-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k-r}\\
&\quad\times\left\{u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{r+1}\cos nu\right\}\mathrm{d}u\\
&= \frac{2(-1)^{k+1}}{\pi\,\Gamma(k-\alpha+1)}\int_0^\pi \mathrm{d}H_\beta(t)\int_t^\pi (u-t)^{k-\alpha}\\
&\quad\times\left\{\sum_{i=0}^{m}\binom{k-r}{i}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{i}u^r\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\right\}\mathrm{d}u\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{where } m=\min(k-r,r)\\
&= \frac{2(-1)^{k+1}}{\pi\,\Gamma(k-\alpha+1)}\sum_{i=0}^{m}\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi \mathrm{d}H_\beta(t)\\
&\quad\times\int_t^\pi (u-t)^{k-\alpha}u^{r-i}\left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{k+1-i}\cos nu\,\mathrm{d}u.
\end{aligned}$$

By the use of Lemmas 20 and 18,

$$\begin{aligned}
&\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w}J_2(n)q_\alpha\left(\frac{n}{w}\right)\right|\\
&\le \frac{2}{\pi\,\Gamma(k-\alpha+1)}\sum_{i=0}^{m}\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi\left|\mathrm{d}H_\beta(t)\right|\int_1^\infty\frac{\mathrm{d}w}{w^2}\\
&\quad\times\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\\
&= \frac{2}{\pi\,\Gamma(k-\alpha+1)}\sum_{i=0}^{m}\binom{k-r}{i}\frac{r!}{(r-i)!}\int_0^\pi\left|\mathrm{d}H_\beta(t)\right|\\
&\quad\times\left\{\int_1^{\pi/t}\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-1}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\right.\\
&\quad\left.+\int_{\pi/t}^\infty\frac{\mathrm{d}w}{w^2}\left|\int_t^\pi u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,\mathrm{d}u\right|\right\}
\end{aligned}$$

$$= \frac{2}{\pi\Gamma(k-\alpha+1)} \sum_{i=0}^{m} \binom{k-i}{i} \frac{r!}{(r-i)!} \int_0^\pi \left|\mathrm{d}H_\beta(t)\right| O\left(\frac{1}{t^{\alpha-r}}\right)$$

by Lemmas 18 and 20

$$= O\left(\sum_{i=0}^{m} \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^\pi \frac{\left|\mathrm{d}H_\beta(t)\right|}{t^\beta}\right) \quad \text{as } \alpha - r = \beta$$

$$= O(1). \tag{4.3}$$

From (4.1), (4.2) and (4.3) it is clear that

$$\sum \alpha_n \in |N_{q_\alpha}|.$$

Let r be an odd number, i.e. $r = 2p+1$, where $p = 0, 1, 2, \ldots$. Then

$$\beta_n = -\frac{2}{\pi} \int_0^\pi \frac{1}{2}\{P(u) + P(-u)\} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2p+1} \sin nu\, \mathrm{d}u$$

$$= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \int_0^\pi \left(\sum_{j=0}^{p} \frac{\theta_{2j} u^{2j}}{(2j)!}\right) \cos nu\, \mathrm{d}u$$

$$= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \sum_{j=0}^{p} \frac{\theta_{2j}}{(2j)!} \int_0^u u^{2j} \cos nu\, \mathrm{d}u$$

$$= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \sum_{j=1}^{p} \frac{\theta_{2j}}{(2j)!} (-1)^n$$

$$\times \left(\sum_{\mu=1}^{j} (-1)^{\mu+1} n^{-2\mu} \pi^{2j-2\mu+1} \frac{(2j)!}{(2j-2\mu)!}\right)$$

$$= 2(-1)^n \sum_{\mu=1}^{p} (-1)^{p+\mu} n^{2p-2\mu+1} \sum_{j=\mu}^{p} \frac{\theta_{2j}}{(2j-2\mu)!} \pi^{2j-2\mu}.$$

Let r be an even number, i.e. $r = 2p$, where $p = 1, 2, \ldots$. Then

$$\beta_n = \frac{2}{\pi} \int_0^\pi \frac{1}{2}\{P(u) - P(-u)\} \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^{2p} \sin nu\, \mathrm{d}u$$

$$= (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^{p} \frac{\theta_{2j-1}}{(2j-1)!} \int_0^\pi u^{2j-1} \sin nu\, \mathrm{d}u$$

$$= (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^{p} \frac{\theta_{2j-1}}{(2j-1)!} (-1)^n$$

$$\times \left(\sum_{\mu=1}^{j} (-1)^{\mu-1} n^{-2\mu+1} \pi^{2j-2\mu+1} \frac{(2j-1)!}{(2j-2\mu)!}\right)$$

$$= 2(-1)^n \sum_{\mu=1}^{p} (-1)^{p+\mu-1} n^{2p-2\mu+1} \sum_{j=\mu}^{p} \frac{\theta_{2j-1}}{(2j-2\mu)!} \pi^{2j-2\mu}.$$

So by the use of Lemma 9,

$$\int_1^\infty \frac{\mathrm{d}w}{w^2}\left|\sum_{n\le w} n\beta_n q_\alpha\left(\frac{n}{w}\right)\right| < \infty,$$

i.e. $\sum \beta_n \in \left|N_{q_\alpha}\right|$. This terminates the proof of Theorem 1.

References

[1] Bosanquet L S, Some extension of Young's criterion for convergence of Fourier series, *Quart. J. Math. Oxford* **6** (1935) 113–123

[2] Hardy G H and Riesz M, The general theory of Dirichlet Series (Cambridge) (1916)

[3] Hyslop J M, On the absolute summability of the successively derived series of a Fourier series and its allied series, *Proc. London Math. Soc. Series 2* **46** (1939) 55–80

[4] Mohanty R and Ray B K, On the behaviour of a series associated with the conjugate series of a Fourier series, *Canad. J. Math.* **21** (1969) 535–551

[5] Moursund A F, On a method of summation of Fourier series, *Ann. Math.* **33(2)** (1932) 773–784

[6] Moursund A F, On a method of summation of Fourier series – II, *Ann. Math.* **34** (1933) 778–798

[7] Nevanlinna F, Über die summation der Fourier Schen Reihen und integrale overskit av Finska Vetensk, *Societeten Forhdlinger* **64A(3)** (1921–1922) 147–156

[8] Sahoo A K, Absolute N_{q_α}-summability of the series conjugate to a Fourier series, *Proc. Indian Acad. Sci. (Math. Sci.)* **108(3)** (1998) 251–271

[9] Samal M, On the absolute N_q-summability of some series associated with Fourier series, *J. Indian Math. Soc.* **50** (1986) 191–209

[10] Samal M, Summability of Fourier series, Ph.D. thesis (Utkal University) (1989)