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# The Jacobian of a nonorientable Klein surface

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**Abstract.** Using divisors, an analog of the Jacobian for a compact connected nonorientable Klein surface $Y$ is constructed. The Jacobian is identified with the dual of the space of all harmonic real one-forms on $Y$ quotiented by the torsion-free part of the first integral homology of $Y$. Denote by $X$ the double cover of $Y$ given by orientation. The Jacobian of $Y$ is identified with the space of all degree zero holomorphic line bundles $L$ over $X$ with the property that $L$ is isomorphic to $\sigma^*\overline{L}$, where $\sigma$ is the involution of $X$.

**Keywords.** Nonorientable surface; divisor; Jacobian

## 1. Introduction

Let $Y$ be a compact connected nonorientable Riemann surface, that is, each transition function is either holomorphic or anti-holomorphic. We consider surfaces without boundary. Let $X$ denote the double cover of $Y$ given by the local orientations. So $X$ is a compact connected Riemann surface.

In §2, we define a morphism from $Y$ to $\overline{\mathbb{H}}$, the closure of the upper half-plane in the Riemann sphere $\widehat{\mathbb{C}}$. Let $\mathrm{Div}_0(Y)$ denote the group defined by all formal finite sums of the form $\sum n_i y_i$, where $n_i \in \mathbb{Z}$ with $\sum n_i = 0$ and $y_i \in Y$. We call such a divisor $D$ to be principal if there is a morphism (see §2 for the definition of morphism) $u$ from $Y$ to $\overline{\mathbb{H}}$ with the property that

$$D = u^{-1}(0) - u^{-1}(\infty).$$

Let $J_0(Y)$ denote the quotient of $\mathrm{Div}_0(Y)$ by its subgroup consisting of all principal divisors. This $J_0(Y)$ is the analog of the Jacobian for a nonorientable Riemann surface.

Harmonic one-forms are defined on $Y$. Let $H^1_{\mathbb{R}}(Y)$ denote the space of all harmonic real one-forms on $Y$. The torsion-free part of $H_1(Y, \mathbb{Z})$ is a subgroup of $\mathcal{H}^1_{\mathbb{R}}(Y)^*$. The quotient is identified with $J_0(Y)$. This is proved by showing that $\mathcal{H}^1_{\mathbb{R}}(Y)$ is identified with the space of all holomorphic one-forms $\omega$ on $X$ satisfying the identity $\overline{\omega} = \sigma^*\omega$, where $\sigma$ is the nontrivial automorphism of the double cover $X$ of $Y$ (Theorem 2.7).

For a holomorphic line bundle $L$ over $X$, the pullback $\sigma^*\overline{L}$ is again a holomorphic line bundle over $X$. We show that $J_0(Y)$ is identified with the group of all holomorphic line bundles $L$ over $X$ for which the holomorphic line bundle $\sigma^*\overline{L}$ is isomorphic to $L$ (Theorem 4.2).

A compact Riemann surface is a smooth projective curve over $\mathbb{C}$. Conversly, every smooth projective curve over $\mathbb{C}$ corresponds to a compact Riemann surface. If we take a smooth projective curve $X_{\mathbb{R}}$ defined over $\mathbb{R}$, then using the inclusion of $\mathbb{R}$ in $\mathbb{C}$ we get a smooth projective curve $X_{\mathbb{C}}$ over $\mathbb{C}$. Now, since the involution of $\mathbb{C}$ defined by conjugation fixes $\mathbb{R}$, the complex curve $X_{\mathbb{C}}$ is equipped with an anti-holomorphic involution that reverses the orientation. Conversely, every complex projective curve equipped with an anti-holomorphic involution is actually defined over $\mathbb{R}$. If the involution does not have any fixed points, that is, the curve does not have any real points, then it is called an imaginary curve.

Therefore, a nonorientable Riemann surface $Y$ (without boundary) corresponds to an imaginary algebraic curve defined over $\mathbb{R}$. The Jacobian of the complexification $Y\mathbb{C}$ is also the complexification of a variety defined over $\mathbb{R}$. The Jacobian $J_0(Y)$ coincides with this variety defined over $\mathbb{R}$.

## 2. Divisors on a nonorientable surface

Let $Y$ be a compact connected nonorientable surface. In other words, $Y$ is a compact connected nonorientable smooth manifold of dimension two, and $Y$ has a covering by smooth coordinate charts such that each transition function is either holomorphic or anti-holomorphic. Any coordinate chart in the maximal atlas satisfying the above condition on transition functions will be called *compatible*. Such a nonorientable surface is called a *Klein surface*.

DEFINITION 2.1

A *divisor* $D$ on $Y$ is a formal sum of type

$$D = \sum_{y\in Y} n_y y ,$$

where $n_y \in \mathbb{Z}$ and $n_y = 0$ except for a finitely many points of $Y$.

DEFINITION 2.2

The *degree* of a divisor $D = \sum_{y\in Y} n_y y$ is defined to be the integer $\deg(D) := \sum_{y\in Y} n_y$.

We will denote by $\mathrm{Div}(Y)$ the set of all divisors on $Y$. Let $\mathrm{Div}_d(Y) \subset \mathrm{Div}(Y)$ be the divisors of degree $d$.

Let $\pi : X \to Y$ be a double cover of $Y$ given by local orientations on $Y$. So for a contractible open subset $U \subset Y$, the inverse image $\pi^{-1}(U)$ is two copies of $U$ with the two possible orientations on $U$ (see [1] for more details on Klein surfaces and their double covers).

Therefore, $X$ is a Riemann surface, and the change of orientation defines an anti-holomorphic involution $\sigma : X \to X$ that commutes with $\pi$.

The involution $\sigma$ induces in a natural way a mapping on the set of divisors on the Riemann surface $X$ as follows

$$\begin{aligned} \sigma^* : \mathrm{Div}(X) &\longrightarrow \mathrm{Div}(X) \\ \sum m_j x_j &\longmapsto \sum m_j \sigma(x_j). \end{aligned}$$

Observe that $\sigma^*$ preserves the degree.

