Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 1, February 2003, pp. 77–86.

When is $f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n)$?

A KŁOPOTOWSKI[1], M G NADKARNI[2]
and K P S BHASKARA RAO[3]

[1] Université Paris XIII, Institut Galilée, 93430 Villetaneuse Cedex, France
[2] Department of Mathematics, University of Mumbai, Kalina, Mumbai 400 098, India
[3] Stat-Math Unit, Indian Statistical Institute, R.V. College Post, Banglore 560 059, India
E-mail: klopot@math.univ-paris13.fr; nadkarni@math.mu.ac.in;
kpsbrao@hotmail.com

MS received 23 March 2001; revised 19 January 2002

Abstract. We discuss subsets S of $\mathbb{R}^n$ such that every real valued function f on S is of the form

$$f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n),$$

and the related concepts and situations in analysis.

Keywords. Good set; sequentially good set; linked component; sequentially good measure; simplicial measure.

Introduction

Let $X_1, X_2, \ldots, X_n$ be non-empty sets. Let $S \subset X_1 \times X_2 \times \cdots \times X_n$. A point $\underline{x} \in S$ will look like $\underline{x} = (x_1, x_2, \ldots, x_n)$. Let $f : S \longrightarrow \mathbb{R}$ be a function. We say that S is *good for* f, if we can write f in the form

$$f(\underline{x}) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), \quad \underline{x} \in S,$$

where for each i, u_i is a function from X_i to $\mathbb{R}$. If this holds for every function in a class $\mathcal{A}$ of functions on S, then we say that S is *good for* $\mathcal{A}$. We call S *good*, if it is good for every $f : S \longrightarrow \mathbb{R}$.

The purpose of this note is to give some descriptions of good sets and comment on the connection of such sets with Kolmogorov's theorem on superposition of functions and related questions in function algebras. Connection with simplicial measures is also discussed (see §5). For $n = 2$ a geometric description of good sets is known, but this description does not immediately generalize for the case $n > 2$ (see §4).

1. Description of good sets

Call a finite set $L = \{\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k\}$ of distinct points in $X_1 \times X_2 \times \cdots \times X_n$ a *loop* if:

(i) there exist non-zero integers $p_1, p_2, \ldots, p_k$ such that

$$p_1\underline{x}^1 + p_2\underline{x}^2 + \cdots + p_k\underline{x}^k = 0, \tag{1}$$

by which we mean that if x_i^j is the ith coordinate of $\underline{x}^j$, then for each i, $1 \le i \le n$, the formal sum $p_1x_i^1 + p_2x_i^2 + \cdots + p_kx_i^k$ vanishes,

(ii) no proper subset of L satisfies (1).

Note that (1) means that $\sum_{j=1}^{k} p_j \mathbf{1}_{\{x_i^j\}} = 0$ for each i.

Remark. For $n = 2$ the integers p_i can be chosen to be $+1$ or -1, but for $n \geq 3$ this fails and there is no universal upper bound (depending on n) on the integers $p_1, p_2, \ldots, p_k$ (see §4).

Theorem 1.1. *Let $S \subset X_1 \times X_2 \times \cdots \times X_n$ and let $f : S \longrightarrow \mathbb{R}$ be such that whenever the formal sum $\sum_{j=1}^{k} p_j \underline{x}^{(j)} = 0$, then $\sum_{j=1}^{k} p_j f(\underline{x}^{(j)}) = 0$. Then there exist real valued functions $u_1, u_2, \ldots, u_n$ defined on $X_1, X_2, \ldots, X_n$ respectively such that*

$$f(\underline{x}) = f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n), \tag{2}$$

for all $(x_1, x_2, \ldots, x_n) \in S$.

Proof. It is clear that if f is of the form (2), then for any loop $L = \{\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k\}$ of points in S the sum $\sum_{j=1}^{k} p_j f(\underline{x}^j)$ vanishes.

