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Limits of rank 4 Azumaya algebras and applications to desingularization

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Abstract. It is shown that the schematic image of the scheme of Azumaya algebra structures on a vector bundle of rank 4 over any base scheme is separated, of finite type, smooth of relative dimension 13 and geometrically irreducible over that base and that this construction base-changes well. This fully generalizes Seshadri's theorem in [16] that the variety of specializations of (2×2)-matrix algebras is smooth in characteristic $\neq 2$. As an application, a construction of Seshadri in [16] is shown in a characteristic-free way to desingularize the moduli space of rank 2 even degree semi-stable vector bundles on a complete curve. As another application, a construction of Nori over $\mathbb{Z}$ (Appendix, [16]) is extended to the case of a normal domain which is a universally Japanese (Nagata) ring and is shown to desingularize the Artin moduli space [1] of invariants of several matrices in rank 2. This desingularization is shown to have a good specialization property if the Artin moduli space has geometrically reduced fibers – for example this happens over $\mathbb{Z}$. Essential use is made of Kneser's concept [8] of 'semi-regular quadratic module'. For any free quadratic module of odd rank, a formula linking the half-discriminant and the values of the quadratic form on its radical is derived.

Keywords. Azumaya algebra; Clifford algebra; desingularization, moduli space; semi-regular quadratic form; simple module; vector bundle.

1. Introduction and overview

The present work consists of two parts: Part A shows the smoothness of the schematic closure of Azumaya algebra structures on a fixed vector bundle of rank 4, while Part B applies the results of A to obtain desingularizations of certain moduli spaces. Further applications to quadratic modules are addressed in [20].

The problems addressed below arose from a study of Seshadri's paper [16] in which the base field k is assumed to be an algebraically closed field of characteristic different from two. In the following it is shown that the results of [16] extend over an arbitrary base scheme, and in fact, the methods used are characteristic-free.

The central result of [16] can be described as follows: Let X be a smooth, irreducible, complete curve of genus $g \geq 2$ over k. Let $\mathcal{U}_X^{SS}(n, 0)$ be the normal projective variety of equivalence classes of semi-stable vector bundles on X of rank n and degree zero [15]. Let $\mathcal{U}_X^{S}(n, 0)$ be the smooth open subvariety of $\mathcal{U}_X^{SS}(n, 0)$ consisting of isomorphism classes of stable vector bundles. This subvariety is precisely the set of smooth points of $\mathcal{U}_X^{SS}(n, 0)$ unless $n = 2$ and $g = 2$ in which case $\mathcal{U}_X^{SS}(n, 0)$ is smooth [13]. Two models describing the desingularization of $\mathcal{U}_X^{SS}(2, 0)$ are known. Narasimhan and Ramanan in [14] describe one model which works in characteristic zero. Seshadri in [16] defines (for any characteristic) a variety $\mathcal{N}_X(4, 0)$ whose closed points are certain stable parabolic vector bundles (in the sense of Mehta–Seshadri [11]) of rank 4 and degree zero on X. He also constructs a map $\pi_2 : \mathcal{N}_X(4, 0) \longrightarrow \mathcal{U}_X^{SS}(2, 0)$. This is seen to be a desingularization in characteristic zero. Section 6 in B of the present work shows that the morphism π_2 may be constructed in positive characteristic as well, and further that it is a desingularization.

In the construction of the above desingularization, one of the crucial steps is to prove that the variety of specializations of (2×2)-matrix algebras on a four-dimensional vector space with a fixed (non-zero) vector for multiplicative identity is smooth over an algebraically closed field k of characteristic 2, extending Seshadri's result ([16], §2, Theorem 1) for $\mathrm{char}(k) \neq 2$. This is proved more generally, i.e., by showing over any base scheme that the schematic image, of the scheme of Azumaya algebra structures on a vector bundle of rank 4 with multiplicative identity a fixed nowhere vanishing global section, is separated, of finite type, smooth and geometrically irreducible over the base, and that it behaves well under base-change. In fact it is shown to be locally isomorphic over the base to relative nine-dimensional affine space (Theorem 5.3). As a further generalization, the schematic image of the scheme of Azumaya algebra structures with multiplicative identities varying is also shown to be separated, of finite type, geometrically irreducible and smooth of relative dimension 13 over the base (Theorem 3.8).

Artin in [1] defines a $\mathbb{Z}$-scheme which is a coarse moduli space for the various module structures over the non-commuting polynomial ring (in a fixed number of indeterminates) on a fixed free finite rank module. This moduli space can be constructed over any commutative noetherian base ring using Seshadri's Geometric Invariant Theory over a general base [18] which further ensures that it has good properties (eg. being of finite type over the base) when the base ring is a universally Japanese (Nagata) ring. Nori (Appendix, [16]) constructs a candidate which would desingularize the Artin moduli space in rank 2 over the integers, and the smoothness of this candidate is a consequence of Theorem 3.8. In fact, in §7, it is shown that a desingularization of the Artin moduli space in rank 2 can be constructed over a normal domain which is a universally Japanese (Nagata) ring. This desingularization is further shown to have a good specialization property provided the Artin moduli space has geometrically reduced fibers, which for example is the case over the integers by the result of Donkin [4].

Seshadri's proof of Theorem 1 in [16] uses the existence of non-singular quadratic forms on a three-dimensional vector space. But in char. 2, such forms do not exist. This can be remedied by considering *semi-regular* quadratic forms, which nevertheless do exist. Generalities on semi-regularity are recalled in §4. The notion of a semi-regular quadratic form was introduced by Kneser [8]. It is defined for a quadratic module of odd rank over any commutative ring, allowing the results of this paper to be formulated over an arbitrary base. Semi-regularity is studied in detail in [9], where it is shown to be the correct analogue of non-singular quadratic form in characteristic two. Therefore, the methods of proof below are characteristic-free.

The author came across another notion called *non-degeneracy*, defined by Dieudonné in [3], which is used by Borel in [2] to study orthogonal groups over fields of characteristic two. While the definition of semi-regularity uses the notion of *half-discriminant*, the definition of non-degeneracy involves the values of the quadratic form on the radical of its associated symmetric bilinear form. Non-degeneracy is recalled and generalized in §4, where further a formula linking the half-discriminant and the values of the quadratic form on its radical (valid for any free quadratic module of finite odd rank over any commutative ring) is derived. This is used to show that the notion of non-degeneracy can be generalized, to the case of quadratic modules that are finitely generated and projective of constant odd rank over any commutative ring, and moreover that this generalized notion coincides with the notion of semi-regularity.

A: Smoothness of limits of rank 4 Azumaya algebras

2. Algebra structures on a vector bundle

We fix a base scheme X and a geometric vector bundle $\mathbf{W}$ over X of constant rank ≥ 2 which by definition is associated to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{W}$ locally free of constant rank ≥ 2. The purpose of this section is to define algebra structures and study the relationship of associative unital algebra structures with those that have a fixed unit element.

DEFINITION 2.1

Given any X-scheme T, by a T-algebra structure on $\mathbf{W}_T := \mathbf{W} \times_X T$ (also referred to as T-algebra bundle), we mean a morphism $\mathbf{W}_T \times_T \mathbf{W}_T \longrightarrow \mathbf{W}_T$ of vector bundles on T arising from a morphism of the associated locally-free sheaves. So this is equivalent to giving a morphism of $\mathcal{O}_T$-modules $\mathcal{W}_T \otimes_T \mathcal{W}_T \longrightarrow \mathcal{W}_T$, i.e., an $\mathcal{O}_T$-algebra structure on the associated locally free sheaf $\mathcal{W}_T$. Given such a T-algebra structure and $T' \longrightarrow T$ an X-morphism, it is clear that one gets by pullback (i.e., by base-change) a canonical T'-algebra structure on $\mathbf{W}_{T'}$. Thus one has a contravariant 'functor of algebra structures on $\mathbf{W}$' from $\{X\text{-}schemes\}$ to $\{Sets\}$ denoted $\mathsf{Alg}_{\mathbf{W}}$ whose set of T-valued points is the set of T-algebra structures on $\mathbf{W}_T$, viz., $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{W}_T \otimes \mathcal{W}_T, \mathcal{W}_T)$.

By Proposition 9.6.1, Chap. I of EGA I [5], it follows that the functor $\mathsf{Alg}_{\mathbf{W}}$ is represented by the X-scheme

$$\mathrm{Alg}_{\mathbf{W}} := \mathrm{Spec}\left(\mathsf{Sym}_X\left[\left({\mathcal{W}_X}^{\vee} \otimes_X {\mathcal{W}_X}^{\vee} \otimes_X \mathcal{W}_X\right)^{\vee}\right]\right).$$

Hence $\mathrm{Alg}_{\mathbf{W}}$ is affine (hence separated), of finite type over X and in fact smooth of relative dimension $\mathrm{rank}_X(\mathbf{W})^3$. If $X' \longrightarrow X$ is an extension of base, then the construction $\mathrm{Alg}_{\mathbf{W}}$ base-changes well, i.e., one may canonically identify $\mathrm{Alg}_{\mathbf{W}} \times_X X'$ with $\mathrm{Alg}_{\mathbf{W}'}$ where $\mathbf{W}' = \mathbf{W} \times_X X'$ (cf. Proposition 9.4.11, Chap. I, EGA I [5]).

We next turn to algebra structures on $\mathbf{W}$ with identity. We call a global section $s \in \Gamma(T, \mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ (locally free of positive rank over T) nowhere vanishing if at each point of the base T, the image of its germ in the fiber over the residue field is non-zero. It can be seen that a section is nowhere vanishing if and only if every one of its pullbacks is non-zero and that the pullback of a nowhere vanishing section is again a nowhere vanishing section.

DEFINITION 2.2

For any X-scheme T, let $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}(T)$ denote the subset of $\mathrm{Alg}_{\mathbf{W}}(T)$ consisting of associative algebra structures with multiplicative identity. Thus we obtain a contravariant subfunctor $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ of $\mathrm{Alg}_{\mathbf{W}}$.

We remark that a multiplicative identity for an associative algebra structure must be a nowhere vanishing section as implied by the following lemma and the implication (2) $\Rightarrow$ (4) of the lemma following it.

Lemma 2.3. *Let B be a ring* (*commutative*, *with* 1) *and A an associative B-algebra with multiplicative identity $e_A \in A$. Suppose that A is finitely generated and projective as a B-module. Then $B \cdot e_A$ is a B-direct summand of A.*

Lemma 2.4. *Let B be a ring* (*commutative, with* 1), *W a finite free B-module, and $w \in W$. Then the following conditions are equivalent:*

(1) *the B-linear map $\varepsilon(w) : B \longrightarrow W$ given by $b \longmapsto b \cdot w$ is a section to a B-linear map $p : W \longrightarrow B$;*
(2) *the map $\varepsilon(w)$ defined above is injective and the short exact sequence*

$$0 \longrightarrow B \xrightarrow{\varepsilon(w)} W \longrightarrow W/B \longrightarrow 0$$

is split exact;
(3) *the map $\varepsilon(w)$ defined above is injective and W/B is projective*;
(4) *for every B-algebra S* (*commutative, with $1_S \neq 0$ in S*) *$w \otimes 1_S \neq 0 \in W \otimes_B S$*;
(5) *if $\{w_j | 1 \leq j \leq n\}$ is a B-basis for W, and if $w = \sum_{j=1}^{n} b_j \cdot w_j$ then $\{b_j \mid 1 \leq j \leq n\}$ generates B.*

The proofs of the above results are elementary and hence omitted. The general linear groupscheme associated to $\mathbf{W}$, viz., $\mathrm{GL}_{\mathbf{W}}$ naturally acts on $\mathrm{Alg}_{\mathbf{W}}$ on the left, so that for each X-scheme T, $\mathrm{Alg}_{\mathbf{W}}(T)$ mod $\mathrm{GL}_{\mathbf{W}}(T)$ is the set of isomorphism classes of T-algebra structures on $\mathbf{W} \times_X T$. It is also clear that $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ is a $\mathrm{GL}_{\mathbf{W}}$-stable subfunctor of $\mathrm{Alg}_{\mathbf{W}}$. It is in fact also representable.

PROPOSITION 2.5

$\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ *is represented by an X-scheme* $\mathrm{Id\text{-}Assoc}_{\mathbf{W}}$ *which is separated and of finite type over X so that the natural inclusion of functors induces a functorially injective* $\mathrm{GL}_{\mathbf{W}}$-*equivariant morphism* $\mathrm{Id\text{-}Assoc}_{\mathbf{W}} \longrightarrow \mathrm{Alg}_{\mathbf{W}}$. *Further the construction* $\mathrm{Id\text{-}Assoc}_{\mathbf{W}} \longrightarrow X$ *behaves well under base-change.*

The functor $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ behaves well under a base-change $X' \longrightarrow X$ because the property of being an associative algebra with identity is preserved under base-change and because $\mathrm{Alg}_{\mathbf{W}}$ itself behaves well under base-change as already noted. Therefore, it only remains to prove the representability of $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ by a scheme $\mathrm{Id\text{-}Assoc}_{\mathbf{W}}$ of finite type over X. We shall achieve this by studying the case of algebras with a fixed identity. Notice that the separatedness of $\mathrm{Id\text{-}Assoc}_{\mathbf{W}}$ over X would follow once $\mathrm{Id\text{-}Assoc}_{\mathbf{W}}$ is shown to exist, for then the valuative criterion for separatedness is true for $\mathrm{Id\text{-}Assoc}_{\mathbf{W}}$ over X since it is true for $\mathrm{Alg}_{\mathbf{W}}$ over X and $\mathrm{Id\text{-}Assoc}_{\mathbf{W}} \longrightarrow \mathrm{Alg}_{\mathbf{W}}$ is a functorially injective morphism.