Similarly, the quotient map $\pi : X \to Y$ induces mappings between the divisors on $X$ and $Y$. To define those mappings we first set up some notation. For any point $y \in Y$ we will denote by $\pi^{-1}(y)$ the divisor given by the inverse image of $y$. In other words, $\pi^{-1}(y) = x + \sigma(x)$, where $x \in X$ is a point satisfying $\pi(x) = y$. Then we can define two mappings as follows:

$$\begin{aligned} &\pi^* : \mathrm{Div}(Y) \to \mathrm{Div}(X), && \pi_* : \mathrm{Div}(X) \to \mathrm{Div}(Y), \\ &\sum_{j=1}^{s} n_j\, y_j \mapsto \sum_{j=1}^{s} n_j\, \pi^{-1}(y_j), && \sum_{j=1}^{s} m_j\, x_j \mapsto \sum_{j=1}^{s} m_j\, \pi(x_j). \end{aligned}$$

Observe that $(\pi_* \circ \pi^*)(D) = 2D$ and $(\pi^* \circ \pi_*)(E) = E + \sigma^*(E)$ for $D \in \mathrm{Div}(Y)$ and $E \in \mathrm{Div}(X)$.

Let $\mathrm{Div}(X)^{\sigma^*}$ denote the set of fixed points of $\sigma^*$ on $\mathrm{Div}(X)$.

The following lemma follows immediately from the above definitions.

*Lemma* 2.3. *The group* $\mathrm{Div}(Y)$ *is identified with* $\mathrm{Div}(X)^{\sigma^*}$. *The isomorphism takes the subgroup* $\mathrm{Div}_0(Y)$ *to* $\mathrm{Div}(X)_0^{\sigma^*} = \mathrm{Div}_0(X) \cap \mathrm{Div}(X)^{\sigma^*}$.

Let $j : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the mapping induced by conjugation on the Riemann sphere $\widehat{\mathbb{C}}$, so that $j(z) = \bar{z}$ and $j(\infty) = \infty$. The quotient space is a surface with boundary, $\overline{\mathbb{H}} = \widehat{\mathbb{C}}/\langle j \rangle$. We can also identify $\overline{\mathbb{H}}$ with the closure of $\mathbb{H}$ (the upper half-plane) in the Riemann sphere. Let

$$p : \widehat{\mathbb{C}} \longrightarrow \overline{\mathbb{H}}$$

denote the quotient map. After identifying $\overline{\mathbb{H}}$ with the closure of $\mathbb{H}$ the map $p$ coincides with the one defined by $p(x + \sqrt{-1}y) = x + \sqrt{-1}|y|$ and $p(\infty) = \infty$.

A *morphism* from $Y$ to $\overline{\mathbb{H}}$ is a continuous mapping

$$u : Y \longrightarrow \overline{\mathbb{H}}$$

such that if $(U, w)$ is a local coordinate function defined on $Y$, compatible with the Riemann surface structure, with $w(U) \subset \mathbb{H}$, then there exists a holomorphic function $F : w(U) \to \mathbb{C}$ that makes the following diagram commutative:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "w"'] & \overline{\mathbb{H}} \\ \mathbb{H} \arrow[r, "F"'] & \mathbb{C} \arrow[u, "p"'] \end{tikzcd},$$

where $p$ is defined above.

Let $u$ be a morphism, as above, from $Y$ to $\overline{\mathbb{H}}$ which is not identically equal to 0 or $\infty$. If $z_0$ is a point of $\overline{\mathbb{H}}$, then by $u^{-1}(z_0)$ we understand the divisor given by the inverse image of $z_0$ under $u$ (so the integers $n_j$ in Definition 2.1 are given by the multiplicities of $u$ at the corresponding points). Since 0 and $\infty$ in $\widehat{\mathbb{C}}$ project to two different points on $\overline{\mathbb{H}}$,

$$\mathrm{div}(u) := u^{-1}(0) - u^{-1}(\infty) \in \mathrm{Div}(Y)$$

is a divisor on $Y$.

DEFINITION 2.4

A divisor $D \in \mathrm{Div}(Y)$ is called *principal* if $D = \mathrm{div}(u)$ for some morphism $u : Y \to \overline{\mathbb{H}}$ of the above type. The set of principal divisors of $Y$ will be denoted by $\mathrm{Div}_P(Y)$.

PROPOSITION 2.5

*A divisor $D$ on $Y$ is principal if and only if there exists a divisor $E \in \mathrm{Div}_P(X) \cap \mathrm{Div}(X)^{\sigma^*}$ with $\pi^* D = E$.*

*Proof.* Let $E = \mathrm{div}(f)$ be a principal divisor in $\mathrm{Div}(X)^{\sigma^*}$, where $f$ is a non-constant meromorphic function on $X$. Consider the function $\psi$ on $X$ defined by $\psi(x) = \overline{f(\sigma(x))}$ on $X$. This function $\psi$ is clearly meromorphic.

Since $E \in \mathrm{Div}(X)^{\sigma^*}$, we have $\mathrm{div}(\psi) = \mathrm{div}(f)$. Consequently, there exists a constant $c \in \mathbb{C} \setminus \{0\}$ such that $\psi = cf$.

Therefore, we have $f(x) = \psi(x)/c = \overline{f(\sigma(x))}/c = \overline{\psi(\sigma(x))}/|c|^2 = f(x)/|c|^2$. Take $c_0 \in \mathbb{C}$ with $c_0^2 = c$. Set $f_0 = c_0 f$.

The divisor for the meromorphic function $f_0$ coincides with $E$. Furthermore, $f_0$ satisfies the condition

$$f_0 \circ \sigma = \overline{f_0}\,.$$

Therefore, it induces a map

$$\hat{f} : Y := X/\sigma \longrightarrow \overline{\mathbb{H}} := \widehat{\mathbb{C}}/\langle j \rangle$$

with $\mathrm{div}(\hat{f}) = D$.