Assume now that for any loop $L = \{\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k\}$ of points in S the sum $\sum_{j=1}^{k} p_j f(\underline{x}^j)$ vanishes. We can suppose without loss of generality that $X_i \cap X_j = \emptyset$ for $i \neq j$. Let $\Omega = X_1 \cup X_2 \cup \cdots \cup X_n$. Every $\underline{x} = (x_1, x_2, \ldots, x_n) \in S$ has associated to it a subset of Ω, namely the set $\{x_1, x_2, \ldots, x_n\}$ with n points. Let

$$\mathcal{C} = \{\{x_1, x_2, \ldots, x_n\} : (x_1, x_2, \ldots, x_n) \in S\}.$$

Then $\mathcal{C}$ is a collection of subsets of Ω. Define on $\mathcal{C}$ the function μ by

$$\mu(\{x_1, x_2, \ldots, x_n\}) = f(x_1, x_2, \ldots, x_n).$$

The class $\mathcal{V}$ of functions of the form $\sum_{j=1}^{l} r_j \mathbf{1}_{C_j}$, r_j rational, $C_j \in \mathcal{C}$, $l \geq 1$, is a vector space over the field of rational numbers and the condition that for any loop $L = \{\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^k\}$ of points in S the sum $\sum_{j=1}^{k} p_j f(\underline{x}^j)$ vanishes, ensures that the map T on $\mathcal{V}$ defined by

$$T\left(\sum_{j=1}^{l} r_j \mathbf{1}_{C_j}\right) = \sum_{j=1}^{l} r_j \mu(C_j)$$

is well defined and linear. We extend this map linearly to the larger class $\mathcal{W}$ of functions of the form $\sum_{j=1}^{l} r_j \mathbf{1}_{C_j}$, r_j rational, $C_j \subset \Omega$, $l \geq 1$, and continue to denote the extended map by T. Let us define $u_i : X_i \longrightarrow \mathbb{R}$ by $u_i(x_i) = T\mathbf{1}_{\{x_i\}}$ for $x_i \in X_i$, $1 \leq i \leq n$. Now, for any $\underline{x} = (x_1, x_2, \ldots x_n) \in S$,

$$\begin{aligned} f(\underline{x}) &= \mu(\{x_1, x_2, \ldots, x_n\}) = T\mathbf{1}_{\{x_1, x_2, \ldots, x_n\}} \\ &= T\mathbf{1}_{\{x_1\}} + \cdots + T\mathbf{1}_{\{x_n\}} = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n). \end{aligned} \qquad \square$$

Theorem 1.2. *A set $S \subset X_1 \times X_2 \times \cdots \times X_n$ is good if and only if S has no loop in it.*

Proof. If $S \subset X_1 \times X_2 \times \cdots \times X_n$ does not admit a loop, then the hypothesis of Theorem 1.1 is vacuously satisfied and so any real valued function on S is of the form

$$f(x_1, x_2, \dots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n),$$

$(x_1, x_2, \dots, x_n) \in S$, where $u_1, u_2, \dots, u_n$ are functions defined on $X_1, X_2, \dots, X_n$ respectively. On the other hand, if S admits a loop then an f violating the condition of Theorem 1.1 can be constructed easily, so Theorem 1.2 follows. □

Remarks

(i) Clearly Theorem 1.2 is also valid for complex-valued functions f. One simply treats real and imaginary parts separately. In the sequel we shall take f to be complex valued.

(ii) If $S \subset \mathbb{R}^n$ is good and the canonical projections of S on the coordinate axes are pairwise disjoint, then clearly we can choose the u_i's all equal. If $S \subset \mathbb{R}^n$ is good, then for any $\underline{c} \in \mathbb{R}^n$ the set $S + \underline{c}$ is also good and, when S is bounded, for a suitable $\underline{c}$ the canonical projections of $S + \underline{c}$ on the coordinate axes are pairwise disjoint, so one can choose the functions u_i, for a given f on such an $S + \underline{c}$, to be the same.

To end this section we shall give a description of good subsets S of $X_1 \times X_2 \times \cdots \times X_n$, when all the sets $X_1, X_2, \dots, X_n$ are finite, i.e, card $X_i = m_i < +\infty,\ 1 \le i \le n$.