DEFINITION 2.6

Let $w \in \Gamma(X, \mathcal{W})$ be a nowhere vanishing section. For any X-scheme T, let $\mathsf{Id\text{-}}w\mathsf{\text{-}Assoc}_{\mathbf{W}}(T)$ denote the subset of $\mathrm{Alg}_{\mathbf{W}}(T)$ consisting of associative algebra structures with multiplicative identity the nowhere vanishing section w_T over T induced from w. Thus we obtain a contravariant subfunctor $\mathsf{Id\text{-}}w\mathsf{\text{-}Assoc}_{\mathbf{W}}$ of $\mathrm{Alg}_{\mathbf{W}}$.

Let $\mathrm{Stab}(w)(T) \subset \mathrm{GL}_{\mathbf{W}}(T)$ denote the stabilizer subgroup of w_T, so that one gets a subfunctor in subgroups $\mathrm{Stab}(w) \subset \mathrm{GL}_{\mathbf{W}}$. It is in fact represented by a closed subgroup scheme (also denoted by) $\mathrm{Stab}(w)$ and further behaves well under base-change relative to X, i.e., $\mathrm{Stab}(w) \times_X T$ can be canonically identified with $\mathrm{Stab}(w_T)$ for any X-scheme T. These follow from para 9.6.6 of Chap. I, EGA I [5]. It is clear that the natural action of $\mathrm{GL}_{\mathbf{W}}$ on $\mathrm{Alg}_{\mathbf{W}}$ induces one of $\mathrm{Stab}(w)$ on $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}}$. It is easy to check that the functor $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}}$ is a sheaf in the big Zariski site over X and further that this functor is represented by a natural closed subscheme of $\mathrm{Alg}_{\mathbf{W}}$ in the case when X is affine; hence by Zariski glueing – Proposition 4.5.4, Corollary 4.5.5, Chap. 0 and 2.4.3, Chap. I of EGA I [5] – it follows that $\mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}}$ is represented by a closed subscheme $\mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}} \hookrightarrow \mathrm{Alg}_{\mathbf{W}}$ which is $\mathrm{Stab}(w)$-invariant. $\mathrm{Stab}(w)$ acts on the fiber product $\mathrm{GL}_{\mathbf{W}} \times_X \mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}}$ by the (right) action given on valued points by $(g, A) \cdot h := (gh, h^{-1} \cdot A)$ and with respect to this action the natural morphism $\mu_w : \mathrm{GL}_{\mathbf{W}} \times_X \mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}} \longrightarrow \mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}}$ (coming from the action of $\mathrm{GL}_{\mathbf{W}}$) is invariant. Let $\mathbf{U}(T) \subset \mathbf{W}(T)$ be the subset corresponding to nowhere vanishing global sections of $\mathbf{W}_T$. Thus we get a subfunctor $\mathbf{U} \subset \mathbf{W}$. It is represented by the complement (also denoted by) $\mathbf{U}$ of the zero section of $\mathbf{W}$ which is of finite type over X: it is easy to check that the functor $\mathbf{U}$ is a sheaf in the big Zariski site over X; hence by Zariski glueing the proof of representability can be reduced to the case when X is affine and $\mathbf{W}$ trivial, in which case, using the implications (4) $\iff$ (5) of Lemma 2.4 it is seen that $\mathbf{U}$ is the union of the complements of the finitely many hyperplanes. Notice that the orbit morphism corresponding to w, viz., $O_w : \mathrm{GL}_{\mathbf{W}} \longrightarrow \mathbf{W}$ factors through $\mathbf{U}$. There is a natural morphism $\phi : \mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}} \longrightarrow \mathbf{U}$ (mapping an algebra to its identity element) such that one has a commutative diagram of morphisms of functors:

$$\begin{tikzcd} \mathrm{GL}_{\mathbf{W}} \times_X \mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}} \arrow[r, "p_1"] \arrow[d, "\mu_w"'] & \mathrm{GL}_{\mathbf{W}} \arrow[d, "O_w"] \\ \mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}} \arrow[r, "\phi"] & \mathbf{U} \end{tikzcd} .$$

The above diagram is in fact a fiber product square because given an X-scheme T, it is easy to see that the natural map

$$\begin{aligned} &(\mathrm{GL}_{\mathbf{W}} \times_X \mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}})(T) \\ &\quad \longrightarrow (\mathrm{GL}_{\mathbf{W}} \times_{\mathbf{U}} \mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}})(T) : (g, A) \longmapsto (g, g \cdot A) \end{aligned}$$

is bijective and functorial in T.

Thus the study of $\mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}}$ reduces to the study of O_w. The next result says that O_w is a Zariski-locally-trivial principal $\mathrm{Stab}(w)$-bundle. Thus O_w has local sections which are closed immersions; from which μ_w therefore has local sections which are representable by closed immersions. It will then follow that $\mathsf{Id}\text{-}\mathsf{Assoc}_{\mathbf{W}}$ is representable by Zariski glueing since it is easily seen to be a sheaf on the big Zariski site over X and it is covered by open subfunctors that are represented by closed subschemes of open subschemes of $\mathrm{GL}_{\mathbf{W}} \times_X \mathrm{Id}\text{-}w\text{-}\mathrm{Assoc}_{\mathbf{W}}$. Further, since O_w is a faithfully-flat quasi-compact morphism, the properties of p_1 in the above cartesian square such as affineness and finite-typeness will descend to ϕ by Proposition 2.7.1 of EGA IV [6]. So $\mathbf{U}$ being of finite type over X would imply that $\mathrm{Id}\text{-}\mathrm{Assoc}_{\mathbf{W}}$ is also of finite type over X. The proof of Proposition 2.5 is thus reduced to the following.

PROPOSITION 2.7

The $\mathrm{Stab}(w)$*-invariant morphism* $O_w : \mathrm{GL}_{\mathbf{W}} \longrightarrow \mathbf{U}$ *is a Zariski-locally-trivial principal* $\mathrm{Stab}(w)$*-bundle.*

We sketch a proof. It is enough to prove the above result for the case when $X = \mathrm{Spec}(R)$ is affine and further when w becomes a part of a global basis. Let W be the free R-module of rank m corresponding to $\mathbf{W}$. Let $\{X_i \mid 0 \le i \le m-1\}$ be the R-basis of $W^\vee$ dual to a chosen R-basis $\{w_i \mid 0 \le i \le m-1\}$ of W with $w = w_0$, so that one gets a canonical identification with affine space over R of dimension m

$$\mathbf{W} := \mathrm{Spec}(\mathrm{Sym}_R(W^\vee)) \cong \mathrm{Spec}\,(R[X_0, \ldots, X_{m-1}]) = A^m_R.$$

For each i, $0 \le i \le m-1$, let $\mathbf{U}_i$ denote the open subscheme of $\mathbf{W}$ corresponding (under the above identification) to $A^m_R - V(X_i)$ where $V(X_i)$ is the closed subscheme defined by the vanishing of X_i. From Lemma 2.4 one sees that $\mathbf{U} = \cup_{i=0}^{m-1}\mathbf{U}_i$. Let S be an R-algebra. Then $(O_w)^{-1}(\mathbf{U}_0)(S)$ (= the set of S-valued points of the open subscheme which is the inverse image of the open subscheme $\mathbf{U}_0$ by O_w) can be identified with the subset of $\mathrm{GL}(m, S)$ consisting of matrices (s_{ij}), $0 \le i, j \le m-1$ such that s_{00} is a unit in S. Given such a matrix (s_{ij}), it is clear that the matrix equation

$$\begin{pmatrix} s_{00} & 0 & 0 & \cdots & 0 \\ s_{10} & 1 & 0 & \cdots & 0 \\ s_{20} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{m-1,0} & 0 & 0 & \cdots & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_{01} & \cdots & x_{0,m-1} \\ 0 & y_{11} & \cdots & y_{1,m-1} \\ 0 & y_{21} & \cdots & y_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_{m-1,1} & \cdots & y_{m-1,m-1} \end{pmatrix} = (s_{ij})$$

can be solved in $\mathrm{GL}(m, S)$ and in fact the solution lies in $\mathrm{Stab}(w)(S)$. We leave it to the reader to verify that this implies that O_w restricted to $\mathbf{U}_0$ is isomorphic to the trivial $\mathrm{Stab}(w)$-bundle and a similar argument gives a $\mathrm{Stab}(w)$-bundle trivialization of $O_w : \mathrm{GL}_{\mathbf{W}} \longrightarrow \mathbf{U}$ over $\mathbf{U}_i$ for each i with $1 \le i \le m-1$. The above proposition along with the discussion preceeding it gives the following result.

COROLLARY 2.8

The morphism $\mu_w : \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Assoc}_{\mathbf{W}} \longrightarrow \text{Id-Assoc}_{\mathbf{W}}$ *is a Zariski-locally-trivial principal* $\mathrm{Stab}(w)$*-bundle.*

3. Sheaves in Azumaya algebras and their limits

From now on we assume that the rank of $\mathbf{W}$ over X is a square. We intend to study subfunctors of $\mathrm{Alg}_{\mathbf{W}}$ which are Azumaya and their specializations. Firstly, we therefore recall the definition of being Azumaya and collect the necessary facts regarding such algebras.

DEFINITION 3.1

An algebra structure $\mathcal{A} \in \text{Id-Assoc}_{\mathbf{W}}(T)$ is said to be *Azumaya* if the natural $\mathcal{O}_T$-algebra homomorphism

$$\mathcal{A} \otimes \mathcal{A}^{op} \longrightarrow \mathcal{E}nd_{\mathcal{O}_T\text{-mod}}(\mathcal{A}) \text{ given on sections by } a \otimes b^{op} \mapsto (x \mapsto (axb))$$

is an isomorphism, where $\mathcal{A}^{op}$ denotes the algebra opposite to $\mathcal{A}$.

We thus obtain subfunctors $\mathsf{Id}\text{-}w\text{-}\mathsf{Azu}_{\mathbf{W}} \hookrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}$ and $\mathsf{Azu}_{\mathbf{W}} \hookrightarrow \text{Id-Assoc}_{\mathbf{W}}$ corresponding to Azumaya algebras. That they are indeed subfunctors follows from property (1) of the next result, which also lists other standard properties of Azumaya algebras that will be used in the sequel.

PROPOSITION 3.2

Let S and S' be noetherian commutative rings with 1 and S' an S-algebra. Further let A be an associative S-algebra with identity.

(1) *If A is an Azumaya S-algebra, then $A \otimes_S S'$ is an Azumaya S'-algebra.*
(2) *If S is an algebraically closed field, then A is Azumaya over S iff it is isomorphic to the algebra $M(n, S)$ of $(n \times n)$-matrices over S for some n, i.e., over algebraically closed fields the only Azumaya algebras are the matrix algebras.*
(3) *A is an Azumaya S-algebra iff $A_{\mathfrak{p}}$ is an Azumaya $S_{\mathfrak{p}}$-algebra for every prime ideal $\mathfrak{p}$ of S and A is finitely generated as an S-module.*
(4) *If A is finitely generated and locally-free, and if $A \otimes_S S'$ is an Azumaya S'-algebra, and further if S' is faithfully-flat over S, then A is an Azumaya S-algebra.*
(5) *If A is locally free of finite positive rank as an S-module, then A is an Azumaya S-algebra iff $A \otimes_S K$ is an Azumaya K-algebra for every algebraically closed field K which is an S-algebra.*
(6) *Let S be a complete local ring with maximal ideal $\mathfrak{m}$. If A is an Azumaya S-algebra such that $A/\mathfrak{m}A \cong M(n, S/\mathfrak{m})$ then $A \cong M(n, S)$.*
(7) *Let P be a finitely generated projective S-module. Then the S-algebra $\mathrm{End}_S(P)$ is Azumaya.*

Properties (1), (3) and (4) are easy. As for the non-trivial part of (2), if A is Azumaya over S, then by [7], Chap. 9, Theorem 9.7, A is isomorphic to an algebra of square matrices of order n (for some n) with entries in a finite-dimensional central division algebra D over S. But since S is an algebraically closed field, $D = S$. Property (5) above can be deduced from (3) and (4) and an application of Nakayama's lemma. Property (6) is Lemma 5.1.16 in Chap. III of [9]. The proof of (7) uses (4), (5) and (2). The following results shall be used in showing the representability of the subfunctors of Azumaya algebras by open subschemes.

PROPOSITION 3.3

(1) *Let T be an X-scheme and $\mathcal{A} \in \text{Id-Assoc}_{\mathbf{W}}(T)$. Then the subset*

$$U(T, \mathcal{A}) := \{t \in T \mid \mathcal{A}_t \text{ is an Azumaya } \mathcal{O}_{T,t}\text{-algebra}\}$$

is an open (possibly empty) subset. When $U(T, \mathcal{A})$ is non-empty, denote by the same symbol the canonical open subscheme structure. Then if $f : T' \longrightarrow T$ is an X-morphism such that the topological image intersects $U(T, \mathcal{A})$, then $U(T', f^*(\mathcal{A})) \cong U(T, \mathcal{A}) \times_T T'$ as open subschemes of T'. Further $U(T, \mathcal{A}) \hookrightarrow T$ is an affine morphism.