Conversely, let $D = \mathrm{div}(u)$ be a principal divisor on $Y$. Consider the composition $u \circ \pi : X \longrightarrow \overline{\mathbb{H}}$. It is straight-forward to see that the function $u \circ p$ lifts to a smooth function

$$f : X \longrightarrow \widehat{\mathbb{C}}$$

such that $p \circ f = u \circ \pi$. There are two such smooth lifts; one is holomorphic and the other is anti-holomorphic ($u \circ p$ also has a continuous lift, defined by the inclusion of $\overline{\mathbb{H}}$ in $\widehat{\mathbb{C}}$ which is not smooth). Let $f$ denote the holomorphic one. Since $\mathrm{div}(f) = \pi^*(D) \in \mathrm{Div}(X)^{\sigma^*}$, the proof of the proposition is complete. $\square$

DEFINITION 2.6

The quotient of $\mathrm{Div}_0(Y)$, the group of all degree zero divisors on $Y$, by the subgroup of all principal divisors on $Y$ is called the *Jacobian* of $Y$. The Jacobian of $Y$ will be denoted by $J_0(Y)$.

From Proposition 2.5, it follows immediately that by sending any divisor $D$ on $Y$ to the divisor $\pi^* D$ on $X$ we obtain an injective homomorphism from $J_0(Y)$ to the Jacobian $J_0(X)$ of $X$. From Lemma 2.3, it follows that $J_0(Y)$ coincides with the fixed point set of the involution of $J_0(X)$ defined by $\sigma$.

A function $f : W \to \mathbb{R}$, defined on an open subset of $Y$ is called *harmonic* if for every point $y \in W$, there exists a compatible coordinate chart $(U, w)$, with

$$y \in U \subseteq W\,,$$

such that the function $f \circ w^{-1}$ is harmonic. Since precomposition with holomorphic and anti-holomorphic functions preserve harmonicity, we conclude that harmonic functions are well-defined on $Y$.

We say that a real one-form $\eta$ on $Y$ is *harmonic* if it is locally given by $\mathrm{d}f$, where $f$ is a harmonic function.

Let $\Omega$ denote the holomorphic cotangent bundle of the Riemann surface $X$. If $\omega \in H^0(X, \Omega)$ is given locally by $\omega = f \mathrm{d}z$, where $f$ is a holomorphic function, then define

$$\overline{\sigma^*\omega} := (\overline{f \circ \sigma})\, \mathrm{d}(\overline{z} \circ \sigma)\,.$$

So if $\omega$ is defined over $U$, then $\overline{\sigma^*\omega}$ is a holomorphic one-form defined over $\sigma(U)$. More generally, for a one-form $\alpha = u\, \mathrm{d}z + v\, \mathrm{d}\bar{z}$, set

$$\sigma^*\alpha = (u \circ \sigma)\, \mathrm{d}(z \circ \sigma) + (v \circ \sigma)\, \mathrm{d}(\overline{z} \circ \sigma)\,.$$

Let $\mathcal{H}^1_{\mathbb{R}}(Y)$ and $\mathcal{H}^1_{\mathbb{R}}(X)$ denote the space of all real harmonic one-forms on $Y$ and $X$ respectively. Using the map $\pi : X \to Y$, we can lift harmonic forms on $Y$ to smooth forms on $X$. It is easy to see that the pullback of a harmonic form on $Y$ is a harmonic form on $X$. Therefore, there is a well-defined injective homomorphism $\pi^* : \mathcal{H}^1_{\mathbb{R}}(Y) \longrightarrow \mathcal{H}^1_{\mathbb{R}}(X)$.

The complex structure on $X$ defines a Hodge-$*$ operator on one-forms on $X$. In local holomorphic coordinates the Hodge-$*$ operator is

$$*(u\, \mathrm{d}z + v\, \mathrm{d}\bar{z}) = -\sqrt{-1}u\, \mathrm{d}z + \sqrt{-1}v \mathrm{d}\bar{z}$$

or $*(a\, \mathrm{d}x + b\, \mathrm{d}y) = -b\, \mathrm{d}x + a\, \mathrm{d}y$.

A holomorphic one-form $\omega$ on $X$ will be called $\sigma$*-invariant* if $\sigma^*\omega = \overline{\omega}$. The space of all $\sigma$-invariant forms on $X$ will be denoted by $H^0(X, \Omega)^{\overline{\sigma^*}}$.

**Theorem 2.7.** *A holomorphic form $\omega \in H^0(X, \Omega)$ is $\sigma$-invariant if and only if there exists a form $\eta \in \mathcal{H}^1_{\mathbb{R}}(Y)$ such that $\omega = \beta + \sqrt{-1}(*\beta)$, where $\beta = \pi^*\eta$.*

*The homomorphism $H^1_{\mathbb{R}}(Y) \longrightarrow H^0(X, \Omega)^{\overline{\sigma^*}}$ defined by*

$$\eta \longmapsto \pi^*\eta + \sqrt{-1}(*\pi^*\eta)$$

*is an isomorphism of real vector spaces.*

*Proof.* Take any $\omega \in H^0(X, \Omega)$. Let $\omega = \beta + \sqrt{-1}(*\beta)$, where $\beta$ is a real one-form. Now the condition $\sigma^*\omega = \overline{\omega}$ immediately implies that $\sigma^*\beta = \beta$. Therefore, $\beta$ is the pullback of a form on $Y$. For any $\eta \in \mathcal{H}^1_{\mathbb{R}}(Y)$, the form $\pi^*\eta + \sqrt{-1}(*\pi^*\eta)$ is a $\sigma$-invariant holomorphic one-form.

Let

$$\varphi : \mathcal{H}^1_{\mathbb{R}}(Y) \longrightarrow H^0(X, \Omega)^{\overline{\sigma^*}}$$

be the homomorphism that sends any harmonic form $\eta \in \mathcal{H}^1_{\mathbb{R}}(Y)$ to the holomorphic form $\pi^*\eta + \sqrt{-1}(*\pi^*\eta)$. This homomorphism is injective since a holomorphic one-form with vanishing real part must be identically zero.