Let $\Pi_i : X_1 \times X_2 \times \cdots \times X_n \longrightarrow X_i$, $1 \le i \le n$, be the canonical projections on X_i. If S is good, then any function $f : S \longrightarrow \mathbb{R},\ f = u_1 + u_2 + \cdots + u_n$, is completely determined by the values of u_i on $\Pi_i S$, $1 \le i \le n$. Hence we can assume in addition that $\Pi_i S = X_i$, $1 \le i \le n$.

Let $X_i = \{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)}\},\ 1 \le i \le n$, and $S = \{s_1, s_2, \dots, s_k\}$, where

$$s_j = (x_{j_1}^{(1)}, x_{j_2}^{(2)}, \dots, x_{j_n}^{(n)}) \qquad 1 \le j \le k, \qquad 1 \le j_i \le m_i.$$

We consider the $k \times (m_1 + m_2 + \cdots + m_n)$-matrix M (called *the matrix of* S) with rows $M_j,\ 1 \le j \le k$, given by

$$M_j = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0),$$

where 1 occurs at the places $j_1, m_1 + j_2, m_1 + m_2 + j_3$, etc. corresponding to the subscripts in the point $s_j = (x_{j_1}^{(1)}, x_{j_2}^{(2)}, \dots, x_{j_n}^{(n)}),\ 1 \le j \le k$. Since S is good,

$$\begin{aligned} f(s_j) &= f(x_{j_1}^{(1)}, x_{j_2}^{(2)}, \dots, x_{j_n}^{(n)}) \\ &= u_1(x_{j_1}^{(1)}) + u_2(x_{j_2}^{(2)}) + \cdots + u_n(x_{j_n}^{(n)}), \quad 1 \le j \le k. \end{aligned}$$

We put

$$\begin{gathered} u_1(x_1^{(1)}) = \alpha_1^{(1)}, \dots, u_1(x_{m_1}^{(1)}) = \alpha_{m_1}^{(1)}, \\ u_2(x_1^{(2)}) = \alpha_1^{(2)}, \dots, u_2(x_{m_2}^{(2)}) = \alpha_{m_2}^{(2)}, \\ \dots \\ u_n(x_1^{(n)}) = \alpha_1^{(n)}, \dots, u_n(x_{m_n}^{(n)}) = \alpha_{m_n}^{(n)}. \end{gathered}$$

The relation (2) gives us k equalities

$$\alpha^{(1)}_{j_1} + \alpha^{(2)}_{j_2} + \cdots + \alpha^{(n)}_{j_n} = f(s_j), \quad 1 \le j \le k.$$

In other words, the column vector

$$(\alpha^{(1)}_1, \ldots, \alpha^{(1)}_{m_1}, \alpha^{(2)}_1, \ldots, \alpha^{(2)}_{m_2}, \ldots, \alpha^{(n)}_1, \ldots, \alpha^{(n)}_{m_n})^t \in \mathbb{R}^{m_1+m_2+\cdots+m_n}$$

is a solution of the matrix equation

$$M\vec{\alpha} = \vec{z}, \tag{3}$$

where $\vec{z} = (f(s_1), f(s_2), \ldots, f(s_k))^t \in \mathbb{R}^k$.

Since S is good, we know that (3) has solution for every $\vec{z}$. Since M has $m_1+m_2+\cdots+m_n$ columns and since the $n-1$ vectors

$$(\underbrace{1, 1, \ldots, 1}_{m_1 \text{ times}}, \underbrace{-1, \ldots, -1}_{m_2 \text{ times}}, 0, 0, 0, \ldots)^t$$

$$(\underbrace{1, 1, \ldots, 1}_{m_1 \text{ times}}, \underbrace{0, \ldots, 0}_{m_2 \text{ times}}, \underbrace{-1, \ldots, -1}_{m_3 \text{ times}}, 0, 0, 0, \ldots)^t$$

$$\ldots$$

$$(\underbrace{1, 1, \ldots, 1}_{m_1 \text{ times}}, 0, \ldots, 0, \underbrace{-1, \ldots, -1}_{m_n \text{ times}})^t$$