(2) *$U(T, \mathcal{A})$ is the maximal open subset restricted to which $\mathcal{A}$ is Azumaya.*
(3) *Further let $f : T' \longrightarrow T$ be a morphism of X-schemes such that $f^*(\mathcal{A}) \in \mathsf{Azu}_{\mathbf{W}}(T')$, i.e., the induced algebra is Azumaya. Then f factors through the open subscheme $U(T, \mathcal{A})$ defined above.*

The proof that $U(T, \mathcal{A})$ is open follows from an application of Nakayama's lemma. The proof of (2) follows from (3) of Proposition 3.2. The proof of (3) uses assertions (1)–(5) of Proposition 3.2. From these the second assertion of (1) follows. Hence for the third assertion of (1), one may as well assume that $T = \mathrm{Spec}(B)$ is affine and that $\mathcal{A}$ is free; in which case $U(T, \mathcal{A})$ is by definition the open subset where a homomorphism of free B-modules of the same finite rank is an isomorphism and is hence a principal open subset.

Theorem 3.4

(1) $\mathsf{Azu}_{\mathbf{W}}$ (*respectively* $\mathsf{Id}\text{-}w\text{-}\mathsf{Azu}_{\mathbf{W}}$) *is represented by a* $\mathrm{GL}_{\mathbf{W}}$*-stable* (*resp.* $\mathrm{Stab}(w)$*-stable*) *open subscheme* $\mathrm{Azu}_{\mathbf{W}} \hookrightarrow \text{Id-Assoc}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}} \hookrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}$) *and the canonical open immersion is an affine morphism.*

(2) $\mathrm{Azu}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}}$) *is separated* (*resp. affine*) *and of finite type over* X, *and the construction* $\mathrm{Azu}_{\mathbf{W}} \longrightarrow X$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}} \longrightarrow X$) *base-changes well.*

(3) *The restriction of* $\mu_w : \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Assoc}_{\mathbf{W}} \longrightarrow \text{Id-Assoc}_{\mathbf{W}}$ *to the open subscheme* $\mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Azu}_{\mathbf{W}}$ *factors by a morphism* μ'_w, *into* $\mathrm{Azu}_{\mathbf{W}}$, *which is a Zariski-locally-trivial principal* $\mathrm{Stab}(w)$*-bundle.*

(4) *Further,* $\mathrm{Azu}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}}$) *is smooth of relative dimension* $m^4 - m^2 + 1$ (*resp. of relative dimension* $(m^2-1)^2$) *and geometrically irreducible over* X, *where* $m^2 := \mathrm{rank}_X(\mathcal{W})$.

Proof. First of all notice that property (7) of Proposition 3.2 shows that the sets $\mathsf{Azu}_W(T)$ (resp. $\mathsf{Id}\text{-}w\text{-}\mathsf{Azu}_{\mathbf{W}}(T)$) are non-empty for any X-scheme T to which the pull-back of $\mathcal{W}$ becomes trivial (resp. and further the pull-back of w becomes part of a global basis). Since $\mathsf{Alg}_{\mathbf{W}}$ is represented by $\mathrm{Alg}_{\mathbf{W}}$, let $\mathcal{B}$ be the universal algebra structure on $\mathcal{W} \otimes_X \mathrm{Alg}_{\mathbf{W}}$ corresponding to the identity morphism of $\mathrm{Alg}_{\mathbf{W}}$. A little bit of writing down shows that the canonical algebra structure on $\mathcal{B}$ corresponds to the diagonal morphism

$$\Delta_{\mathrm{Alg}_{\mathbf{W}}/X} : \mathrm{Alg}_{\mathbf{W}} \hookrightarrow \mathrm{Alg}_{\mathbf{W}} \times_X \mathrm{Alg}_{\mathbf{W}}.$$

Then the representability of $\mathsf{Id\text{-}Assoc}_{\mathbf{W}}$ by $\text{Id-Assoc}_{\mathbf{W}}$ (Proposition 2.5) shows that the pull-back $\mathcal{B}'$ of $\mathcal{B}$ to $\text{Id-Assoc}_{\mathbf{W}}$ (resp. $\mathcal{B}_w$ of $\mathcal{B}$ to $\text{Id-}w\text{-Assoc}_{\mathbf{W}}$) is the universal associative algebra structure with identity (resp. with identity $w \otimes_X \text{Id-}w\text{-Assoc}_{\mathbf{W}}$). With the notations of Proposition 3.3, it is routine using the assertions of that proposition to verify that $\mathrm{Azu}_{\mathbf{W}} := U(\text{Id-Assoc}_{\mathbf{W}}, \mathcal{B}')$ (resp. $\text{Id-}w\text{-Azu}_{\mathbf{W}} := U(\text{Id-}w\text{-Assoc}_{\mathbf{W}}, \mathcal{B}_w)$) represents $\mathsf{Azu}_{\mathbf{W}}$ (resp. $\mathsf{Id}\text{-}w\text{-}\mathsf{Azu}_{\mathbf{W}}$) and the rest of the assertions of the theorem in the first statement. Note therefore that the pull-back of $\mathcal{B}'$ (resp. of $\mathcal{B}_w$) to $\mathrm{Azu}_{\mathbf{W}}$ (resp. to $\text{Id-}w\text{-Azu}_{\mathbf{W}}$) is the universal Azumaya algebra structure (resp. also with identity w). The functors $\mathsf{Azu}_{\mathbf{W}}$ and $\mathsf{Id}\text{-}w\text{-}\mathsf{Azu}_{\mathbf{W}}$ base-change well since the property of being Azumaya is preserved under base-change (property (1) of Proposition 3.2) and since $\mathrm{Alg}_{\mathbf{W}}$ and $\text{Id-}w\text{-Assoc}_{\mathbf{W}}$ base-change well; further $\mathrm{Alg}_{\mathbf{W}}$ is affine and of finite type over X, $\text{Id-Assoc}_{\mathbf{W}}$ is separated and of finite type over X by Proposition 2.5 and $\text{Id-}w\text{-Assoc}_{\mathbf{W}} \hookrightarrow \mathrm{Alg}_{\mathbf{W}}$ is a closed immersion. From these facts the assertions in the second statement of the theorem follow. As for the third statement, it is easy to check functorially that the restriction of $\mu_w : \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Assoc}_{\mathbf{W}} \longrightarrow \text{Id-Assoc}_{\mathbf{W}}$ to the open subscheme $\mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Azu}_{\mathbf{W}}$ factors by a morphism μ'_w, into $\mathrm{Azu}_{\mathbf{W}}$, which is in fact the base-change of μ_w to $\mathrm{Azu}_{\mathbf{W}}$. Hence by Corollary 2.8 which says that μ_w is a Zariski-locally-trivial principal $\mathrm{Stab}(w)$-bundle, one may conclude the same of $\mu'_w : \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Azu}_{\mathbf{W}} \longrightarrow \mathrm{Azu}_{\mathbf{W}}$. Given this and the easy fact that $\mathrm{Stab}(w)$ is smooth, surjective, affine and geometrically irreducible of relative dimension $m^4 - m^2$ over X, it is clear that in order to prove the assertions in the last statement of the theorem, it is enough to prove only those concerning Id-w-Azu$_{\mathbf{W}}$. Since Id-w-Azu$_{\mathbf{W}}$ base-changes well, going to geometric points of X, in view of property (2) of Proposition 3.2 the use of the theorem of Skolem–Noether shows that Id-w-Azu$_{\mathbf{W}}$ is geometrically irreducible and has geometric fibers of the claimed dimension. When the base X is integral, by considering the orbit morphism Stab$(w) \longrightarrow$ Id-w-Azu$_{\mathbf{W}}$ corresponding to the universal Azumaya algebra structure in Id-w-$\mathsf{Azu}_{\mathbf{W}}$(Id-$w$-Azu$_{\mathbf{W}}$) and noting that this morphism is surjective and that Stab(w) is integral, one gets that in this case Id-w-Azu$_{\mathbf{W}}$ is irreducible. Now using properties (6) and (2) of Proposition 3.2, since Id-w-Azu$_{\mathbf{W}}$ is of finite type over X, it is easy to check that Id-w-Azu$_{\mathbf{W}} \longrightarrow X$ is geometrically regular by verifying the formal smoothness criterion at any closed point of any geometric fiber. It follows that the fibers of this morphism are integral smooth varieties of the claimed dimension. Finally only the flatness of Id-w-Azu$_{\mathbf{W}}$ over X remains to be checked. Since X can be covered by affine opens restricted to each of which $\mathbf{W}$ becomes trivial and w part of a global basis, and Id-w-Azu$_{\mathbf{W}}$ base-changes well, we may assume $X = \operatorname{Spec}(\mathbb{Z})$. Now we observe that the structure morphism of Id-w-Azu$_{\mathbf{W}}$ is equidimensional (13.2.2 EGA IV, Err. IV.34, [6] – the smoothness, irreducibility and equidimensionality of the fibers is used here); using this and applying Chevalley's criterion (ii) of Corollary 14.4.4 of EGA IV [6] shows that the structure morphism is universally open. Feeding this into Theorem 15.2.2 of EGA IV [6] shows that the structure morphism is actually flat. Q.E.D.

We are interested in studying the specializations of Azumaya algebra structures. In the topological sense these are points of the closure of the space underlying Azu$_{\mathbf{W}}$ (resp. Id-w-Azu$_{\mathbf{W}}$) in Id-Assoc$_{\mathbf{W}}$ (resp. in Id-w-Assoc$_{\mathbf{W}}$). To give a scheme-theoretic interpretation for these spaces of limits, one naturally turns to the notion of schematic image. This notion and its properties are recalled next, after which comes the theorem that the limiting schemes of Azumaya algebras are smooth and base-change well in the case when $\mathbf{W}$ is of rank 4 over X.

DEFINITION 3.5 (Definitions 6.10.1–2, Chap. I, EGA I [5])

Let $f : X \longrightarrow Y$ be a morphism of schemes. If there exists the smallest closed subscheme $Y' \hookrightarrow Y$ such that the inverse image scheme $f^{-1}(Y') := Y' \times_Y ({}_f X)$ is equal to X, one calls Y' the *schematic image* of f (or of X in Y under f). If X were a subscheme of Y and f the canonical immersion, and if f has a schematic image Y', then Y' is called the *schematic limit* or the *limiting scheme* of the subscheme $X \overset{f}{\hookrightarrow} Y$.

PROPOSITION 3.6 (Proposition 6.10.5, Chap. I, EGA I)

The schematic image Y' of X by a morphism $f : X \longrightarrow Y$ exists in the following two cases: (1) $f_*(\mathcal{O}_X)$ *is a quasi-coherent $\mathcal{O}_Y$-module, which is for example the case when f is quasi-compact and quasi-separated*; (2) X *is reduced.*

PROPOSITION 3.7

In each of the following statements whenever a schematic image is mentioned, we assume that one of the two hypotheses of the above proposition is true so that the schematic image does exist.

(1) *Let Y' be the schematic image of X under a morphism $f : X \longrightarrow Y$ and let f factor as $X \xrightarrow{g} Y' \overset{j}{\hookrightarrow} Y$. Then Y' is topologically the closure of $f(X)$ in Y, the*

morphism g is schematically dominant (*i.e.*, $g^{\#}:\mathcal{O}_{Y'}\longrightarrow g_*(\mathcal{O}_X)$ *is injective*) *and the schematic image of X in Y'* (*under g*) *is Y' itself. If X is reduced* (*respectively integral*) *then the same is true of Y'.*

(2) *The schematic image of a closed subscheme under its canonical closed immersion is itself.*

(3) (*Transitivity of schematic image*). *Let there be given morphisms $X\xrightarrow{f}Y\xrightarrow{g}Z$, such that the schematic image Y' of X under f exists, and further such that if g' is the restriction of g to Y', the schematic image Z' of Y' by g' exists. Then the schematic image of X under $g\circ f$ exists and equals Z'.*

(4) *Let $f:X\longrightarrow Y$ be a morphism which factors through a closed subscheme Y_1 of Y by a morphism $f_1:X\longrightarrow Y_1$. Then the scheme-theoretic image Y' of X in Y is the same as the scheme-theoretic image Y_1' of X in Y_1 considered canonically as closed subscheme of Y.*

(5) *If $f:X\longrightarrow Y$ has a schematic image Y' then f is schematically dominant iff $Y'=Y$.*

(6) *The formation of schematic image commutes with flat base-change*: *i.e., if $f:X\longrightarrow Y$ is a morphism of S-schemes which has a schematic image Y' then for a flat morphism $S'\longrightarrow S$, one has that the induced S'-morphism $f\times_S S':X\times_S S'\longrightarrow Y\times_S S'$ has a schematic image and it may be canonically identified with $Y'\times_S S'$. In particular this means that the formation of schematic image is local over the base.*

Assertions (1) and (3) are respectively Propositions 6.10.5 and 6.10.3 in EGA I. The defining property of schematic image gives (2), while (4) can be deduced from the first three. As for (5), from (1) it follows that $Y'=Y$ implies $f=g$ is schematically dominant. For the other way around, one uses the following characterization of a schematically dominant morphism in Proposition 5.4.1 of EGA I: if $f:X\longrightarrow Y$ is a morphism of schemes, then f is schematically dominant iff for every open subscheme U of Y and every closed subscheme Y_1 of U such that there exists a factorization $f^{-1}(U)\xrightarrow{g_1}Y_1\overset{j_1}{\hookrightarrow}U$, of the restriction $f^{-1}(U)\longrightarrow U$ of f (where j_1 is the canonical closed immersion), one has $Y_1=U$ – given f is schematically dominant, one just has to take $U=Y$, $Y_1=Y'$ and $g_1=g$. Assertion (6) follows from statement (ii) (a) of Theorem 11.10.5 of EGA IV [6].