The inverse homomorphism

$$H^0(X, \Omega)^{\overline{\sigma^*}} \longrightarrow \mathcal{H}^1_{\mathbb{R}}(Y)$$

sends a $\sigma$-invariant form $\omega$ on $Y$ to $\eta$ with the property

$$\pi^*\eta = \frac{\omega + \overline{\omega}}{2}.$$

This completes the proof of the theorem. $\square$

## 3. The Jacobian

A closed oriented smooth path $\gamma$ on $X$ gives an element $L_\gamma \in H^0(X, \Omega)^*$ defined by

$$L_\gamma(\omega) = \int_\gamma \omega,$$

where $\omega \in H^0(X, \Omega)$. Using Stokes' theorem we get a mapping from $H_1(X, \mathbb{Z})$ to $H^0(X, \Omega)^*$. The quotient space $H^0(X, \Omega)^*/H_1(X, \mathbb{Z})$ will be denoted by $J_1(X)$.

As we saw in the previous section, for a holomorphic one-form $\omega$ on $X$, the form $\overline{\sigma^*\omega}$ is again a holomorphic one-form. This involution of $H^0(X, \Omega)$ induces an involution

$$\sigma_1 : H^0(X, \Omega)^* \longrightarrow H^0(X, \Omega)^*.$$

In other words, $(\sigma_1(L))(\omega) = \overline{L(\overline{\sigma^*(\omega)})}$. It is easy to check that for any closed smooth-oriented path $\gamma$ on $X$, the identity

$$\sigma_1(L_\gamma) = L_{\sigma(\gamma)}$$

is valid. So, the involution $\sigma_1$ preserves the subgroup $H_1(X, \mathbb{Z}) \subset H^0(X, \Omega)^*$.

Consequently, the involution $\sigma_1$ of $H^0(X, \Omega)^*$ induces an involution on the quotient space $J_1(X)$. The involution of $J_1(X)$ obtained this way will also be denoted by $\sigma_1$.

Let $g$ be the genus of the compact connected Riemann surface $X$. Suppose we have a canonical basis of $H_1(X, \mathbb{Z})$, say $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$. This means that the corresponding intersection matrix is

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ is the identity matrix of rank $g$. Then there exists a unique basis of $H^0(X, \Omega)$, say $\{\omega_1, \dots, \omega_g\}$, such that $\int_{\alpha_k} \omega_j = \delta_{jk}$ ([2], Proposition III.2.8). We say that this basis is *adapted* to the given basis of homology.

Using this adapted basis we can identify $H^0(X, \Omega)^*$ with $\mathbb{C}^g$ by sending the element $L$ of $H^0(X, \Omega)^*$ to the vector $(L(\omega_1), \dots, L(\omega_g))$.

Therefore, for any $\gamma \in H_1(X, \mathbb{Z})$, we may identify the element $L_\gamma \in H^0(X, \Omega)^*$ with

$$(L_\gamma(\omega_1), \dots, L_\gamma(\omega_g)) \in \mathbb{C}^g.$$

Denote by $\mathcal{L}$ the lattice in $\mathbb{C}^g$ defined by $H_1(X, \mathbb{Z})$ using this identification. The quotient space $J_1(X)$ defined earlier is clearly identified with the quotient $\mathbb{C}^g/\mathcal{L}$.

Assume that the basis $\{\omega_j\}$ is $\sigma$-invariant, that is, $\overline{\sigma^*(\omega_j)} = \omega_j$ for each $j \in [1, g]$. It is easy to check that by the above isomorphism of $H^0(X, \Omega)^*$ with $\mathbb{C}^g$ the involution $\sigma_1$ of

$H^0(X, \Omega)^*$ (defined earlier) coincides with the conjugation defined as $(z_1, \ldots, z_g) \longmapsto (\overline{z_1}, \ldots, \overline{z_g})$.

We will denote by $\sigma_\#$ the involution of $H_1(X, \mathbb{Z})$ induced by the involution $\sigma$ of $X$. Let

$$\{\gamma_1, \ldots, \gamma_g, \delta_1, \ldots, \delta_g\}$$

be a canonical basis of $H_1(X, \mathbb{Z})$ satisfying the condition $\sigma_\#(\gamma_j) = \gamma_j$ for all $j \in [1, g]$. Let $\{\omega_1, \ldots, \omega_g\}$ denote the corresponding adapted basis.

PROPOSITION 3.1

*The above adapted basis* $\{\omega_1, \ldots, \omega_g\}$ *is* $\sigma$*-invariant.*

*Proof.* Since

$$\int_{\sigma_\#\gamma} \omega = \int_\gamma \sigma^*\omega = \overline{\int_\gamma \sigma^*\overline{\omega}}$$

(as $\sigma$ is an involution), the proposition follows immediately. □

As in §2, let $J_0(X)$ denote the quotient $\mathrm{Div}_0(X)/\mathrm{Div}_P(X)$. For a meromorphic function $f$ we have $\sigma^*(\mathrm{div}(f)) = \mathrm{div}(\overline{f \circ \sigma})$. So $\sigma^*$ induces an involution on $J_0(X)$. This involution of $J_0(X)$ will be denoted by $\sigma_0$.

Let $\{\omega_1 \ldots, \omega_g\}$ be the basis in Proposition 3.1. Recall the quotient $J_1(X)$ of $H^0(X, \Omega)^*$ defined earlier. The Abel–Jacobi map $A : X \to J_1(X)$ is defined as follows: choose a point $x_0$ of $X$ and set $A(x) = \left[\int_{x_0}^x \omega_1, \ldots, \int_{x_0}^x \omega_g\right]$, where the brackets denote the equivalence class in $J_1(X)$. We have

$$\begin{aligned}
A(\sigma(x)) &= \left[\int_{x_0}^{\sigma(x)} \omega_1, \ldots, \int_{x_0}^{\sigma(x)} \omega_g\right] = \left[\int_{x_0}^{\sigma(x_0)} \omega_1, \ldots, \int_{x_0}^{\sigma(x_0)} \omega_g\right] \\
&\quad + \left[\int_{\sigma(x_0)}^{\sigma(x)} \omega_1, \ldots, \int_{\sigma(x_0)}^{\sigma(x)} \omega_g\right] \\
&= c_0 + \left[\int_{\sigma(x_0)}^{\sigma(x)} \sigma^*(\omega_1), \ldots, \int_{\sigma(x_0)}^{\sigma(x)} \sigma^*(\omega_1)\right] \\
&= c_0 + \left[\int_{x_0}^{x} \overline{\omega_1}, \ldots, \int_{x_0}^{x} \overline{\omega_g}\right] = c_0 + \overline{A(x)},
\end{aligned}$$