are linearly independent solutions of the homogeneous equation $M\vec{\alpha} = \vec{0}$, we see that the rank of M is at most $m_1 + m_2 + \cdots + m_n - (n-1)$. Clearly k cannot exceed the rank of M. On the other hand the union of n sets

$$(X_1 \times \{x_2\} \times \cdots \times \{x_n\}) \cup (\{x_1\} \times X_2 \times \{x_3\} \times \cdots \\ \times \{x_n\}) \cup \cdots \cup (\{x_1\} \times \cdots \times \{x_{n-1}\} \times X_n)$$

is a good subset of $X_1 \times X_2 \times \cdots \times X_n$ of cardinality $m_1 + m_2 + \cdots + m_n - (n-1)$. It is clear that if the rank of M is k and $k \le m_1 + m_2 + \cdots + m_n - (n-1)$ then S is good. We have proved:

Theorem 1.3. *Let S be a finite subset of $X_1 \times X_2 \times \cdots \times X_n$ of cardinality k and let m_i denote the cardinality of $\Pi_i S$, the canonical projection of S on X_i. Then S is good if and only if $k \le m_1 + m_2 + \cdots + m_n - (n-1)$ and the matrix M of S defined above has* rank *k. There always exist a good set of cardinality $k \le m_1 + m_2 + \cdots + m_n - (n-1)$.*

Let us remark also that the procedure described in Proposition 2.7 of [5] does not work even in the three-dimensional case.

2. Sequentially good sets

We say that S is *sequentially good for a complex valued function f* defined on S if

$$f(x_1, x_2, \ldots, x_n) = \lim_{k\to\infty} (u_{1,k}(x_1) + u_{2,k}(x_2) + \cdots + u_{n,k}(x_n)),$$

where $(x_1, x_2, \dots, x_n) \in S$ and $u_{1,k}, u_{2,k}, \dots, u_{n,k},\ k = 1, 2, 3, \dots$ are functions on $X_1, X_2, \dots, X_n$ respectively. If S is sequentially good for every function on S, then we say that S is *sequentially good*. It is clear that if a set S is good for f, then it is sequentially good for f. The converse holds in view of Theorem 1.2. Indeed, if S is sequentially good for f, but not good for f, then there exists a loop $L = \{\underline{x}^1, \underline{x}^2, \dots, \underline{x}^k\}$ of points in S such that the sum $\sum_{j=1}^{k} p_j f(\underline{x}^j)$ does not vanish, and at the same time f is the pointwise limit of a sequence of functions $g_n,\ n = 1, 2, \dots$ such that for each g_n, $\sum_{j=1}^{k} p_j g_n(\underline{x}^j)$ vanishes. The contradiction shows that S is good for f.

Say that a subset S of $X_1 \times X_2 \times \cdots \times X_n$ is *sequentially good for a collection* $\mathcal{F}$ of functions on S, if every $f \in \mathcal{F}$ is of the form

$$f(x_1, x_2, \dots, x_n) = \lim_{k\to\infty} (u_{1,k}(x_1) + u_{2,k}(x_2) + \cdots + u_{n,k}(x_n)),$$

$(x_1, x_2, \dots, x_n) \in S,\ u_{1,k}, u_{2,k}, \dots, u_{n,k},\ k = 1, 2, \dots$ being functions on $X_1, X_2, \dots,$ X_n respectively.