Theorem 3.8

(1) *The open immersion* $\mathrm{Azu}_{\mathbf{W}}\hookrightarrow\text{Id-Assoc}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}}\hookrightarrow\text{Id-}w\text{-Assoc}_{\mathbf{W}}$) *has a schematic image denoted* $\text{Sp-Azu}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$) *which is separated* (*resp. affine*) *and of finite type over X and is naturally a* $\mathrm{GL}_{\mathbf{W}}$*-stable* (*resp.* $\mathrm{Stab}(w)$*-stable*) *closed subscheme of* $\text{Id-Assoc}_{\mathbf{W}}$ (*resp. of* $\text{Id-}w\text{-Assoc}_{\mathbf{W}}$), *the action extending the natural one on the open subscheme* $\mathrm{Azu}_{\mathbf{W}}$ (*resp.* $\text{Id-}w\text{-Azu}_{\mathbf{W}}$).

(2a) $\text{Sp-Azu}_{\mathbf{W}}$ *is the schematic image of* $\mathrm{GL}_{\mathbf{W}}\times_X\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ *under the composition of the canonical closed immersion into* $\mathrm{GL}_{\mathbf{W}}\times_X\text{Id-}w\text{-Assoc}_{\mathbf{W}}$ *followed by μ_w.*

(2b) *The induced morphism* $\mu_w'':\mathrm{GL}_{\mathbf{W}}\times_X\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}\longrightarrow\text{Sp-Azu}_{\mathbf{W}}$ *is in fact the base-change of μ_w and is hence a Zariski-locally-trivial principal* $\mathrm{Stab}(w)$*-bundle.*

(3a) *When the rank of* $\mathbf{W}$ *over X is* 4, $\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ *is locally* (*over X*) *isomorphic to relative nine-dimensional affine space*; *in fact over every open affine subscheme U of X where* $\mathbf{W}$ *becomes trivial and w becomes part of a global basis,* Id-w-Sp-Azu$_{\mathbf{W}}|_U \cong A^9_U$. *Thus* Id-$w$-Sp-Azu$_{\mathbf{W}}$ *is smooth of relative dimension* 9 *and geometrically irreducible over* X.

(3b) *When* $\mathrm{rank}_X(\mathbf{W}) = 4$, *the construction* Id-$w$-Sp-Azu$_{\mathbf{W}} \longrightarrow X$ *base-changes well.*

(4a) *When* $\mathrm{rank}_X(\mathbf{W}) = 4$, Sp-Azu$_{\mathbf{W}}$ *is smooth*$/X$ *of relative dimension* 13 *and geometrically irreducible*$/X$.

(4b) *When* $\mathrm{rank}_X(\mathbf{W}) = 4$, *the construction* Sp-Azu$_{\mathbf{W}} \longrightarrow X$ *base-changes well.*

We remark that due to the fact that the formation of the schematic image is local on the base (property (6) of Proposition 3.7), it is enough to prove property (3a) for the case when X is affine, $\mathbf{W}$ is trivial over X and w becomes part of a global basis. This will require the use of semi-regular quadratic forms which are recalled in the next section and will be the goal of §5. Notice that (3b) (resp. (4b)) is a consequence of the smoothness and geometric irreducibility asserted in (3a) (resp. (4a)), the defining property of schematic image and the fact that the corresponding scheme of Azumaya algebras base-changes well (statement (2) of Theorem 3.4). Further note that given the fact that $\mathrm{Stab}(w)$ is affine, geometrically irreducible and smooth of relative dimension 12 over X in the case $\mathbf{W}$ is of rank 4, (4a) follows from (3a) and (2b). So, for now we shall only prove (1), (2a) and (2b), and only (3a) will need to be proved for the affine case as noted above. Since an affine morphism is quasi-compact and separated, the existence of the schematic images in (1) of the theorem follow from (1) of Theorem 3.4 and case (1) of Proposition 3.6. The rest of the properties like separatedness/affineness/finite-typeness now follow from (2) of Theorem 3.4, while the assertions on the extension of the natural action on the open subscheme (by the relevant groupscheme) to an action on the limiting scheme may be verified using the defining property of the schematic image involved. In effect one may show that an automorphism of a scheme T which leaves an open subscheme U stable will also leave stable the limiting scheme of U in T (of course here one assumes that the canonical open immersion $U \hookrightarrow T$ is a quasi-compact open immersion, which ensures the existence of the limiting scheme). The assertion of (2a) follows by using the properties (1)–(5) of the schematic image given in Proposition 3.7. As for (2b), one immediately sees that there is a natural morphism of X-schemes

$$\begin{aligned}&\mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}\\ &\quad\longrightarrow \text{Sp-Azu}_{\mathbf{W}} \times_{\text{Id-Assoc}_{\mathbf{W}}} ({}_{\mu_w}(\mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Assoc}_{\mathbf{W}}))\end{aligned}$$

which is seen to be a closed immersion and which needs to be shown to be an isomorphism. Hence it is enough to show that this morphism is functorially surjective. We shall deduce it from the following more general result, which simply put, says that for a locally-trivial principal G-bundle, G-stable closed subschemes of the top space descend, and the natural candidate, viz., the schematic image under the restriction of the bundle projection, fits the bill.

Theorem 3.9. *Let S be a base scheme, G an S-group scheme which is flat of finite type and separated over S, and $f : B \longrightarrow T$ an S-morphism which is also a Zariski-locally-trivial principal G-bundle* (*with the G-action on B on the right*). *Let $Q \overset{\iota}{\hookrightarrow} B$ be a G-stable closed subscheme. Then the schematic image $Z \overset{\iota'}{\hookrightarrow} T$ of Q under the composition $f \circ i$ is the descent of Q under f, i.e., there exists a natural isomorphism $\beta : Q \xrightarrow{\cong} (Z_{\iota'}) \times_T \left({}_f B\right)$*

such that $\iota = p_B \circ \beta$. Thus Q is naturally identified with the locally-trivial principal G-bundle given by the pull back of f to Z. Moreover, when G is smooth over S, it follows that Q is smooth over S iff Z is smooth over S.

Proof. Note that $f \circ i$ is quasi-compact and separated and so the schematic image Z of Q exists by case (1) of Proposition 3.6. Also note that under the given assumptions on G, f is universally submersive and so the G-stability of Q and the G-invariance of f imply that Q is topologically the full inverse image of the closed set below which by property (1) of Proposition 3.7 is seen to be the underlying topological space of the schematic image Z. There is an obvious natural closed immersion $\beta : Q \hookrightarrow (Z_{\iota'}) \times_T ({}_f B)$ which we need to show is functorially surjective. Since this morphism is functorially injective, and since the formation of schematic image is local over the base (property (6) of Proposition 3.7), it can be seen that one may reduce to the case of a trivial principal G-bundle, i.e., $B := T \times_S G$. Let $s : T \longrightarrow B$ be the section to f induced by the identity section of G over S. Define Z_1 to be the scheme-theoretic intersection of the closed subschemes Q and T in B, i.e., $Z_1 := (Q_\iota) \times_B ({}_s T)$. Let $s' : Z_1 \hookrightarrow Q$ be the base-change of s. Define $\alpha : Z_1 \times_S G \longrightarrow Q$ to be the composition $\mu_Q \circ (s' \times id_G)$ where $\mu_Q : Q \times_S G \longrightarrow Q$ is the canonical right action of G on Q induced from that on $B = T \times_S G$. Now using the language of valued points one checks that α is functorially bijective, hence an isomorphism. Since $Z_1 \times_S G \cong Z_1 \times_T B$ canonically, a little bit of routine writing-down shows that $f \circ i$ factors through Z_1. Due to the defining property of the schematic image Z, this induces a closed immersion $\iota'_1 : Z \hookrightarrow Z_1$ which in turn induces a closed immersion $\iota''_1 : (Z_{\iota'}) \times_T ({}_f B) \hookrightarrow Z_1 \times_T B$. Now using the facts that $Z_1 \times_T B \longrightarrow Z_1$ is functorially surjective (being a trivial bundle) and that $Z_1 \times_T B \cong Q$ via α, one checks that ι'_1 is functorially surjective and therefore an isomorphism. So ι''_1 is also an isomorphism and from this one gets that β is functorially surjective and hence an isomorphism as wanted. Q.E.D. for Theorem 3.9

Now if in the above result, one takes $B := \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Assoc}_{\mathbf{W}}$, $T := \text{Id-Assoc}_{\mathbf{W}}$, $f := \mu_w$, $Q := \mathrm{GL}_{\mathbf{W}} \times_X \text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ (and whence $Z = \text{Sp-Azu}_{\mathbf{W}}$), $G := \mathrm{Stab}(w)$ and bears in mind that μ_w is a locally-trivial principal $\mathrm{Stab}(w)$-bundle (Corollary 2.8), one immediately gets that μ''_w is indeed the base-change of μ_w as wanted. We remind the reader now that we only need to prove assertion (3a) of Theorem 3.8 for the case when $X = \mathrm{Spec}(R)$ is an affine scheme, $\mathcal{W}$ corresponds to a trivial R-module W of rank 4, and w becomes part of a global basis for W. We shall show in this case that $\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ is isomorphic to the nine-dimensional affine space given by the fiber-product of the six-dimensional affine space of quadratic forms on a free rank 3 R-module and a suitable commutative affine subgroupscheme of $\mathrm{Stab}(w)$ isomorphic to three-dimensional affine space. This involves the use of the notion of semi-regular quadratic form, generalities on which we shall deal with in the next section.

4. Generalities on semi-regular quadratic forms

As indicated in the last section, one needs to bring in quadratic forms for the proof of (3a) of Theorem 3.8. Seshadri's method of proving (3a) for the case $X = \mathrm{Spec}(k)$, k an algebraically closed field, involves firstly defining a morphism from the space of quadratic forms on a three-dimensional vector space into $\text{Id-}w\text{-Assoc}_{\mathbf{W}}$. This essentially associates a quadratic form to its even Clifford algebra. That this morphism takes values in Id-w-Sp-Azu$_{\mathbf{W}}$ depends on the fact that the even Clifford algebra of a regular quadratic form is isomorphic to the algebra of (2×2)-matrices with entries in k. Further using this morphism, another morphism is defined to establish (3a) and to conclude that this latter morphism is proper and functorially injective, Seshadri computes this morphism. In this computation the bilinear form associated to the quadratic form, and not the quadratic form itself, is involved, and therefore some terms in the computation (crucial for concluding the properness and functorial injectivity) involve a factor of 2, and hence vanish in char. 2. Other fundamental problems encountered in char. 2 arise from the facts that the mapping that associates a quadratic form to its symmetric bilinear form is no longer bijective and that when the quadratic module is of odd rank, there do not exist regular quadratic forms. The remedy for all this is to consider semi-regular quadratic forms, a concept of Kneser [8] and elaborated upon by Knus in [9], which in fact works over an arbitrary base ring (and hence in a characteristic-free way) and further reduces to the usual notion of regular form in characteristics $\neq 2$. We therefore devote this section to recall this notion and its properties. We also use this opportunity to show how a non-degenerate form in the sense of Dieudonné [3] is the same as a semi-regular form.

Throughout this section, R denotes a commutative ring. A pair (V, q) consisting of a module V over R and a quadratic form q on V will be referred to as a quadratic module. Recall that a quadratic form q is by definition a map $q : V \longrightarrow R$ satisfying (1) $q(r \cdot v) = r^2 \cdot q(v),\ v \in V,\ r \in R$ and (2) $b_q : V \times V \longrightarrow R$ given by $(u, v) \mapsto q(u + v) - q(u) - q(v)$ is R-bilinear. As usual, we call b_q the bilinear form 'associated' to q. Before proceeding further, let us recall the usual definition of regularity (also called non-singularity). Let (V, q) be a quadratic module with V finitely generated and projective of constant rank n over R. For any n-tuple $\{f_i \mid 1 \le i \le n\}$ of elements of V, the element $\Delta_q(\{f_i\}) := \det(b_q(f_i, f_j))$ is an element of R; in the case when the module is free and the chosen n-tuple is an R-basis, this element is called the *discriminant* of (V, q) with respect to this basis. Its class modulo $(R^*)^2$ is independent of the choice of the basis. Let $\Delta_q(V)$ denote the *discriminant-ideal* in R generated by the elements $\Delta_q(\{f_i\})$ for all possible n-tuples. (V, q) is said to be a *regular quadratic module* and q a *regular quadratic form* if $\Delta_q(V)$ is all of R. In order to define semi-regularity on the other hand, one needs the following fundamental result.