where $c_0 = A(\sigma(x_0))$. For a divisor $D = \sum_{j=1}^r n_j\, x_j$, we define

$$A(D) = \sum_{j=1}^r n_j\, A(x_j)\,.$$

If $D$ has degree equal to 0 then we can write it as $D = \sum_{j=1}^s x_j - \sum_{j=1}^s y_j$, where $x_j \neq y_k$ (though we can have repetitions among the $x_j$s or the $y_k$s). Then it is easy to check that

$$A(\sigma_0(D)) = \overline{A(D)} = \sigma_1(A(D))\,, \tag{1}$$

where $\sigma_1$ and $\sigma_0$ are the earlier defined involutions of $H^0(X, \Omega)^*$ and $J_0(X)$ respectively.

By Abel's theorem, the map $A$ can be extended to a map from $J_0(X)$ to $J_1(X)$. By the Abel–Jacobi inversion problem, the map $A : J_0(X) \to J_1(X)$ is surjective. Thus (1) says that $\sigma_0$ and $\sigma_1$ are equivalent under $A$, that is, the following diagram commutes:

$$\begin{array}{ccc} J_0(X) & \xrightarrow{A} & J_1(X) \\ {\scriptstyle\sigma_0}\downarrow & & \downarrow{\scriptstyle\sigma_1} \\ J_0(X) & \xrightarrow[A]{} & J_1(X). \end{array} \tag{2}$$

In the paragraph following Definition 2.6 we noted that the Jacobian $J_0(Y)$ coincides with the fixed point set of $J_0(X)$ for the action of the involution $\sigma_0$. Let $J_1(X)^{\sigma_1} \subset J_1(X)$ be the fixed point set for the action of the involution $\sigma_1$ on $J_1(X)$. From the commutativity of the diagram in (2) it follows immediately that $J_0(Y)$ is identified with $J_1(X)^{\sigma_1}$. Finally using Theorem 2.7, the Jacobian $J_0(Y)$ is identified with the quotient of $\mathcal{H}^1_{\mathbb{R}}(Y)$ by the torsion-free part of $H_1(Y, \mathbb{Z})$.

## 4. Line bundles on a Klein surface

Let $L$ be a holomorphic line bundle over a Riemann surface $X$. By $\overline{L}$ we will mean the $C^\infty$ complex line bundle over $X$ whose transition functions are the conjugations of the transition functions for $L$. To explain this, let $U_i$, $i \in I$, be an open covering of $X$ and assume that over each $U_i$ we are given a holomorphic trivialization of $L$. So for any ordered pair $i, j \in I$, we have the corresponding transition function

$$f_{i,j} : U_i \cap U_j \longrightarrow \mathbb{C}^*$$

which is holomorphic. The $C^\infty$ complex line bundle $\overline{L}$ has $C^\infty$ trivializations over each $U_i$, $i \in I$, and for any ordered pair $i, j \in I$ the corresponding transition function is $\overline{f_{i,j}}$. It is easy to see that the collection $\{\overline{f_{i,j}}\}_{i,j \in I}$ satisfy the cocycle condition to define a $C^\infty$ complex line bundle.

The line bundle $\overline{L}$ can also be defined without using local trivializations. A $C^\infty$ complex line bundle is a $C^\infty$ real vector bundle of rank two together with a smoothly varying complex structure on the fibers (which are real vector spaces of dimension two). The underlying real vector bundle of rank two for $\overline{L}$ coincides with the one for $L$. For any $x \in X$, if $J_x$ is the complex structure on the fiber $L_x$, then the complex structure of the fiber $\overline{L}_x$ is $-J_x$.

As in §2, let $Y$ be a nonorientable Klein surface and $X$ its double cover, which is a connected Riemann surface of genus $g$.

Let $L$ be a holomorphic line bundle over $X$. The complex line bundle $\sigma^*\overline{L}$ has a natural holomorphic structure, where $\sigma$, as before, is the involution of $X$. To construct the holomorphic structure on $\sigma^*\overline{L}$, observe that if $f$ is a holomorphic function on an open subset $U$ of $X$, then $\overline{f \circ \sigma}$ is a holomorphic function of $\sigma(U)$. We can choose the above open subsets $U_i$ (sets over which $L$ is trivialized) in such a way that $\sigma(U_i) = U_i$. Now, since each $\overline{f_{i,j} \circ \sigma}$ is a holomorphic function on $U_i \cap U_j$, the complex line bundle $\sigma^*\overline{L}$ gets equipped with a holomorphic structure.

PROPOSITION 4.1.

*Let $D$ be a divisor on $X$ of degree $d$ and $L$ the corresponding holomorphic line bundle $\mathcal{O}_X(D)$ over $X$ of degree $d$. Then the holomorphic line bundle $\sigma^*\overline{L}$ corresponds to the divisor $\sigma(D)$, that is, $\sigma^*\overline{L} \cong \mathcal{O}_X(\sigma(D))$.*

*Proof.* Since $L \cong \mathcal{O}_X(D)$, we have a meromorphic section $s$ of $L$ with the positive part of $D$ as the zeros of $s$ (of order given by multiplicity) and the negative part of $D$ as the poles of $s$ (of order given by multiplicity). Since $L$ and $\overline{L}$ are identified as real rank two vector bundles, the pullback $\sigma^*s$ defines a smooth section of $\sigma^*\overline{L}$ over the complement (in $X$) of the support of $D$.