Assume now that S is sequentially good for an algebra $\mathcal{F}$ of functions on S which is closed under conjugation, separates points and contains constants. Then in fact S is sequentially good (hence good). For otherwise S will admit a loop L. The restriction of functions in $\mathcal{F}$ to L (denoted by $\mathcal{F}|_L$) is an algebra of functions on L, closed under conjugation, separating points and containing constants. Since L is a finite set (hence compact in the discrete topology), by Stone–Weierstrass theorem, the algebra $\mathcal{F}|_L$ is dense in the collection of functions on L, hence actually equal to the collection of all functions on the finite set L. Since L is sequentially good for all functions on L, we see by our earlier conclusion that L is good and so not a loop. The contradiction shows that S is good. We have proved:

Theorem 2.1. *The following are equivalent for a set* $S \subset X_1 \times X_2 \times \cdots \times X_n$:

(i) *S is good,*
(ii) *S is sequentially good,*
(iii) *every finite subset of S is good,*
(iv) *S is sequentially good for an algebra of functions on S, which is closed under conjugation, separates points of S and contains constants.*

3. Sequentially good measures

Let $X_1, X_2, \dots, X_n$ be Polish spaces. Call a probability measure μ on Borel subsets of $\Omega = X_1 \times X_2 \times \cdots \times X_n$ *sequentially good for a collection* $\mathcal{F}$ of complex-valued functions on Ω if every function $f \in \mathcal{F}$ is of the form

$$f(x_1, x_2, \dots, x_n) = \lim_{k\to\infty} (u_{1,k}(x_1) + u_{2,k}(x_2) + \cdots + u_{n,k}(x_n)), \quad \mu -\ \text{a.e.},$$

where $u_{1,k}, u_{2,k}, \dots, u_{n,k},\ k = 1, 2, \dots$ are Borel measurable.

Let $A_1, A_2, A_3, \dots$ be a countable collection of Borel subsets of Ω which is closed under finite unions and compliments and separates points of Ω. Let μ be a sequentially good probability measure for the countable collection of functions $\mathbf{1}_{A_i},\ i = 1, 2, 3, \dots$. Then there is a Borel subset S of full μ measure which is sequentially good for the collection

$\mathbf{1}_{A_i}, i = 1, 2, 3, \ldots$. The set S continues to be sequentially good for the algebra $\mathcal{A}$ of finite linear combinations of $\mathbf{1}_{A_i},\ i = 1, 2, 3, \ldots$ with complex coefficients, an algebra which is closed under conjugation, separates points and contains constants. By Theorem 2.1 the set S is sequentially good, hence a good set. We have proved:

Theorem 3.1. *If μ is sequentially good for the countable collection of indicator functions $\mathbf{1}_{A_i},\ i = 1, 2, 3, \ldots$ of sets in a countable field of Borel sets which separate points of $X_1 \times X_2 \times \cdots \times X_n$, then μ admits a Borel support S which is good.*

4. Cases $n = 2$ and $n > 2$

A good subset of $\mathbb{R}^2$ has a geometric description which does not seem to be available for $n > 2$.

Two arbitrary points $(x, y), (z, w)$ in $S \subseteq X \times Y$ (S is not necessarily good) are said to be *linked* (and we write $(x, y)L(z, w)$), if there exists a finite sequence of points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ in S (called *a link* of *length n* joining (x, y) to (z, w)) such that:

(i) $(x_1, y_1) = (x, y),\ (x_n, y_n) = (z, w)$;
(ii) for any i, $1 \le i \le n-1$ *exactly one* of the following equalities holds:

$$x_i = x_{i+1}, \quad y_i = y_{i+1};$$

(iii) for any i, $1 \le i \le n-2$, it is not possible to have $x_i = x_{i+1} = x_{i+2}$ or $y_i = y_{i+1} = y_{i+2}$.

Note that L is an equivalence relation. An equivalence class of L is called *a linked component* of S. If $(x, y) \in S$, then the equivalence class to which (x, y) belongs is called the *linked component* of (x, y). Two points $(x, y), (z, w) \in S$ are said to be *uniquely linked*, if there is a unique link joining (x, y) to (z, w). A linked component of $S \subseteq X \times Y$ is said to be *uniquely linked* if any two points in it are uniquely linked.

One can prove (see [5,7]) that a subset $S \subset X \times Y$ is good if and only if each of its linked components is uniquely linked. See [8,9] for more discussion on good sets for $n = 2$.

A geometric description of good subsets S of $X \times Y \times Z$, and more generally of $X_1 \times X_2 \times \cdots \times X_n$ is not available. We only have a partial answer. We consider here the case $n = 3$. For $n > 3$ the notion of a link and linked component can be similarly defined.