Lemma 4.1 (Lemma 3.1.2, Chap. IV, [9]). *Consider the quadratic module (R_0^n, q_0) over $R_0 := \mathbb{Z}[\zeta_i, \zeta_{ij}]$ where $(1 \le i, j \le n,\ i < j)$ with the standard basis $\{e_i \mid 1 \le i \le n\}$ where $q_0(e_i) := \zeta_i,\ b_{q_0}(e_i, e_j) := \zeta_{ij}\ (i < j)$. In other words, $q_0(\Sigma_i x_i e_i = \Sigma_i \zeta_i x_i^2 + \Sigma_{i<j} \zeta_{ij} x_i x_j$. Note that $b_{q_0}(e_i, e_i) = 2q_0(e_i) = 2\zeta_i$. Then the discriminant $d(\{e_i\}) :=$ det $(b_{q_0}(e_i, e_j))$ of the matrix of b_{q_0} equals $2\, P_n(\zeta_i, \zeta_{ij})$ for a uniquely determined polynomial P_n in R_0.*

DEFINITION 4.2 (§3, Chap. IV [9])

Let (V, q) be a quadratic module with the underlying R-module projective of constant odd rank n. For any n-tuple $\{f_i \mid 1 \le i \le n\}$ of elements of V, the element $d_q(\{f_i\}) := P_n(q(f_i), b_q(f_i, f_j))$ is an element of R; in the case when the module is free and the chosen n-tuple is an R-basis, this element is called the *half-discriminant* of (V, q) with respect to this basis. Let $d_0(V, q)$ denote the *half-discriminant-ideal* in R generated by the elements $d_q(\{f_i\})$ for all possible n-tuples. (V, q) is said to be a *semi-regular quadratic module* and q a *semi-regular quadratic form* if $d_0(V, q)$ is all of R.

PROPOSITION 4.3

Let (V, q) be a quadratic module with the underlying R-module projective of constant odd rank.

(1) *When V is free, q is semi-regular if and only if there exists a basis $\{f_i\}$ with respect to which the half-discriminant is a unit. If this is the case then the half-discriminant with respect to any basis is a unit.*
(2) *When 2 is a unit in R and V is free, q is semi-regular if and only if it is regular.*
(3) *The orthogonal direct sum of a semi-regular quadratic module and a regular quadratic module is again a semi-regular quadratic module.*
(4) *For a quadratic module (V, q) of odd rank, regularity is a very strong condition: it implies that 2 is a unit of R. Hence there are no regular quadratic forms over modules of odd rank in char. 2. However, semi-regular quadratic forms do exist in all ranks in all characteristics.*

Statement (1) is proved in §3, Chap. IV of [9]. Statement (2) follows from the fact that relative to any basis, the half-discriminant and the discriminant differ by the factor of the unit $2 \in R^*$; note in this situation also that the associated bilinear form may be used to identify V with its dual for any fixed basis. Assertion (3) is in §3, Ch. IV of [9], and essentially follows from the observation that in the free case, the half-discriminant of the orthogonal sum is the product of the half-discriminant of the semi-regular summand and the discriminant of the regular summand. The first assertion of (4) essentially boils down to realizing that an alternating matrix in char. 2 of odd rank is of determinant zero. As for the last assertion of (4), first let V be free of rank 3 over R with basis $\{e_1, e_2, e_3\}$ and S a commutative R-algebra. Then the quadratic form $q : V \otimes_R S \longrightarrow S$ given by

$$x(e_1 \otimes 1) + y(e_2 \otimes 1) + z(e_3 \otimes 1) \longmapsto yz - x^2$$

is easily checked to be semi-regular. Now using (3) and the fact that regular quadratic forms exist for modules of even rank in all characteristics, one gets (4). The following is Proposition 3.1.5, §3, Chap. IV, [9], and shows that semi-regularity is well-behaved.

PROPOSITION 4.4

Let (V, q) be a quadratic module of odd rank over a commutative ring R. The following properties are equivalent:

(1) *q is semi-regular.*
(2) *$q \otimes (R/\mathfrak{m})$ is semi-regular over $R/\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R.*
(3) *$q \otimes R_{\mathfrak{m}}$ is semi-regular over $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of R.*
(4) *$q \otimes S$ is semi-regular over S for some faithfully-flat R-algebra S.*

Thus semi-regularity is preserved under extension of scalars.

We now recall the notion of non-degeneracy and show that it is the same as semi-regularity. To begin with, given a quadratic R-module (V, q), its *radical*, denoted $V(q)$, is defined to be the subset of V defined by

$$\{v \in V \mid b_q(v, v') = 0 \; \forall v' \in V\}.$$

Then we have the following elementary results.

Lemma 4.5. *With the above notations,* (1) *the radical of* (V, q) *is a submodule of* V (*it is the kernel of the linear map* $V \longrightarrow V^*$ *which sends* $v \in V$ *to the linear form* $v' \mapsto b_q(v, v')$ *and the radical is also the left-* (*and right-*) *kernel of the bilinear form* b_q). (2) *When* V *is a finite dimensional vector space over a field* R, *the quadratic form* q *is regular if and only if its radical is zero.* (*So in general, if* q *is regular and* V *is of odd rank, it follows that* 2 *is necessarily a unit in* R. *Thus for a free module in char.* 2 *it follows that the radical is non-zero and hence every quadratic form is necessarily non-regular.*)

Recall the following notion from Chap. V, (23.5), of Borel [2] (who in turn quotes Dieudonné [3]): for a quadratic module (V, q) of odd dimension over a field k of char. 2, q is said to be *non-degenerate* iff $V(q)$ is one-dimensional and further for each $v \in V(q) - \{0\}$, $q(v)$ is non-zero. Now extend this definition to the case char$(k) \neq 2$ as follows: q is non-degenerate iff it is regular (i.e., iff $V(q) = 0$).

DEFINITION 4.6

Let R be any commutative ring and let (V, q) be a quadratic module such that the R-module V is finitely generated and projective of constant odd rank. Call q *non-degenerate* if for each maximal ideal $\mathfrak{m}$ of R, the quadratic form $q \otimes (R/\mathfrak{m})$ is non-degenerate in the above sense.

It can be seen that for any quadratic module (V, q) of odd rank ≥ 3 over a *perfect* field of characteristic 2, the rank of b_q cannot be zero (i.e., $V(q)$ cannot be all of V) under the hypothesis that q does not vanish on non-zero elements of its radical. In particular, in the above definition, when $R = k$ a perfect field of characteristic 2 and $n = 3$, the requirement that $V(q)$ be one-dimensional is redundant.

Theorem 4.7. *Let* (R_0^n, q_0) *be the generic quadratic module of Lemma* 4.1. *Let* $\{x_1, \ldots, x_n\}$ *be indeterminates and let* I_0 *be the ideal in* $R_0[x_1, \ldots, x_n]$ *generated by the expressions that are the rows of the column vector defined by*

$$V_0 := (b_{q_0}(e_i, e_j)) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

i.e., if we denote the i'th row by $V_{0,i}$ *then* $I_0 := (V_{0,1}, \ldots, V_{0,n})$. *Further, for each* k, $1 \leq k \leq n$, *let* $I_{0,k}$ *denote the ideal generated by* $\{V_{0,i} \mid i \neq k\}$ *so that* $I_{0,k} \subset I_0$. *Then in the ring* $R_0[x_1, \ldots, x_n]/I_{0,k}$ (*and hence also in the ring* $R_0[x_1, \ldots, x_n]/I_0$) *one has the following identity*:

$$x_k^2 P_n(\zeta_i, \zeta_{ij}) = \left(\sum_{i=1}^{n} x_i^2 \zeta_i + \sum_{1 \leq i < j \leq n} x_i x_j \zeta_{ij} \right) (M_{kk}(b_{q_0}(e_i, e_j)))$$

where $2P_n = \det(b_{q_0}(e_i, e_j))$ *as noted just before Definition* 4.2 *and where* M_{kk} *denotes the* (k, k)*-minor of* $(b_q(e_i, e_j))$ *with the convention that for* $n = 1$, $M_{11}(b_{q_0}(e_1, e_1)) := 1$.

Proof. For $n = 1$ the formula reads $x_1^2 P_1(\zeta_1) = x_1^2 \zeta_1$ which follows from the very definition $P_1(\zeta_1) := \zeta_1$. Hence assume $n \geq 3$. One then has, say for $k = 1$:

$$2x_1^2 P_n(\zeta_i, \zeta_{ij}) = x_1^2 \det(b_{q_0}(e_i, e_j)) = x_1^2 \begin{vmatrix} 2\zeta_1 & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{12} & 2\zeta_2 & \cdots & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{1n} & \zeta_{2n} & \cdots & 2\zeta_n \end{vmatrix}.$$

Now one performs the following elementary operations on the determinant in the right side of the above identity:

(1) push an x_1 into the first column of the determinant;
(2) if C_i denotes the i'th column, replace C_1 by $C_1 + \Sigma_{i\neq 1} x_i C_i$;
(3) push the remaining x_1 into the first row of the determinant;
(4) if R_i denotes the i'th row, replace R_1 by $R_1 + \Sigma_{i\neq 1} x_i R_i$.

After all this the above identity reads

$$2x_1^2 P_n(\zeta_i, \zeta_{ij}) = \begin{vmatrix} 2\Big(\sum_i x_i^2\zeta_i + \sum_{i<j} x_i x_j \zeta_{ij}\Big) & V_{0,2} & \cdots & V_{0,n} \\ V_{0,2} & 2\zeta_2 & \cdots & \zeta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{0,n} & \zeta_{2n} & \cdots & 2\zeta_n \end{vmatrix}.$$

Reading the identity modulo $I_{0,1}$ and cancelling off the factor 2, one gets the required result. Q.E.D.

Theorem 4.7 may be applied as follows to obtain a formula linking the half-discriminant and the values of the quadratic form on its radical, valid for any free quadratic module of finite odd rank over any commutative ring.

COROLLARY 4.8

Let R be any commutative ring, V a free R-module of odd rank n and q a quadratic form on V. Let $\{f_1, \ldots, f_n\}$ be an R-basis for V. Then one has for each $v = \sum_k y_k f_k$ in the radical $V(q)$, and for every k with $1 \leq k \leq n$, the following identity in R:

$$y_k^2 d_q\,(f_1, \ldots, f_n) = q(v) M_{kk}(b_q(f_i, f_j)).$$

To prove this, define the ring homomorphism $R_0[x_1, \ldots, x_n] \longrightarrow R$ by

$$\zeta_i \mapsto q(f_i),\ \ \zeta_{ij} \mapsto b_q(f_i, f_j) \text{ and } x_i \mapsto y_i.$$

Therefore, $P_n(\zeta_i, \zeta_{ij}) \mapsto d_q(\{f_j\})$ and $(\sum_i x_i^2 \zeta_i + \sum_{i<j} x_i x_j \zeta_{ij}) \mapsto q(v)$. But since $v \in V(q)$, this homomorphism factors through $R_0[x_1, \ldots, x_n]/I_0$. Now apply the previous theorem.

COROLLARY 4.9

The quadratic form q is semi-regular in the sense of 4.2 *if and only if it is non-degenerate in the sense of* 4.6.

Proof of Corollary 4.9. In view of (1) $\Leftrightarrow$ (2) of Proposition 4.4 and Definition 4.6, it is enough to consider the case when R is a field K. If char(K) is not 2, the proof follows from (3), Lemma 4.5 and (2), Prop. 4.3. So assume that char(K) is 2.

Semi-regularity $\Rightarrow$ *Non-degeneracy*: Apply the above corollary with $R = K$. By (1), Prop. 4.3, $d_q\,(f_1, \ldots, f_n)$ is non-zero. Hence for any $v \neq 0 \in V(q)$, choosing any k such that $y_k \neq 0$, the formula of the corollary implies that both $q(v)$ and $M_{kk}(b_q(f_i, f_j))$ are non-zero. Hence q is non-degenerate.

Non-degeneracy $\Rightarrow$ *Semi-regularity*: Continuing with the above notations, one sees that since $V(q)$ is one dimensional, one may choose the basis $\{f_j\}$ in such a way that (i) when $n \geq 3$, b_q is non-singular on the subspace generated by $\{f_j | j \neq n\}$ and (ii) $V(q)$ is generated by f_n. Apply the previous corollary with this choice. Taking $v = f_n$ in the formula there, one gets that $d_q\,(f_1, \ldots, f_n)$ is non-zero. Now use (1), Prop. 4.3. Q.E.D.

Thus the theory of semi-regular quadratic modules over a commutative ring developed in [9] holds good for non-degenerate quadratic modules. For example, the word 'semi-regular' may be replaced by 'non-degenerate' in Proposition 4.4, showing that the notion of non-degeneracy is preserved under extension of scalars.

5. Smoothness of limiting algebras with fixed identity

In this section, we shall prove that when $X = \mathrm{Spec}(R)$ is affine and the rank 4 vector bundle $\mathbf{W}$ on X is free with the nowhere vanishing global section w part of a global basis, then Id-w-Sp-Azu$_{\mathbf{W}} \cong A^9_R$ from which the assertion (3a) of Theorem 3.8 will follow as explained earlier. This isomorphism will be natural in the sense that firstly, the nine-dimensional affine space shall be the fiber product of a suitable commutative subgroup scheme Λ_w of $\mathrm{Stab}(w)$ isomorphic to A^3_R with the scheme $\Phi_V \cong A^6_R$ of quadratic forms on a free rank 3 module V over R; secondly the isomorphism shall be given by the morphism $\Theta : \Phi_V \times_R \Lambda_w \longrightarrow$ Id-w-Sp-Azu$_{\mathbf{W}}$ that associates a quadratic form q and an automorphism g to the algebra $g \cdot A$ where A is the algebra structure induced from the even Clifford algebra of q after identifying its underlying module as coming from $\mathbf{W}$. The notion of semi-regularity of the previous section allows us to work over any commutative ring R and in a characteristic-free way.

