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Some approximation theorems

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Abstract. The general theme of this note is illustrated by the following theorem:

Theorem 1. *Suppose K is a compact set in the complex plane and 0 belongs to the boundary ∂K. Let $\mathcal{A}(K)$ denote the space of all functions f on K such that f is holomorphic in a neighborhood of K and $f(0) = 0$. Also for any given positive integer m, let $\mathcal{A}(m, K)$ denote the space of all f such that f is holomorphic in a neighborhood of K and $f(0) = f'(0) = \cdots = f^{(m)}(0) = 0$. Then $\mathcal{A}(m, K)$ is dense in $\mathcal{A}(K)$ under the supremum norm on K provided that there exists a sector $W = \{re^{i\theta};\ 0 \leq r \leq \delta, \alpha \leq \theta \leq \beta\}$ such that $W \cap K = \{0\}$. (This is the well-known Poincare's external cone condition).*

We present various generalizations of this result in the context of higher dimensions replacing holomorphic with harmonic.

Keywords.

1. Introduction

Axler and Ramey (personal communication) have obtained the following interesting result: Let $L^2(S^n)$ denote the usual Lebesgue space on the unit sphere S^n with respect to the surface area measure on S^n; x_0 be a fixed point in R^n. Let $\mathcal{P}(x_0, m)$ denote the space of all harmonic polynomials which vanish at x_0 together with all their derivatives of order less than or equal to m, and $m > 0$. Then

Theorem AR. *$\mathcal{P}(x_0, m)$ is dense in $L^2(S^n)$ if and only if $|x_0| \geq 1$.*
They also posed the following questions:

(1) Does the above result remain valid if $L^2(S^n)$ is replaced by any $L^p(S^n)$ with $p > 2$?
(2) Could S^n be replaced by more general surfaces?

We shall show here that the answer to the 1st question is yes. Let $\mathcal{C}(x_0, S^n)$ denote the space of all continuous functions on S^n that vanish at x_0 and the space of all continuous functions on S^n.

Theorem 2. *For any positive integer m, $\mathcal{P}(x_0, m)$ is dense in $\mathcal{C}(x_0, S^n)$ with the sup norm if and only if $|x_0| \geq 1$. When x_0 is not on the sphere, then $\mathcal{C}(x_0, S^n)$ is the same as $\mathcal{C}(S^n)$.*

Dedicated to Prof. Ashoke Roy on his 62nd birthday.

Remark. We shall not prove that the density fails when $|x_0| < 1$ since it is rather obvious and we shall not explicitly deal with the case when $|x_0| > 1$, because the proof for $|x_0| = 1$ can be imitated without any problems.

We will derive Theorem 2 as a corollary of a more general result for which we need to introduce some more notation. Let K be any compact set in R^n, ∂K its boundary. We define a notion called ECC. (This is the well-known Poincare's external cone condition.) We say that *K satisfies* ECC *at a point x_0 if there exists a closed solid truncated cone W with vertex at x_0 such that $W \cap K = \{x_0\}$.*

It is clear that to satisfy ECC at x_0, x_0 must be on the boundary of K and also that the set of points where K satisfies ECC is dense in the boundary of K. In order to see this, take any point ξ on the boundary of K and a ball of radius r with center at ξ, where r is arbitrary and positive. There must exist a point η outside K such that $|\eta - \xi| < r/2$ for otherwise ξ would be an interior point of K. Now choose a nearest point to η in K, say λ. Clearly $|\lambda - \eta| = \delta < r/2$ and the ball of radius δ with center at η is entirely contained in the ball of radius r with center at ξ. Now λ must belong to the boundary of K, must lie within a distance of r from ξ and satisfies ECC for K.

Let $\mathcal{H}(x_0, K)$ denote the space of all functions f on K such that f vanishes at x_0 and is the restriction to K of a function harmonic in a neighborhood of K. Let $\mathcal{H}(m, x_0, K)$ denote the space of all functions f on K such that f is the restriction to K of a function, harmonic in a neighborhood of K and it, together with all its derivatives of order $\leq m$ vanish at x_0.