DEFINITION

Two arbitrary points $(x, y, z), (p, q, r) \in S \subseteq X \times Y \times Z$ are said to be *linked* (and we write $(x, y, z)L(p, q, r)$), if there exists a finite sequence of points $\{(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots,$ $(x_n, y_n, z_n)\}$ in S (called a *link* joining (x, y, z) to (p, q, r)) such that:

(i) $(x_1, y_1, z_1) = (x, y, z),\ (x_n, y_n, z_n) = (p, q, r)$,
(ii) for any $1 \le i \le n-1$ *exactly one* of the following holds

$$x_i \ne x_{i+1}, \quad y_i \ne y_{i+1}, \quad z_i \ne z_{i+1},$$

(iii) for any i, $1 \le i \le n-2$, none of the following holds:

$$(x_i \ne x_{i+1} \quad \text{and} \quad x_{i+1} \ne x_{i+2}),$$
$$(y_i \ne y_{i+1} \quad \text{and} \quad y_{i+1} \ne y_{i+2}),$$
$$(z_i \ne z_{i+1} \quad \text{and} \quad z_{i+1} \ne z_{i+2}).$$

As before L is an equivalence relation. A uniquely linked set is similarly defined. An equivalence class of L is called *a linked component* of S. We call S *linked*, if it has only one linked component. As in the case of two-dimensional sets, one can prove:

A linked set $S \subset X \times Y \times Z$ *is good if and only if it is uniquely linked.*

However, it is not true that a subset $S \subset \mathbb{R}^3$ is good if each linked component is uniquely linked, as the following example shows:

The set $\{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\}$ has two uniquely linked components, namely, $\{(0, 0, 0), (0, 0, 1)\}$ and $\{(1, 1, 0), (1, 1, 1)\}$, but it is not a good set, as can be seen by writing four linear equations in six unknowns $u(0), v(0), w(0), u(1), v(1), w(1)$ onto $\mathbb{R}^4$.

In case $n = 2$, the coefficients p_i in the definition of a loop can be chosen to be $+1$ or -1. However, for $n > 2$ the coefficients p_i do not have a universal bound (depending only on n). Here are two examples: The set

$$\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}$$

is not a good subset of $\{0, 1\}^3$. It is also a loop, because the formal sum

$$2(0, 0, 0) - (0, 0, 1) - (0, 1, 0) - (1, 0, 0) + (1, 1, 1)$$

is equal to 0. This loop is minimal (i.e. each of its proper subset is good) and one cannot have the formal sum above vanish with all the coefficients equal to $+1$ or -1. For the second example, let $X_1 = X_2 = X_3 = \mathbb{R}$. For the obvious loop described by the following expression not all $p_i's$ can be chosen less than five.

$5\ (1\ 1\ 1)$		
$-(2\ 3\ 1)$	$-(12, 1, 13)$	$-(1, 22, 23)$
$-(4\ 5\ 1)$	$-(14, 1, 15)$	$-(1, 24, 25)$
$-(6, 7, 1)$	$-(16, 1, 17)$	$-(1, 26, 27)$
$-(8, 9, 1)$	$-(18, 1, 19)$	$-(1, 28, 29)$
$-(10, 11, 1)$	$-(20, 1, 21)$	$-(1, 30, 31)$
$+(2, 5, 13)$	$+(12, 22, 25)$	
$+(4, 7, 15)$	$+(14, 24, 27)$	
$+(6, 9, 17)$	$+(16, 26, 29)$	
$+(8, 11, 19)$	$+(18, 28, 31)$	
$+(10, 3, 21)$	$+(20, 30, 23)$	

The above example can be modified so that at least one p_i is bigger than P, a preassigned positive integer ≥ 2.