The affine scheme of quadratic forms Φ_V. Let Φ_V be the six-dimensional affine space over R corresponding to the rank 6 free R-module $S^2(V^\vee)$ which is the degree 2 part of the symmetric algebra $S(V^\vee)$ of $V^\vee$ over R. Let V have R-basis $\{e_1, e_2, e_3\}$ and let $\{X_1, X_2, X_3\}$ be the dual basis for $V^\vee = \mathrm{Hom}_R(V, R)$. Then

$$\{Z_{ij} := X_i.X_j,\ Z_k := X_k^2,\ 1 \leq i < j \leq 3,\ 1 \leq k \leq 3\}$$

is an R-basis for $S^2(V^\vee)$ so that the natural algebra homomorphism from the symmetric algebra over R of $S^2(V) = (S^2(V^\vee))^\vee$ to the polynomial algebra

$$R[Y_i, Y_{ij}] := R[Y_1, Y_2, Y_3, Y_{12}, Y_{13}, Y_{23}]$$

given by $Z_k^\vee \mapsto Y_k$, $Z_{ij}^\vee \mapsto Y_{ij}$ is an R-algebra isomorphism. Thus $\Phi_V \cong \mathrm{Spec}(R[Y_i, Y_{ij}])$. This isomorphism can be used to interpret Φ_V as the R-scheme of quadratic forms on V as follows. For a commutative R-algebra S with 1, we define a bijective map by associating to $(\lambda_i, \lambda_{ij}) \in \mathrm{Spec}(R[Y_i, Y_{ij}])(S)$ the quadratic form on $V_S := V \otimes_R S$ given by $\sum_i x_i(e_i \otimes 1) \mapsto \sum_i \lambda_i x_i^2 + \sum_{i<j} \lambda_{ij} x_i x_j$. We see that

this is functorial in S as well. Consider the quadratic module $(V \otimes_R R[Y_i, Y_{ij}], \mathbf{q})$ where

$$\mathbf{q} : V \otimes_R R[Y_i, Y_{ij}] \longrightarrow R[Y_i, Y_{ij}]$$

is defined by $\sum_i x_i(e_i \otimes 1) \mapsto \sum_i Y_i x_i^2 + \sum_{i<j} Y_{ij} x_i x_j$. We then see that the pair $(\Phi_V, \mathbf{q})$ represents the functor of quadratic forms on V so that $\mathbf{q}$ is the universal quadratic form. It is worth noting that under the usual identification of quadratic forms with symmetric bilinear forms valid when $2 \in R^*$, the quadratic form corresponding to $(\lambda_i, \lambda_{ij})$ would be identified with the tuple $(2 \cdot \lambda_i, \lambda_{ij})$ and this becomes a bad mapping in char. 2. It is obvious that $\Phi_V \longrightarrow X = \mathrm{Spec}(R)$ base-changes well.

The open subscheme Φ_V^{sr} of semiregular forms. For any commutative R-algebra S with 1, let $\underline{\Phi}_V^{sr}(S)$ be the subset of $\Phi_V(S)$ consisting of semi-regular forms (Definition 4.2). It is non-empty by (4), Proposition 4.3. Since semi-regularity is preserved under base-change (Proposition 4.4), one sees that $S \mapsto \underline{\Phi}_V^{sr}(S)$ is functorial. A direct computation shows that the polynomial $P_n(\zeta_i, \zeta_{ij})$ of Lemma 7.21 for $n = 3$ is given by

$$4\zeta_1\zeta_2\zeta_3 + \zeta_{12}\zeta_{13}\zeta_{23} - (\zeta_1\zeta_{23}^2 + \zeta_2\zeta_{13}^2 + \zeta_3\zeta_{12}^2).$$

Then $P_3(Y_i, Y_{ij})$ is a polynomial function on Φ_V. Since 1 is a coefficient of $P_3 \in R[\Phi_V]$, P_3 is not a zero divisor. One sees from (1) of Proposition 4.3 that $\underline{\Phi}_V^{sr}$ is an open subfunctor of Φ_V and in fact that $\underline{\Phi}_V^{sr}$ is represented by the open subscheme given by

$$\Phi_V^{sr} := \mathrm{Spec}(R[Y_i, Y_{ij}]_{P_3(Y_i, Y_{ij})})$$

where $R[Y_i, Y_{ij}]_{P_3(Y_i, Y_{ij})}$ denotes localization. The universal quadratic form $\mathbf{q}$ induces a semi-regular quadratic form $\mathbf{q}^{sr}$ on $V \otimes_R R[Y_i, Y_{ij}]_{P_3(Y_i, Y_{ij})}$. It is obvious that $\Phi_V^{sr} \longrightarrow X = \mathrm{Spec}(R)$ base-changes well.

Preliminaries on Clifford algebras. For $q \in \Phi_V(S)$ let $\mathsf{Cliff}(V_S, q)$ denote the *Clifford algebra* of the quadratic module (V_S, q). It is by definition a unital associative S-algebra with a homomorphism $i : V_S \longrightarrow \mathsf{Cliff}(V_S, q)$ of S-modules which are universal with respect to the property $i(x) \cdot i(x) = q(x) \cdot 1_{\mathsf{Cliff}(V,q)}\ \forall\, x \in V_S$. The Clifford algebra exists by Theorem (1.1.2), § 1, Chap. IV, [9]. Further $\mathsf{Cliff}(V_S, q) = \mathsf{Cliff}_0(V_S, q) \oplus \mathsf{Cliff}_1(V_S, q)$ is a $(\mathbb{Z}/2)$-graded S-algebra, with $\mathsf{Cliff}_0(V_S, q)$ consisting of even degree (or zero degree) elements and $\mathsf{Cliff}_1(V_S, q)$ consisting of odd degree (or positive degree) elements. Thus $\mathsf{Cliff}_0(V_S, q)$ is an S-subalgebra, called the *even Clifford algebra* of (V_S, q). The Clifford algebra behaves well under base-change, i.e., if S' is a commutative S-algebra, then one has a canonical identification of $(\mathbb{Z}/2)$-graded S'-algebras

$$\mathsf{Cliff}(V_S \otimes_S S', q \otimes_S S') = \mathsf{Cliff}(V_S, q) \otimes_S S'.$$

The morphism θ. Let S be a commutative R-algebra with 1, and $q \in \Phi_V(S)$. The Poincaré–Birkhoff–Witt theorem (Theorem 1.5.1, §1, Chap. IV, [9]) asserts that $i : V_S \longrightarrow \mathsf{Cliff}(V_S, q)$ identifies V_S as a submodule of $\mathsf{Cliff}(V_S, q)$, and with this identification further asserts that, since V is free of rank 3 with R-basis $\{e_i | 1 \leq i \leq 3\}$, $\mathsf{Cliff}(V_S, q)$ is free of rank $8/S$, and that $\mathsf{Cliff}_0(V_S, q)$ and $\mathsf{Cliff}_1(V_S, q)$ are free of rank $4/S$ with bases $\{1_{\mathsf{Cliff}}; (e_1 \otimes 1) \cdot (e_2 \otimes 1), (e_2 \otimes 1) \cdot (e_3 \otimes 1), (e_3 \otimes 1) \cdot (e_1 \otimes 1)\}$ and $\{e_1, e_2, e_3, (e_1 \otimes 1) \cdot (e_2 \otimes 1) \cdot (e_3 \otimes 1)\}$ respectively. Let W be the R-module corresponding to $\mathbf{W}$. Since W is also of rank

$4/R$ with w part of a basis, one completes to an R-basis $w = w_0, w_1, w_2, w_3$, and defines the S-linear isomorphism $\Psi_S : W \otimes_R S \cong \mathsf{Cliff}_0(V_S, q)$ that maps $w_0 \otimes 1 \mapsto 1_{\mathsf{Cliff}_0} = 1_{\mathsf{Cliff}}$, $w_1 \otimes 1 \mapsto (e_1 \otimes 1)\cdot(e_2 \otimes 1)$, $w_2 \otimes 1 \mapsto (e_2 \otimes 1)\cdot(e_3 \otimes 1)$, $w_3 \otimes 1 \mapsto (e_3 \otimes 1)\cdot(e_1 \otimes 1)$. In particular taking $S = R[Y_i, Y_{ij}]$, one gets an associative $R[Y_i, Y_{ij}]$-algebra structure on $W \otimes_R R[Y_i, Y_{ij}]$ with unit $w \otimes_R 1$ and hence a morphism

$$\theta : \Phi_V \longrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}.$$

The morphism Θ. Continuing with the above notations, if one identifies W with R^4 by mapping the chosen basis onto the standard basis (column) vectors, then one sees that for any commutative R-algebra S with 1, the subgroup $\mathrm{Stab}(w)(S) \subset \mathrm{GL}_{\mathbf{W}}(S)$ may be identified with the subgroup of $GL(4, S)$ consisting of matrices of the form

$$\begin{pmatrix} 1 & t_1 \; t_2 \; t_3 \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \text{ with } \mathbf{B} \in \mathrm{GL}(3, S).$$

Let $\underline{\Lambda}_w$ be the subgroup-subfunctor of $\mathrm{Stab}(w)$ defined as follows: Let $\underline{\Lambda}_w(S)$ be the subgroup of $\mathrm{Stab}(w)(S)$ consisting of matrices of the above form and further such that $\mathbf{B} = \mathbf{I}_3$ is the 3×3-identity matrix in $\mathrm{GL}(3, S)$. Then $\underline{\Lambda}_w$ is represented by a closed normal subgroupscheme Λ_w of $\mathrm{Stab}(w)$ and $\Lambda_w \cong A^3_R$. To understand the relevance of Λ_w with θ, let $\underline{0}_R \in \Phi_V(R)$ denote the *zero* quadratic form and consider the associative R-algebra structure

$$(A_0)_R := \theta(R)(\underline{0}_R)$$

which is induced from the even Clifford algebra of the zero quadratic R-form on W. It is commutative, since all products $w_i \cdot w_j = 0$ for $1 \le i, j \le 3$. Let $\mathsf{Stab}_{\mathrm{Stab}(w)}((A_0)_R)$ denote the *stabilizer subgroup functor* of $(A_0)_R$ in $\mathrm{Stab}(w)$. Then a straightforward computation gives the following:

Lemma 5.1

(1) $\mathsf{Stab}_{\mathrm{Stab}(w)}((A_0)_R)$ *is represented by a closed subgroupscheme* $\mathsf{Stab}_{\mathrm{Stab}(w)}((A_0)_R)$ *of* $\mathrm{Stab}(w)$ *whose set of* S*-valued points is given by*

$$\left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \mid \mathbf{B} \in \mathrm{GL}(3, S) \right\}.$$

(2) $\mathrm{Stab}(w)$ *is the semi-direct product of* Λ_w *with* $\mathsf{Stab}_{\mathrm{Stab}(w)}((A_0)_R)$.

DEFINITION 5.2

Let $\Theta : \Phi_V \times \Lambda_w \longrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}$ be the R-morphism given by the composition

$$\begin{aligned} \Phi_V \times \Lambda_w &\xrightarrow{\theta \times Id} \mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}} \times \Lambda_w \\ &\xrightarrow{\cong} \Lambda_w \times \mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}} \xrightarrow{\mu} \mathsf{Id}\text{-}w\text{-}\mathsf{Assoc}_{\mathbf{W}} \end{aligned}$$

where μ is the action morphism induced from that of $\mathrm{Stab}(w)$ and the arrow in the middle is the ‘swapping isomorphism’. Thus, for $q \in \Phi_V(S)$ and $(\underline{t}) := \left(\begin{smallmatrix} 1 & t_1 \; t_2 \; t_3 \\ \mathbf{0} & \mathbf{I}_3 \end{smallmatrix}\right) \in \Lambda_w(S)$, one has by definition: $\Theta(q, (\underline{t})) = (\underline{t}) \cdot \theta(q)$.

Theorem 5.3. *The schematic image of the morphism* $\Theta : \Phi_V \times \Lambda_w \longrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}$ *is* $\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ *and the induced morphism into* $\text{Id-}w\text{-Sp-Azu}_{\mathbf{W}}$ *is an isomorphism which maps the open subscheme* $\Phi_V^{sr} \times \Lambda_w$ *onto the open subscheme* $\text{Id-}w\text{-Azu}_{\mathbf{W}}$.