We shall assume the following well-known result:

Lemma A. *Let K be any closed ball in R^n and x_0 belong to K. Then $\mathcal{P}(x_0, m)$ is dense in $\mathcal{H}(m, x_0, K)$.*

Also we need

Theorem 3. *Assume K satisfies* ECC *at x_0. Then for any positive integer m, $\overline{\mathcal{H}(m, x_0, K)} \supset \mathcal{H}(x_0, K)$ with the sup norm.*

We shall supply a proof of this later.

Proof of Theorem 2. Let K be the closed unit ball in R^n and x_0 belong to $S^n = \partial K$. Certainly K satisfies ECC at x_0. Let f belong to $\mathcal{C}(x_0, S^n)$. Let ε be any positive number. It is well-known that there exists a harmonic polynomial P such that

$$|f(x) - P(x)| < \varepsilon \quad \text{on} \quad S^n.$$

Let $h(x) = P(x) - P(x_0)$. Then $|f(x) - h(x)| \leq |f(x) - P(x)| + |P(x_0)| < 2\varepsilon$ on S^n and also $h(x)$ belongs to $\mathcal{H}(x_0, K)$. But by Theorem 3, there exists a g in $\mathcal{H}(m, x_0, K)$ such that $|h(x) - g(x)| < \varepsilon$ and so $|f(x) - g(x)| < 3\varepsilon$. This proves Theorem 2 in view of Lemma A. $\square$

Since $\mathcal{C}(x_0, S^n)$ is dense in all $L^p(S^n)$ for $0 < p < \infty$, from Theorem 2, we have

COROLLARY 4

For any p, $0 < p < \infty$; for any positive integer m, and any point x_0 on S^n, the space $\mathcal{P}(m, x_0)$ of harmonic polynomials that vanish together with all their derivatives of order less than or equal to m is dense in $L^p(S^n)$.

Proof of Theorem 3. Let $G(x) = \ln|x|$, if $n = 2$ and $|x|^{2-n}$, if $n > 2$. We may assume without loss of generality that $x_0 = 0$, $W = \{z; |z| \leq \rho, z/|z| \in \text{a spherical cap } D\}$, and $W \cap K = \{0\}$.

Fix a z outside K. Then $G(x-z)$ is harmonic as a function of x in a neighborhood of K and in a neighborhood of the origin can be expanded in an absolutely convergent power series

$$G(x-z) = \sum a_\alpha(z) x^\alpha$$

where α is a multi-index $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and indices are allowed to run through all non-negative integers. Let $|\alpha|$ denote $\alpha_1 + \alpha_2 + \cdots + \alpha_n$. Further we notice that for any fixed non-negative integer k, the polynomial $\sum_{|\alpha|=k} a_\alpha(z) x^\alpha$ is harmonic in x and for any fixed α, $a_\alpha(z)$ is harmonic in z except at the origin and if we set $z = |z|\omega$ where ω varies on the unit sphere,

$$a_\alpha(z) = |z|^{2-n-|\alpha|} a_\alpha(\omega). \tag{1}$$

We note that $a_\alpha(\omega)$ is real-analytic on the unit sphere. Now let

$$G(m, x, z) = G(x-z) - \sum_{|\alpha| \leq m} a_\alpha(z) x^\alpha.$$

Clearly for any fixed $z \neq o$, $G(m, x, z)$ is harmonic in a neighborhood of K and vanishes together with its derivatives of order less than or equal to m. Then if μ is any finite Borel measure on ∂K orthogonal to $\mathcal{H}(m, 0, K)$, it follows that

$$\int G(x-z)\,\mathrm{d}\mu(x) = \sum_{|\alpha| \leq m} a_\alpha(z) \int x^\alpha\,\mathrm{d}\,\mu(x) \quad \text{for all } z \text{ in the complement of } K. \tag{0}$$