It is straight-forward to check that the section $\sigma^*s$ of $\sigma^*\overline{L}$ is meromorphic. The divisor defined by the meromorphic section $\sigma^*s$ clearly coincides with $\sigma(D)$. Consequently, $\sigma^*\overline{L}$ is holomorphically isomorphic to the line bundle over $X$ defined by the divisor $\sigma(D)$. This completes the proof of the proposition. □

Recall the quotient space $J_0(X) := \mathrm{Div}_0(X)/\mathrm{Div}_P(X)$ considered in §2. The Jacobian $J_0(X)$ is identified with the space of all isomorphism classes of degree zero holomorphic line bundles over $X$. The isomorphism sends any divisor $D$ to the line bundle $\mathcal{O}_X(D)$. As in §3, let $\sigma_0$ denote the involution of $J_0(X)$ defined by $\sigma$. From Proposition 4.1, it follows immediately that the above identification of $J_0(X)$ with degree zero line bundles takes the involution $\sigma_0$ to the involution defined by $L \longmapsto \sigma^*\overline{L}$ on the space of all isomorphism classes of degree zero line bundles.

Let $D$ be a divisor of degree zero on the nonorientable Klein surface $Y$. From Proposition 2.5, it follows immediately that $D$ is principal if and only if $\pi^*D$ is principal. Therefore, we have an injective homomorphism

$$\rho : \frac{\mathrm{Div}_0(Y)}{\mathrm{Div}_P(Y)} \longrightarrow \frac{\mathrm{Div}_0(X)}{\mathrm{Div}_P(X)} = J_0(X) \tag{3}$$

defined by $D \longmapsto \pi^*D$, where $\mathrm{Div}_P(Y)$ denotes the group of principal divisors on $Y$ (as before, $\mathrm{Div}_0$ denotes degree zero divisors).

**Theorem 4.2.** *The image of the homomorphism $\rho$ in* (3) *coincides with the subgroup of $J_0(X)$ defined by all holomorphic line bundle $L$ with $\sigma^*\overline{L}$ holomorphically isomorphic to $L$.*

*Proof.* Let $D$ be a divisor on $Y$ of degree zero. The divisor $\pi^*D$ on $X$ is left invariant by the action of the involution $\sigma$. From the above remark that the involution $\sigma_0$ is taken into the involution defined by $L \longmapsto \sigma^*\overline{L}$, it follows immediately that the holomorphic line bundle $L = \mathcal{O}_X(\pi^*D)$ over $X$ corresponding to the divisor $\pi^*D$ satisfies the condition $L \cong \sigma^*\overline{L}$.

For the converse direction, take a holomorphic line bundle $L$ over $X$ which has the property that $\sigma^*\overline{L}$ is isomorphic to $L$. Let $s$ be a nonzero meromorphic section of $L$. If the divisor div($s$) is left invariant by the involution $\sigma$, then $L$ is in the image of $\rho$.

If div($s$) is *not* left invariant by the involution $\sigma$, then consider the meromorphic section of $\sigma^*\overline{L}$ defined by $\sigma^*s$. (Recall that $\sigma^*\overline{L}$ and $\sigma^*L$ are identified as real rank two $C^\infty$ bundles, and the section of $\sigma^*\overline{L}$ defined by $\sigma^*s$ using this identification is meromorphic.)

Now, fix a holomorphic isomorphism

$$\alpha : L \longrightarrow \sigma^*\overline{L} \tag{4}$$

such that the composition

$$L \xrightarrow{\alpha} \sigma^*\overline{L} \xrightarrow{\sigma^*\overline{\alpha}} \overline{\sigma^*\sigma^*\overline{L}} = L \tag{5}$$

is the identity automorphism of $L$, where $\overline{\alpha}$ is the isomorphism of $\overline{L}$ with $\overline{\sigma^*\overline{L}}$ induced by $\alpha$. Note that such an isomorphism exists. Indeed, if

$$\alpha' : L \longrightarrow \sigma^*\overline{L}$$

is any isomorphism, then the automorphism $\sigma^*\overline{\alpha'} \circ \alpha'$ of $L$ (defined as in (5)) is the multiplication by a nonzero scalar $c \in \mathbb{C}$. Take any $c_0 \in \mathbb{C}$ such that $c_0^2 = c$. Now the isomorphism $\alpha = \alpha'/c_0$ satisfies the condition that the composition in (5) is the identity automorphism of $L$.

Let $s'$ be the meromorphic section of $L$ defined by the above section $\sigma^*s$ using this isomorphism. Consider the meromorphic section $s'+s$ of $L$. Since $\mathrm{div}(s)$ is not left invariant by $\sigma$, this meromorphic section $s'+s$ is not identically zero. The divisor $\mathrm{div}(s+s')$ is clearly left invariant by the involution $\sigma$. Hence $L \in J_0(X)$ is in the image of $\rho$. This completes the proof of the theorem. $\square$

## 5. Nonorientable line bundle

In this section we will define a line bundle on $Y$ intrinsically without using $X$.

Let $\{U_i\}_{i\in I}$ be a covering of $Y$ by open subsets and for each $U_i$,

$$\phi_i : U_i \longrightarrow \mathbb{R}^2,$$

a $C^\infty$ coordinate chart. Consider the trivial (real) line bundle $U_i \times \mathbb{R}$ on each $U_i$. Using

$$\frac{\det \mathrm{d}(\phi_j \circ \phi_i^{-1})}{|\det \mathrm{d}(\phi_j \circ \phi_i^{-1})|} \in \pm 1 \subset \mathrm{Aut}(\mathbb{R})$$

as the transition function over $U_i \cap U_j$ for the pair $(i, j)$, we get a real line bundle over $Y$. This line bundle will be denoted by $\xi$. Since the transition functions are $\pm 1$, the line bundle $\xi^{\otimes 2}$ has a natural isomorphism with the trivial line bundle $Y \times \mathbb{R}$. Let

$$\lambda : \xi^{\otimes 2} \longrightarrow Y \times \mathbb{R} \tag{6}$$

be the isomorphism.