5. Discussions

As a solution to Hilbert's 13th problem, Kolmogorov (see [11,12,14]) proved that one can imbed the unit cube $E^n = [0, 1]^n$ in $\mathbb{R}^{2n+1}$ homeomorphically by a map of the type $\psi : (x_1, \dots, x_n) \longrightarrow (\sum_{p=1}^n \psi_{1,p}(x_p), \dots, \sum_{p=1}^n \psi_{2n+1,p}(x_p))$, with $\psi_{q,p}$ continuous and monotonic increasing on $[0, 1]$, such that every continuous function g on $\psi(E^n)$ is of the form

$$g(y_1, \dots, y_{2n+1}) = \sum_{q=1}^{2n+1} g_q(y_q).$$

In particular this implies that $\psi(E^n)$ is a good set for complex valued continuous functions, and since such functions form an algebra closed under conjugation, contain constants, and separate points, we see by Theorem 2.1 that $\psi(E^n)$ is a good set. It has been observed by Lorentz [14] that ψ can be chosen so that $g_1, \ldots, g_n$ are all equal. Remark (ii) following Theorem 1.2 shows how this may be arranged.

Two questions naturally arise:
(A) describe compact subsets of $C \subset \mathbb{R}^n$ such that every continuous function g on C is of the form

$$g(y_1, \ldots, y_n) = \sum_{q=1}^{n} g_q(y_q),$$

with $g_1, \ldots, g_n$ continuous,
(B) describe compact subsets of $C \subset \mathbb{R}^n$ such that every continuous function g on C is of the form

$$g(y_1, \ldots, y_n) = \lim_{l \to \infty} \sum_{q=1}^{n} g_{q,l}(y_q),$$

with $g_{q,l},\ 1 \le q \le n,\ l = 1, 2, \ldots$ continuous.

For $n = 2$ these questions are well discussed in the literature. For question (A) a necessary and sufficient condition on C is that it be loopfree (i.e., a good set) and the lengths of links in C be bounded [15,17,18]. For question (B) a sufficient condition is that C be loopfree and that linked components be closed [16] or more generally that linked components admit a Borel cross-section [10].

For $n > 2$ natural analogues of these are not known since a good definition of linked component is not available (see also [19,21]). Theorem 2.1 however shows that a necessary condition on C for both question (A) and (B) is that C be loopfree.

Let $X_1, X_2, \ldots, X_n$ be Polish spaces and let $\Omega = X_1 \times X_2 \times \cdots \times X_n$. A probability measure μ on Ω is said to be *simplicial*, if μ is an extreme point of the convex set of all probability measures λ on Ω, whose one-dimensional marginals are the same as those of μ. Let μ be a simplicial measure and let $\mu_1, \mu_2, \ldots, \mu_n$ denote the one-dimensional marginals of μ. A theorem of Lindenstrauss [13] and Douglas [6] states that:

A probability measure μ on Ω is simplicial if and only if the collection of functions of the form

$$f(x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n),$$

where $u_i \in L^1(X_i, \mu_i),\ 1 \le i \le n$, is dense in $L^1(\Omega, \mu)$.

This theorem is usually proved for $n = 2$, but the same proof holds for any n. It is clear from this theorem that a simplicial measure is sequentially good for the functions $\mathbf{1}_{A_i},\ i = 1, 2, 3, \ldots$, where $\{A_i : i = 1, 2, 3, \ldots\}$ form a countable field of Borel sets which separate points of Ω and so by Theorem 3.1 admits a Borel support which is a good set. We have proved:

Theorem 5.1. *A simplicial measure admits a good Borel set as support.*

For $n = 2$ this result is due to Beneš and Štěpán [3,4].

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are continuous probability measures on $X_1, X_2, \ldots, X_n$ respectively, then it is an easy consequence of Fubini theorem that any Borel set of positive $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$ measure contains a loop of the type $B_1 \times B_2 \times \cdots \times B_n$ with each B_i a two point set. Since a simplicial measure admits a good Borel set as support, we see that *a simplicial measure is singular to* $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$ *for any choice of continuous* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *on* $X_1, X_2, \ldots, X_n$ *respectively* (see [13,20] for the case $n = 2$).