Proof. We first recall the following crucial fact (see (1), Prop. 3.2.4, Chap. IV [9]): *The even Clifford algebra of a semi-regular quadratic form is an Azumaya algebra.* Using this fact and the definition of Θ, one sees that the morphism Θ restricted to $\Phi_V^{sr} \times \Lambda_w$ factors through $\text{Id-}w\text{-Azu}_{\mathbf{W}}$ by a morphism Θ^{sr} such that the following diagram is commutative

$$\begin{array}{ccc}
\Phi_V \times \Lambda_w & \xrightarrow{\Theta} & \text{Id-}w\text{-Assoc}_{\mathbf{W}} \\
\uparrow & & \uparrow \\
\Phi_V^{sr} \times \Lambda_w & \xrightarrow{\Theta^{sr}} & \text{Id-}w\text{-Azu}_{\mathbf{W}}
\end{array}$$

where the vertical arrows are the canonical open immersions. The above diagram base-changes well in view of (2) of Theorem 3.4. Notice that since the base $X = \text{Spec}(R)$ is affine, Θ is a morphism of affine schemes and therefore has a schematic image by Case (1) of Proposition 3.6. The same is true of each of the two vertical arrows and of Θ^{sr} since Φ_V^{sr} is a principal open subset of Φ_V (by definition) and since $\text{Id-}w\text{-Azu}_{\mathbf{W}} \hookrightarrow \text{Id-}w\text{-Assoc}_{\mathbf{W}}$ is affine by (1) of Theorem 3.4. As noted earlier $P_3 \in R[\Phi_V] = R[Y_i, Y_{ij}]$ is not a zero divisor. Hence $R[\Phi_V^{sr}] = R[\Phi_V]_{P_3}$ shows that $\Phi_V^{sr} \hookrightarrow \Phi_V$ is schematically dominant, i.e., the limiting scheme of Φ_V^{sr} is Φ_V (cf. assertion (5), Proposition 3.7). Now from assertion (6) of Proposition 3.7, the flatness of Λ_w over $X = \text{Spec}(R)$ implies that $\Phi_V^{sr} \times \Lambda_w \hookrightarrow \Phi_V \times \Lambda_w$ is also schematically dominant. So using the commutative diagram above, and the transitivity of the schematic image (assertion (3), Proposition 3.7), we see that in order to prove the theorem, it is enough to show that

(*) Θ^{sr} is schematically dominant and surjective, and Θ is a closed immersion.

We now claim that the above properties are equivalent to

(**) Θ^{sr} is proper and Θ is a closed immersion.

Suppose **(**)** holds. To show **(*)**, we only need to show that Θ^{sr} is surjective and schematically dominant. From **(**)** it follows that $\Theta_K^{sr} := \Theta^{sr} \otimes_R K$ is functorially injective and proper for each algebraically closed field K which is an R-algebra. That both the K-schemes $(\Phi_V^{sr} \times \Lambda_w) \otimes_R K$ and $(\text{Id-}w\text{-Azu}_{\mathbf{W}}) \otimes_R K$ are integral and smooth of the same dimension follows from the smoothness of relative dimension nine and geometric irreducibility $/R$ of $\Phi_V^{sr} \times \Lambda_w$ (which is obvious), and of $\text{Id-}w\text{-Azu}_{\mathbf{W}}$ from statement (4), Theorem 3.4. Since Θ_K^{sr} is differentially injective at each closed point, it has to be a smooth morphism by Theorem 17.11.1 of EGA IV [6] and thus has to be an open map. But by **(**)** it is also proper and hence a closed map. Hence Θ_K^{sr} is bijective etale, and hence an isomorphism. This also gives that Θ^{sr} is surjective. Now from Cor. 11.3.11 of EGA IV and from the flatness of $\Phi_V^{sr} \times \Lambda_w$ over R, it follows that Θ^{sr} is itself flat, and hence schematically dominant since it is faithfully flat (being already surjective). Therefore **(**)** $\Longrightarrow$ **(*)**.

We shall establish **(**)** by computing Θ. For this we shall have to first compute θ which was used to define Θ. The following outlines the method to compute the multiplication

$*_q$ in the algebra $\theta(q)$ for $q \in \Phi_V(S)$, S a commutative R-algebra with 1. For ease of reading, we shall write x° for $x \otimes 1$ in the following. Let q correspond to the point $(\lambda_1, \lambda_2, \lambda_3, \lambda_{12}, \lambda_{13}, \lambda_{23}) \in S^6$, i.e., $q(e_i^\circ) = \lambda_i$ and $b_q(e_i^\circ, e_j^\circ) = \lambda_{ij}$, $(1 \le i < j \le 3)$. Then by definition, $\theta(q)$ is the associative S-algebra structure on W_S with identity element $w^\circ = w_0^\circ$ induced from the isomorphism $\Psi_S : W_S \cong \mathsf{Cliff}_0(V_S, q)$. Since the w_i° are an S-basis for W_S, it is enough to compute $w_i^\circ *_q w_j^\circ$ for $(1 \le i, j \le 3)$. Let $\theta_{ijk}(q)$ denote the coefficient of w_k° for $0 \le k \le 3$ in the expression for $w_i^\circ *_q w_j^\circ$, for each pair of indices (i, j) with $1 \le i, j \le 3$. These $\theta_{ijk}(q)$ are polynomial functions of the λ_i and the λ_{jk} which may be computed explicitly as follows. For example, suppose one wants to compute the product $w_2^\circ *_q w_1^\circ$. Using the properties of the multiplication in $\mathsf{Cliff}_0(V_S, q)$, one gets the following:

$$\begin{aligned}
w_2^\circ *_q w_1^\circ := \Psi_S^{-1}\{(e_2^\circ \cdot e_3^\circ)\cdot(e_1^\circ \cdot e_2^\circ)\} &= \Psi_S^{-1}\{(\lambda_{23}(1^\circ) \\
&\quad - e_3^\circ \cdot e_2^\circ)\cdot(\lambda_{12}(1^\circ) - e_2^\circ \cdot e_1^\circ)\} \\
&= \Psi_S^{-1}\{\lambda_{23}\lambda_{12}(1^\circ) - \lambda_{23}e_2^\circ \cdot e_1^\circ - \lambda_{12}e_3^\circ \cdot e_2^\circ \\
&\quad + (e_3^\circ \cdot e_2^\circ)\cdot(e_2^\circ \cdot e_1^\circ)\} \\
&= \Psi_S^{-1}\{\lambda_{23}\lambda_{12}(1^\circ) - \lambda_{23}(\lambda_{12}(1^\circ) - e_1^\circ \cdot e_2^\circ) - \lambda_{12}(\lambda_{23}(1^\circ) \\
&\quad - e_2^\circ \cdot e_3^\circ) + e_3^\circ \cdot (e_2^\circ \cdot e_2^\circ)\cdot e_1^\circ\} \\
&= -\lambda_{12}\lambda_{23}(w^\circ) + \lambda_{23}(w_1^\circ) + \lambda_{12}(w_2^\circ) + \lambda_2(w_3^\circ).
\end{aligned}$$

Thus $\theta_{210}(q) = -\lambda_{12}\lambda_{23}$, $\theta_{211}(q) = \lambda_{23}$, $\theta_{212}(q) = \lambda_{12}$, and $\theta_{213}(q) = \lambda_2$. In a similar fashion, the other products may be computed. The following result is needed to compute Θ from θ.

Lemma 5.4. *Let $q \in \Phi_V(S)$ and $(\underline{t}) := \begin{pmatrix} 1 & t_1 \; t_2 \; t_3 \\ \mathbf{0} & \mathbf{I}_3 \end{pmatrix} \in \Lambda_w(S)$, for a commutative R-algebra S with 1. Let $*_{(q,t)}$ denote the multiplication in the algebra $\Theta(q, (\underline{t})) = (\underline{t}) \cdot \theta(q)$ and as before, $*_q$ denote the multiplication in $\theta(q)$. Then one has*

(1) $(\underline{t})(w_i^\circ) = t_i w^\circ + w_i^\circ$ *for* $(1 \le i \le 3)$;

(2) $(\underline{t})^{-1}(w_i^\circ) = -t_i w^\circ + w_i^\circ$ *for* $(1 \le i \le 3)$;

(3) $w_i^\circ *_{(q,t)} w_j^\circ = (\underline{t})(w_i^\circ *_q w_j^\circ) - t_j w_i^\circ - t_i w_j^\circ - t_i t_j w^\circ$.

The first two formulae follow easily by direct computation. To prove the third formula, one uses the first two formulae along with the following one:

$$(w_i^\circ) *_{(q,t)} (w_j^\circ) = (\underline{t})\{((\underline{t})^{-1}(w_i^\circ)) *_q ((\underline{t})^{-1}(w_j^\circ))\}.$$

One may now compute the multiplication in the algebra $(\underline{t}) \cdot \theta(q)$ by making use of the formulae listed in the above lemma since the method for computing the products of the form $w_i^\circ *_q w_j^\circ$ had already been illustrated before the lemma. Let $\Theta_{ijk}(q, (\underline{t}))$ denote the coefficient of w_k° for $0 \le k \le 3$ in the expression for $w_i^\circ *_{(q,t)} w_j^\circ$, for each pair of indices (i, j) with $1 \le i, j \le 3$. These $\Theta_{ijk}(q, (\underline{t}))$ are polynomial functions of the λ's and the t's. Computations give in particular the following:

$$\begin{array}{lll}
\Theta_{132}(q,(\underline{t})) = \lambda_1\,; & \Theta_{213}(q,(\underline{t})) = \lambda_2\,; & \Theta_{231}(q,(\underline{t})) = -\lambda_3\,; \\
\Theta_{122}(q,(\underline{t})) = -t_1\,; & \Theta_{133}(q,(\underline{t})) = \lambda_{12} - t_1\,; & \Theta_{121}(q,(\underline{t})) = -t_2\,; \\
\Theta_{211}(q,(\underline{t})) = \lambda_{23} - t_2\,; & \Theta_{232}(q,(\underline{t})) = -t_3\,; & \Theta_{131}(q,(\underline{t})) = \lambda_{13} - t_3.
\end{array}$$

The upshot of the above computations is that, if one denotes by Θ_1 the composition of the following two morphisms

$$\Phi_V \times \Lambda_w \xrightarrow{\Theta} \text{Id-}w\text{-Assoc}_{\mathbf{W}} \hookrightarrow \text{Alg}_{\mathbf{W}}$$

where the second one is the canonical closed immersion, so that the R-algebra homomorphism $\Theta_1^{\#}$ of coordinate rings corresponding to Θ_1 is given by the composition

$$\begin{aligned} R[Z_{ijk}] \cong R[\text{Alg}_{\mathbf{W}}] \twoheadrightarrow R[\text{Id-}w\text{-Assoc}_{\mathbf{W}}] &\xrightarrow{\Theta^{\#}} R[\Phi_V \times \Lambda_w] \\ &\cong R[L_1, L_2, L_3, L_{12}, L_{23}, L_{13}, T_1, T_2, T_3] \end{aligned}$$

then under $\Theta_1^{\#}$ we have shown that

$$\begin{array}{lll} Z_{132} \mapsto L_1\ ; & Z_{213} \mapsto L_2\ ; & Z_{231} \mapsto -L_3\ ; \\ Z_{122} \mapsto -T_1\ ; & Z_{133} \mapsto L_{12} - T_1\ ; & Z_{121} \mapsto -T_2\ ; \\ Z_{211} \mapsto L_{23} - T_2\ ; & Z_{232} \mapsto -T_3\ ; & Z_{131} \mapsto L_{13} - T_3. \end{array}$$

Therefore we see that $\Theta_1^{\#}$ is surjective, which implies that $\Theta^{\#}$ is surjective, i.e., Θ is a closed immersion. Further the above table shows that both Θ and Θ^{sr} are proper since they satisfy the valuative criterion for properness. Thus the conditions (**) are verified.

Q.E.D. for Theorem 5.3

B: Applications to desingularization

6. Application 1: The Seshadri desingularization in positive characteristic

Notations for this section

X: a smooth, irreducible, complete curve of genus $g \geq 2$ over an algebraically closed field k.

$\mathcal{U}_X^{SS}(n,d)$: the normal projective variety of equivalence classes of *semi-stable* vector bundles on X of rank n and degree d [15].

$\mathcal{U}_X^{S}(n,d)$: the smooth open subvariety of $\mathcal{U}_X^{SS}(n,d)$ consisting of *stable* vector bundles. If n is coprime to d, semi-stability is the same as stability. When $d = 0$, this subvariety is precisely the set of smooth points of $\mathcal{U}_X^{SS}(n,0)$ unless $n = 2$ and $g = 2$ in which case $\mathcal{U}_X^{SS}(n,0)$ is smooth (see [13]).

$\mathcal{V}_X(n,0)$: the category of rank n and degree 0 vector bundles on X.

$\mathcal{V}_X^{SS}(n,0)$: the subcategory of $\mathcal{V}_X(n,0)$ consisting of semi-stable vector bundles.

$\mathcal{V}_X^{S}(n,0)$: the subcategory of $\mathcal{V}_X^{SS}(n,0)$ consisting of stable vector bundles.

The problem is to desingularize $\mathcal{U}_X^{SS}(2,0)$. Seshadri's solution in [16] is based on the smoothness of the variety of specializations of (2×2)-matrix algebras over algebraically closed fields of characteristics $\neq 2$. Since we have extended this smoothness to an arbitrary base scheme (the smoothness of Id-w-Sp-Azu$_{\mathbf{W}}$ of Theorem 3.8), we are able to extend

Seshadri's methods to char. 2 as well. The birationality of the desingularizing morphism over the open subscheme of stable bundles in positive characteristics is mentioned though not explicitly proved in [16], and even in the more elaborate account [17], this birationality is arrived at from the claim that the morphism $\mathcal{U}_X^S(n,d) \longrightarrow \mathcal{U}_X^{SS}(n^2, d\cdot n)$ given on points by $[E] \longmapsto [n\cdot E]$ is an isomorphism over the image – a claim which is again not explicitly proved. Some work is done in §6.2 to show the generic smoothness from which the birationality is deduced (in the case of zero characteristic this would of course follow from general considerations). The rest of the proof is more or less on the same lines as in [16] or [17] except that we make some local simplifications – in particular we are able to do without the notion of rigidified parabolic family and hence avoid going into questions of descent etc. that are involved in the existence of universal objects for such families. Section 6.5 announces the generalization of the above result over an arbitrary base.