Let b_α denote $\int x^\alpha\,\mathrm{d}\mu(x)$ and $p(k, z)$ denote $\sum_{|\alpha|=k} a_\alpha(z) b_\alpha$. By (1) it follows that $p(k, z) = |z|^{2-n-k} p(k, \omega)$ and $p(k, \omega)$ is real-analytic on the unit sphere. We claim that

$$p(k, z) \equiv 0 \quad \text{for all } \ k, 1 \leq k \leq m. \tag{2}$$

Suppose not. Then there would exist a positive integer l such that $p(j, z) \equiv 0$ for $j > l$ and $p(l, z) \not\equiv 0$. Because $p(l, z)$ is homogeneous and is real-analytic, the set of its zeroes on the unit sphere would be a closed set without any interior. Hence there would exist sub-cone V of W and a positive number δ such that

$$|p(l, z)| \geq \delta |z|^{2-n-l} \quad \text{for all } \ z \in V \tag{3}$$

and further by choosing a sufficiently small $\beta < \rho$, we have

$$\left| \sigma(z) = \sum_{0 \leq k \leq l} p(k, z) \right| \geq \frac{\delta}{2} |z|^{2-n-l} \quad \text{on } U = V \cap \{|z| \leq \beta\}. \tag{4}$$

Choose a hyper-plane section S of U through the origin and integrate $\sigma(z)$ on S with respect to the surface measure on it. Since $\sigma(z)$ stays away from 0 on U, it has the same sign everywhere and so from (4) it follows that

$$\left| \int_S \sigma(z)\,\mathrm{d}z \right| \geq \int |\sigma(z)|\,\mathrm{d}z \geq \frac{\delta}{2} \int_S |z|^{2-n-l}\,\mathrm{d}z. \tag{5}$$

But the last integral is infinite for $l > 0$. But on the other hand $\int_S |G(x-z)|\,\mathrm{d}z$ is uniformly bounded and so $\int_S |\int G(x-z)\,\mathrm{d}\mu(x)|\,\mathrm{d}z$ is finite. This, (5), and (0) lead to a contradiction establishing (2). Hence

$$\int G(x-z)\,\mathrm{d}\mu(x) = b_0 G(z) \quad \text{for every } z \text{ outside } K. \tag{6}$$

If $\nu = \mu - b_0\delta_0$ where δ_0 is the Dirac measure at the origin, (6) can be restated as

$$\int G(x-z)\,\mathrm{d}\nu(x) = 0 \quad \text{for all } z \text{ outside } K. \tag{7}$$

(7) implies ν is orthogonal to any function f which is the restriction to K of a function harmonic in a neighborhood of K. This is a rather standard Runge argument and we omit the proof. Hence for any f in $\mathcal{H}(0, K)$, $\int f(x)\,\mathrm{d}\nu(x) = \int f(x)\,\mathrm{d}\mu(x) - b_0 f(0) = 0$ and so $\int f(x)\,\mathrm{d}\mu(x) = 0$. Now by Hahn–Banach, we have Theorem 3. □

Proof of Theorem 1. Fix a z outside K and write the Taylor formula of order m for the Cauchy kernel:

$$\frac{1}{x-z} = -\sum_{0\le k\le m} \frac{x^n}{z^{n+1}} + \frac{x^{m+1}}{z^{m+1}(x-z)}. \tag{8}$$

Let μ be any finite Borel measure on ∂K such that

$$\int f(x)\,\mathrm{d}\mu(x) = 0 \quad \text{for any } f \in \mathcal{A}(m, K). \tag{9}$$

So $\int x^{m+1}/z^{m+1}(x-z)\,\mathrm{d}\mu(x) = 0$ and consequently

$$\int \frac{1}{(x-z)} = -\sum_{0\le k\le m} \frac{\int x^n\,\mathrm{d}\mu(x)}{z^{n+1}}. \tag{10}$$

Let $a_k = \int x^k\,\mathrm{d}\mu(x), 0 \le k \le m$. Arguing as in the proof of Theorem 3, we find that $a_k = 0, 1 \le k \le m$ and $\mu - a_0\delta_0$ is orthogonal to all functions holomorphic in a neighborhood of K and so to $\mathcal{A}(0, K)$. But δ_0 is orthogonal to $\mathcal{A}(K)$ and hence follows the theorem. □

2. Conclusion

Several problems remain. One of them is whether ECC is really necessary. Another one is what is the capacity of the set of points where the conclusion of either Theorem 3 or Theorem 1 holds in analogy with the set of regular points for the Dirichlet problem? Lastly, what would be an analogue of this Theorem 1 in the context of several complex variables?