Let us briefly return to question (B) above and let C be a compact subset of $\mathbb{R}^n$ such that every continuous function on C is approximable as described there. Then every probability measure on C is simplicial. For, if μ_1 and μ_2 are two distinct probability measures on Borel subsets of C with the same one-dimensional marginals then $\mu_1 - \mu_2$ is a non-trivial signed measure which integrates all continuous functions on C to zero, which is not possible.

Remark. For a discussion of Hilbert's 13th problem from algebraic point of view see [1,2].

References

[1] Abhyankar S S, Hilbert's 13th problem (preprint)

[2] Arnold V I and Shimura G, Superposition of algebraic functions, mathematical developements arising from Hilbert problems, *Proc. Symposia in Pure Mathematics*, Am. Math. Soc. **XXVIII(1)** (1976) 45–46

[3] Beneš V and Štěpán J, The support of extremal probability measures with given marginals, in: Mathematical Statistics and Probability Theory (Eds) M L Puri *et al* (D. Reidel Publishing Company) (1987) vol. A, pp. 33–41

[4] Beneš V and Štěpán J, Extremal solutions in the marginal problem, in: Advances in probability distributions with given marginals (Dordrecht: Kluwer Academic Publishers) (1991) pp. 189–207

[5] Cowsik R C, Kłopotowski A and Nadkarni M G, When is $f(x, y) = u(x) + v(y)$? *Proc. Indian Acad. Sci. (Math. Sci.)* **109** (1999) 57–64

[6] Douglas R G, On extremal measures and subspace density, *Michigan Math. J.* **11** (1964) 243–246

[7] Hestir K and Williams S, Supports of doubly stochastic measures, *Bernoulli* **1(3)** (1995) 217–243

[8] Kłopotowski A and Nadkarni M G, On transformations with simple Lebesgue spectrum, *Proc. Indian Acad. Sci. (Math. Sci.)* **109** (1999) 47–55

[9] Kłopotowski A and Nadkarni M G, Shift invariant measure and simple spectrum, *Colloquium Mathematicum* **84/85** (2000) 385–394

[10] Kłopotowski A and Nadkarni M G, Sets with doubleton sections, good sets and ergodic theory, *Fund. Math.* **173** (2002) 133–158

[11] Kahane J-P, Sur le Théorème de Superposition de Kolmogorov, *J. Approximation Theory* **13** (1975) 229–234

[12] Kolmogorov A N, On the representation of continuous functions of several variables as superposition of continuous function of one variable and addition, *Dokl. Acad. Nauk SSSR* **114** (1957) 679–681; Am. Math. Soc. Transl. **28** (1963) 55–59

[13] Lindenstrauss J, A remark on doubly stochastic measures, *Am. Math. Monthly* **72** (1965) 379–382

[14] Lorentz G G, The 13th problem of Hilbert, mathematical developements arising from Hilbert problems, *Proc. Symposia in Pure Mathematics*, Am. Math. Soc. **XXVIII(2)** (1976) 419–430

[15] Marshall D E and O'Farrell A G, Uniform approximation by real functions, *Fund. Math.* **104(3)** (1979) 203–211

[16] Marshall D E and O'Farrell A G, Approximation by sums of two algebras: The lightening bolt principle, *J. Funct. Anal.* **52** (1983) 353–368
[17] Mehta R D and Vasavada M H, Algebra direct sum decomposition of $C_R(X)$, *Proc. Am. Math. Soc.* **98** (1986) 71–74
[18] Mehta R D and Vasavada M H, Algebra direct sum decomposition of $C_R(X)$, II, *Proc. Am. Math. Soc.* **100** (1987) 123–126
[19] Sproston J P and Strauss D, Sums subalgebras of $C(X)$, *J. Lond. Math. Soc.* **45(2)** (1992) 265–278
[20] Štěpán J, Simplicial measures and sets of uniqueness in the marginal problem, in: Statistics and decision (München: R. Oldenbourg Verlag) (1993) vol. 11, pp. 289–299
[21] Sternfeld Y, Uniform separation of points and measures and representation of sums of algebras, *Israel J. Math.* **55** (1986) 350–363