We will give a construction of the line bundle $\xi$ without using coordinate charts. Consider the complement $\bigwedge^2 TY \setminus \{0_Y\}$ of the zero section of the real line bundle $\bigwedge^2 TY$, where $TY$ is the real tangent bundle of $Y$. The multiplicative group

$$\mathbb{R}^+ := \{c \in \mathbb{R} \mid c > 0\}$$

acts on $\bigwedge^2 TY \setminus \{0\}$. The action of any $c \in \mathbb{R}^*$ sends any $v \in \bigwedge^2 TY \setminus \{0\}$ to $cv$. Also, the multiplicative group $\pm 1$ acts on $\bigwedge^2 TY \setminus \{0\}$ by sending any $v$ to $\pm v$. Since these two actions commute, we have an action of the multiplicative group $\pm 1$ on

$$Z := \frac{\bigwedge^2 TY \setminus \{0_Y\}}{\mathbb{R}^+}.$$

Now, we have

$$\xi = \frac{Z \times \mathbb{R}}{\pm 1},$$

where $\pm 1$ acts diagonally and it acts on $\mathbb{R}$ as multiplication by $\pm 1$.

We will show that the Klein surface (nonorientable complex) structure on $Y$ gives an isomorphism of $TY$ with $TY \otimes \xi$, where $TY$ as before is the (real) tangent bundle of $Y$. To construct the isomorphism, take a compatible coordinate chart

$$\phi_i : U_i \longrightarrow \mathbb{C}$$

compatible with the nonorientable complex structure. The orientation of the complex line $\mathbb{C}$ induces an orientation of $U_i$ using $\phi_i$. This gives a trivialization of $\xi$ over $U_i$ (this induced trivialization is also clear from the first construction of $\xi$). Using $\phi_i$ we have a complex structure on $U_i$ obtained from the complex structure of $\mathbb{C}$. Let

$$\gamma_i : TU_i \longrightarrow TU_i \otimes \xi|_{U_i}$$

be the isomorphism defined by the almost complex structure of $U_i$ and the trivialization of $\xi|_{U_i}$. If $\phi_j$ is another compatible coordinate chart then the function $\phi_i \circ \phi_j^{-1}$ is either holomorphic or anti-holomorphic. This immediately implies that the isomorphism

$$\gamma_j : TU_j \longrightarrow TU_j \otimes \xi|_{U_j}$$

(obtained by repeating the construction of $\gamma_i$ for the new compatible coordinate chart) coincides with $\gamma_i$ over $U_i \cap U_j$. Consequently, the locally defined isomorphisms $\{\gamma_i\}$ patch together compatibly to give a global isomorphism

$$\gamma : TY \longrightarrow TY \otimes \xi \tag{7}$$

over $Y$.

A *nonorientable complex line bundle* over $Y$ is a $C^\infty$ real vector bundle of rank two over $Y$ together with a $C^\infty$ isomorphism of vector bundles

$$\tau : E \longrightarrow E \otimes \xi \tag{8}$$

satisfying the condition that the composition

$$E \xrightarrow{\tau} E \otimes \xi \xrightarrow{\tau \otimes \mathrm{Id}_\xi} (E \otimes \xi) \otimes \xi = E \otimes \xi^{\otimes 2} \xrightarrow{\mathrm{Id}_E \otimes \lambda} E \tag{9}$$

coincides with the automorphism of $E$ defined by multiplication with $-1$, where $\lambda$ is defined in (6).

Therefore, if for a point $y \in Y$ we fix $w \in \xi_y$ with $\lambda(w \otimes w) = 1$, then the automorphism of the fiber $E_y$ defined by

$$v \longmapsto \langle \tau(v), w^* \rangle$$

is an almost complex structure on $E_y$, where $\langle - , - \rangle$ denotes the contraction of $\xi_y$ with its dual line $\xi_y^*$ and $w^* \in \xi_y^*$ is the dual element of $w$, that is, $\langle w, w^* \rangle = 1$.

Let $(E, \tau)$ be a nonorientable complex line bundle over $Y$ as above. It is easy to see that the $C^\infty$ vector bundle $E$ is *not* orientable. Indeed, the two orientations on a two-dimensional real vector space $V$ defined by $J$ and $-J$, where $J$ is an almost complex structure on $V$, are opposite to each other. To explain this, note that an orientation of the tangent space $T_yY$, where $y \in Y$, induces an orientation of the fiber $E_y$ and conversely. Indeed, giving an orientation of $T_yY$ is equivalent to giving a vector in $w \in \xi_y$ with $\lambda(w \otimes w) = 1$. As it was shown above, such an element $w$ gives an almost complex structure on $E_y$. Hence $E_y$ gets an orientation. Conversely, if we have an orientation of the fiber $E_y$, then choose the element $w \in \xi_y$, with $\lambda(w \otimes w) = 1$, that induces this orientation using $\tau$. Now, $w$ gives an orientation of $T_yY$. Therefore, giving an orientation of $E_y$ is equivalent to giving an orientation of $T_yY$. Since the tangent bundle $TY$ is not orientable, we conclude that the vector bundle $E$ is not orientable.

The total space of the vector bundle $E$ will also be denoted by $E$. Let

$$f : E \longrightarrow Y$$

be the natural projection. Note that the relative tangent bundle for $f$ (that is, the kernel of the differential $\mathrm{d}f$) is identified with $f^*E$. So we have the following exact sequence of vector bundle

$$0 \longrightarrow f^*E \longrightarrow TE \longrightarrow f^*TY \longrightarrow 0 \tag{10}$$

over the manifold $E$.

The line bundle $f^*\xi$ will be denoted by $\hat{\xi}$. Let

$$J : TE \longrightarrow TE \otimes \hat{\xi}$$

be an isomorphism such that the composition

$$TE \xrightarrow{J} TE \otimes \hat{\xi} \xrightarrow{J \otimes \mathrm{Id}_{\hat{\xi}}} (TE \otimes \hat{\xi}) \otimes \hat{\xi} = TE \otimes \hat{\xi}^{\otimes 2} \xrightarrow{\mathrm{Id}_E \otimes f^*\lambda} TE$$

coincides with the automorphism of $E$ defined by multiplication with $-1$. Assume that the isomorphism $J$ satisfies the following further conditions:

(1) The subbundle $f^*E$ in (10) is preserved by $J$ and $J|_{f^*E}$ coincides with the isomorphism $f^*\tau$, where $\tau$ is defined in (8).
(2) The action of $J$ on the quotient $f^*TY$ in (10) coincides with the isomorphism $f^*\gamma$, where $\gamma$ is constructed in (7).

A holomorphic structure on the nonorientable complex line bundle $E$ is an isomorphism $J$ as above satisfying the following conditions (apart from the above conditions) described below.

If we take a coordinate chart $(U, \phi)$ on $Y$ compatible with the nonorientable Riemann surface structure, then as we saw before, the restriction $\xi|_U$ gets a trivialization. This in turn gives a trivialization of $\hat{\xi}$ over $f^{-1}(U)$. Using this trivialization of $\hat{\xi}|_{f^{-1}(U)}$, the isomorphism $J|_{f^{-1}(U)}$ becomes an automorphism $J_\phi$ of $(TE)_{f^{-1}(U)}$ with the property that $J_\phi \circ J_\phi$ coincides with the automorphism of $(TE)_{f^{-1}(U)}$ given by multiplication with $-1$. In other words, $J_\phi$ is an almost complex structure on $f^{-1}(U)$.

A *holomorphic structure* on the nonorientable complex line bundle $E$ is an isomorphism $J$ satisfying the following two conditions (apart from the earlier conditions):

(1) The almost complex structure $J_\phi$ on $f^{-1}(U)$ is integrable for every compatible coordinate chart.
(2) There is a homomorphic isomorphism

$$f_\phi \,:\, f^{-1}(U) \longrightarrow \phi(U) \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$$

that fits in a commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f_\phi} & \phi(U) \times \mathbb{C} \\ \downarrow f & & \downarrow \\ U & \xrightarrow{\phi} & \phi(U) \end{array}$$

(the right vertical arrow is the projection to the first coordinate), and the restriction of $f_\phi$ to any fiber of $f$ is a complex linear isomorphism with $\mathbb{C}$.

A *holomorphic line bundle* over $Y$ is defined to be a complex line bundle equipped with a holomorphic structure.

As in §2, let $\pi : X \longrightarrow Y$ be the double cover of the nonorientable Riemann surface $Y$ given by local orientations. As before, let $\sigma$ denote the anti-holomorphic involution of the Riemann surface $X$.

**Theorem 5.1.** *The space of all holomorphic line bundles over $Y$ are in bijective correspondence with the holomorphic line bundles $L$ over $X$ with the property that $\sigma^*\overline{L}$ is holomorphically isomorphic to $L$.*

*Proof.* Let $L$ be a holomorphic line bundle over $X$ such that $\sigma^*\overline{L}$ is holomorphically isomorphic to $L$. Fix an isomorphism

$$\alpha \,:\, L \longrightarrow \sigma^*\overline{L}$$

as in (4) such that the composition in (5) is the identity automorphism of $L$.

Since the underlying $C^\infty$ line bundle for $\overline{L}$ is identified with that of $L$, the isomorphism $\alpha$ gives a $C^\infty$ isomorphism of $L$ with $\sigma^*L$ whose composition with itself is the identity automorphism of $L$. In other words, $\alpha$ is a $C^\infty$ lift to $L$ of the involution $\sigma$ of $X$. Therefore, the quotient $L/\alpha$ is a real vector bundle of rank two over $X/\sigma = Y$. This real vector bundle of rank two over $Y$ will be denoted by $E$.

To construct a complex structure on $E$, first note that the (real) line bundle $\pi^*\xi$ over $X$ is canonically trivialized, i.e., there is a natural isomorphism of $\pi^*\xi$ with the trivial line bundle $X \times \mathbb{R}$ over $X$. Indeed, this follows immediately from the definitions of $X$ and $\xi$. The complex structure on the fibers on $L$ give an isomorphism

$$L \longrightarrow L$$

defined by multiplication by $\sqrt{-1}$. Consider the composition

$$L \longrightarrow L \longrightarrow L \otimes_{\mathbb{R}} (X \times \mathbb{R}) \longrightarrow L \otimes_{\mathbb{R}} \pi^*\xi$$

which we denote by $J_0$. Since $\pi^*\xi$ is the pullback of a line bundle over $Y$, there is a natural lift of the involution $\sigma$ to $\pi^*\xi$. On the other hand, $\alpha$ is a $C^\infty$ lift of the involution $\sigma$ to

$L$. Therefore, we have a lift of the involution $\sigma$ to $L \otimes_{\mathbb{R}} \pi^*\xi$. It is straight-forward to check that the isomorphism $J_0$ defined above commutes with the lifts of the involution $\sigma$ to $L$ and $L \otimes_{\mathbb{R}} \pi^*\xi$. This immediately implies that the isomorphism $J_0$ descends to an isomorphism of $E$ with $E \otimes_{\mathbb{R}} \xi$ over $Y$. This isomorphism of $E$ with $E \otimes_{\mathbb{R}} \xi$, which we denote by $J$, clearly satisfies the condition that the composition in (9) is multiplication by $-1$. Therefore, $(E, J)$ is a nonorientable complex line bundle.

It is easy to see that $J$ defines a holomorphic structure on $E$. Indeed, this is an immediate consequence of the fact that the almost complex structure on the total space of $L$ is integrable.

For the converse direction, take a holomorphic line bundle $(E, J)$ over $Y$. Consider the (real) rank two $C^\infty$ vector bundle $\pi^*E$ over $X$. Since $\pi^*\xi$ is identified with the trivial line bundle, the complex structure $\tau$ on $E$ (defined in (8)) gives a complex structure on $\pi^*E$. For the same reason, $J$ defines an integrable complex structure on the total space of $\pi^*E$. Using the conditions on $J$ the vector bundle $\pi^*E$ gets the structure of a holomorphic line bundle over $X$.

Since $\pi^*E$ is the pullback of a vector bundle over $Y$, the involution $\sigma$ of $X$ has a natural $C^\infty$ lift to $\pi^*E$. The isomorphism of $\pi^*E$ with $\sigma^*\pi^*E$ defined by this lift gives a holomorphic isomorphism of the holomorphic line bundle $\pi^*E$ with $\sigma^*\overline{\pi^*E}$. This completes the proof of the theorem.

## References


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