6.1 *Preliminaries on the Seshadri Construction*

For the easy reference of the reader, in this subsection we recall the main facts underlying Seshadri's construction in [16]. The ideas and notations introduced are essential for the rest of the discussion.

6.1.1 *Facts on parabolic bundles.* Throughout this section one works with parabolic vector bundles of a certain type, which is recalled next. The reader may consult [11] for a general discussion.

DEFINITION 6.1

Fix a (closed) point $P \in X(k)$ and a pair of real numbers (α_1, α_2) such that $0 < \alpha_1 < \alpha_2 < 1$. A *parabolic structure at P with weights* (α_1, α_2) on an object $V \in \mathcal{V}_X(4,0)$ is a codimension 1 subspace Δ of the fiber V_P of V at P. The pair (V, Δ) is called a *parabolic bundle*. The *parabolic degree* of (V, Δ) (denoted $\mathrm{pardeg}(V, \Delta)$) is the number $\alpha_1 + 3\cdot\alpha_2$ and the *parabolic slope* of (V, Δ) (denoted $\mathrm{par}\mu(V, \Delta)$) is $\mathrm{pardeg}(V, \Delta)/4$. Let W be a proper sub-bundle of V. Then given a parabolic structure on V, W acquires the structure of a parabolic sub-bundle $(W, \Delta|W)$ of (V, Δ) which is defined as follows:

Case 1. If W_P is not a subspace of Δ, then $\Delta|W := W_P \cap \Delta$ and

$$\mathrm{pardeg}(W, \Delta|W) := \mathrm{degree}(W) + \alpha_1 + (\mathrm{rank}(W) - 1)\cdot\alpha_2,$$
$$\mathrm{par}\mu(W, \Delta|W) := \mathrm{pardeg}(W, \Delta|W)/\mathrm{rank}(W).$$

Case 2. If W_P is a subspace of Δ, then $\Delta|W := (0)$ and

$$\mathrm{pardeg}(W, \Delta|W) := \mathrm{degree}(W) + \mathrm{rank}(W)\cdot\alpha_2,$$
$$\mathrm{par}\mu(W, \Delta|W) := \mathrm{pardeg}(W, \Delta|W)/\mathrm{rank}(W).$$

Given a parabolic structure on V, it is called a *parabolic semi-stable* (resp. *parabolic stable*) structure if the following condition is satisfied: for every proper sub-bundle W of V given the structure of a parabolic sub-bundle of V as explained above, one has $\mathrm{par}\mu(W) \leq \mathrm{par}\mu(V)$ (resp. $\mathrm{par}\mu(W) < \mathrm{par}\mu(V)$).

The symbols $\mathcal{PV}_X(4,0)$, $\mathcal{PV}_X^{SS}(4,0)$ and $\mathcal{PV}_X^{S}(4,0)$ will respectively denote the category of parabolic, parabolic semi-stable and parabolic stable vector bundles on X as defined above with the underlying vector bundles being of rank 4 and degree 0. We have the following elementary result relating parabolic semi-stability with the usual semi-stability in relation to the choice of weights.

Lemma 6.2. *The real numbers α_i may be chosen such that $\mathcal{PV}_X^{SS}(4,0) = \mathcal{PV}_X^{S}(4,0)$ and such that $(V,\Delta) \in \mathcal{PV}_X^{SS}(4,0) \implies$ the underlying vector bundle V is semi-stable.*

A little writing-down shows that if one chooses $\alpha_2 - \alpha_1$ to be a positive irrational number less than 1, then the parabolic slope of any proper sub-bundle of V (given the canonical parabolic sub-bundle structure explained above) can never equal the parabolic slope of V. If, further, one chooses α_2 such that $\alpha_2 < 1/4$, then one sees that the claims of the above lemma are satisfied. We make such a choice of weights and fix it for the rest of the discussion. We next recall the notion of families of parabolic bundles from §3 of [16].

DEFINITION 6.3

Let T be any k-scheme. The following data determine a family (V,D) in $\mathcal{PV}_X^{S}(4,0)$ parametrized by T:

(1) *Underlying family V in $\mathcal{V}_X(4,0)$.* V is a vector bundle of rank 4 on $X_T := X \times_k T$ such that for every point $t \in T(K)$, K any algebraically closed extension field of k, if V_t is defined to be the base-change of V to $X_K := X \times_k \operatorname{Spec}(K)$ (via t), then $V_t \in \mathcal{V}_{X_K}(4,0)$ – the category of rank 4 degree zero vector bundles on X_K.

(2) *Underlying quasi-parabolic family (V,D) in $\mathcal{PV}_X(4,0)$.* Let V_P denote the base-change of V to the reduced closed subscheme $T \cong \{P\} \times T \hookrightarrow X_T$ (via the closed immersion of the point $P \in X(k)$). Then D is a global section of the projective bundle $\mathbb{P}(V_P)$ associated to the locally-free sheaf (associated to) V_P. Note that D corresponds to a quotient line bundle of V_P whose kernel at each point $t \in T(K)$ ($k \subset K = \overline{K}$) defines a codimension 1 subspace D_t of the fiber of V_t at $P_K := \{P\} \times_k K \in X_K(K)$.

(3) With the above notations, for all $t \in T(K)$, one has that $(V_t, D_t) \in \mathcal{PV}_{X_K}^{S}(4,0)$ – the category of parabolic stable vector bundles on X_K defined in a manner similar to Definition 6.1 above.

PROPOSITION 6.4

(1) *There exists a parabolic structure $(W \oplus W, \Delta) \in \mathcal{PV}_X^{S}(4,0)$ for each $W \in \mathcal{V}_X^{S}(2,0)$.* (2) *For any two such parabolic structures Δ_1 and Δ_2, the objects $(W \oplus W, \Delta_1)$ and $(W \oplus W, \Delta_2)$ are isomorphic.* (3) *If R is a reduced noetherian local k-algebra with residue field k, then the analogues of* (1) *and* (2) *above hold for families parametrized by* $\operatorname{Spec}(R)$.

For proofs of (1)–(3) see [16]. Assertion (3) uses the fact that parabolic semi-stability/stability is an open condition on the parameter space (§4 of [11]). Before proceeding, one needs the following facts about the moduli space of parabolic semi-stable vector bundles on X in $\mathcal{PV}_X^{SS}(4,0)$ from the paper of Mehta–Seshadri [11].

Theorem 6.5. *With the notations of* 6.1 *above one has the following*:

(1) *on* $\mathcal{PV}_X^{SS}(4,0)$ *there is defined a natural equivalence of objects such that the set of equivalence classes is the set of* (*closed*) *points of a normal projective integral scheme of finite type* $\mathcal{PU}_X^{SS}(4,0)$ *of dimension* $4g$ *where* g *is the genus of* X;

(2) *the above equivalence reduces to isomorphism on the subcategory* $\mathcal{PV}_X^{S}(4,0)$ *and the set of isomorphism classes is precisely the set of* (*closed*) *points of a smooth open subscheme* $\mathcal{PU}_X^{S}(4,0)$ *of* $\mathcal{PU}_X^{SS}(4,0)$;

(3) $\mathcal{PU}_X^{SS}(4,0)$ *has the universal mapping property for families of parabolic semi-stable vector bundles on* X *in* $\mathcal{PV}_X^{SS}(4,0)$ *parametrized by noetherian* k*-schemes*;

(4) *for each noetherian* k*-scheme* T, *let* $\mathcal{F}(T)$ *denote the set of isomorphism classes of families in* $\mathcal{PV}_X^{S}(4,0)$ *parametrized by* T. *Thus one gets a contravariant functor* $\mathcal{F}:\{k\text{-schemes}\}\longrightarrow\{\text{Sets}\}$. *Then* $\mathcal{PU}_X^{S}(4,0)$ *is a coarse moduli space for the functor* $\mathcal{F}$ *defined above. In other words, there exists a morphism of functors*

$$\Phi:\mathcal{F}\longrightarrow \mathrm{Mor}_k(-,\mathcal{PU}_X^{S}(4,0))$$

such that (a) *the pair* $(\Phi,\mathrm{Mor}_k(-,\mathcal{PU}_X^{S}(4,0)))$ *is universally repelling and* (b) *for every algebraically closed extension field* K *of* k *the map* $\Phi(K)$ *is bijective.*

6.1.2 *The identification of the smooth locus of stable bundles*

For each $W\in\mathcal{V}_X^S(2,0)$, we now fix one parabolic structure Δ as in statement (1) of Proposition 6.4 above, and denote it by $\Delta(W)$. Then one has

PROPOSITION 6.6

(1) *The association* $[W]\mapsto[(W\oplus W,\Delta(W))]$ *is a well-defined injective set-theoretic map which is the underlying map (on closed points) of a morphism of finite type of* k*-schemes*

$$\zeta_{2,k}^S:\mathcal{U}_X^S(2,0)\longrightarrow\mathcal{PU}_X^S(4,0).$$

(2) *Let* K *be an algebraically closed extension field of* k *and let* $X_K:=X\otimes_k K$. *Then the corresponding morphism of* K*-schemes*

$$\zeta_{2,K}^S:\mathcal{U}_{X_K}^S(2,0)\longrightarrow\mathcal{PU}_{X_K}^S(4,0)$$

is simply the base-change of $\zeta_{2,k}^S$. *In particular, the topological map underlying the morphism* $\zeta_{2,k}^S$ *is injective.*

By Proposition 6.4, for a family of stable rank 2 degree 0 vector bundles on X parametrized by T, one gets a morphism $T\longrightarrow\mathcal{PU}_X^S(4,0)$. Hence assertion (1) is a consequence of the fact that $\mathcal{U}_X^S(2,0)$ is a coarse moduli space. Assertion (2) uses the following fact: $\mathcal{PU}_X^S(4,0)$ is a geometric quotient under a free action [11] and the same is true of $\mathcal{U}_X^S(2,0)$ [15]. Therefore, by Prop. 0.9 of Mumford *et al* [12], each of these moduli spaces is the base space over which the geometric quotient is a principal fiber bundle with structure group the corresponding reductive algebraic group; hence these moduli spaces are well-behaved under base-change, viz., $\mathcal{U}_X^S(2,0)\otimes_k K = \mathcal{U}_{X_K}^S(2,0)$ and $\mathcal{PU}_X^S(4,0)\otimes_k K = \mathcal{PU}_{X_K}^S(4,0)$. Further if one were to work with the corresponding categories of vector bundles over X_K, then the analogue of Prop. 6.4 over K also holds. From these, (2) easily follows. Henceforth we denote $\zeta_{2,k}^S$ simply by ζ_2^S.

Properness of the scheme-theoretic image of ζ_2^S. Since the parabolic weights have been chosen such that $\mathcal{PV}_X^{SS}(4,0) = \mathcal{PV}_X^S(4,0)$ (see hypothesis following Lemma 6.2), cases (1) and (2) of Theorem 6.5 imply that $\mathcal{PU}_X^S(4,0)$ is a smooth projective scheme. Further, since $\mathcal{U}_X^S(2,0)$ is integral, the scheme-theoretic image of ζ_2^S is an integral closed subscheme of $\mathcal{PU}_X^S(4,0)$ and hence in particular it is a projective integral scheme of finite type. The scheme-theoretic image of ζ_2^S is the candidate for desingularizing $\mathcal{U}_X^{SS}(2,0)$. It will be shown that the desingularization is an isomorphism precisely over $\mathcal{U}_X^S(2,0)$, with this isomorphism being given by the inverse of ζ_2^S.

The subscheme $\mathcal{N}_X^S(4,0)$ *of* $\mathcal{PU}_X^S(4,0)$. The next thing is to define a subscheme $\mathcal{N}_X^S(4,0)$ of $\mathcal{PU}_X^S(4,0)$ (which will later turn out to be isomorphic to $\mathcal{U}_X^S(2,0)$ via ζ_2^S) and to show that ζ_2^S factors through $\mathcal{N}_X^S(4,0)$. The definition of $\mathcal{N}_X^S(4,0)$ will require three steps: (A) Proving that $(V,\Delta)\in\mathcal{PV}_X^S(4,0)\Rightarrow \dim(\mathrm{End}(V))\le 4$. (B) Determination of a reduced closed subscheme $\mathcal{QU}_X^S(4,0)$ of $\mathcal{PU}_X^S(4,0)$. (C) Determination of $\mathcal{N}_X^S(4,0)$ as an open subscheme of $\mathcal{QU}_X^S(4,0)$. Step (A) follows from Prop. 1(c) of [16].

PROPOSITION 6.7

Let $(V,\Delta)\in\mathcal{PV}_X^S(4,0)$. *Then* $\dim_k(\mathrm{End}(V))\le 4$. *If* $k\subset K$ *is an algebraically closed extension field, then a similar inequality holds for* $\mathcal{PV}_{X_K}^S(4,0)$ *where* $X_K := X\otimes_k K$.

Before proceeding, one recalls that $\mathcal{PU}_X^S(4,0)$ is the geometric quotient under a free action of $PGL(n,k)$ (for a suitable n) on a certain reduced scheme of finite type PR_X^S. With the notations of (4), Theorem 6.5, $\exists$ a locally universal family $(\mathbf{V},\mathcal{D})$ whose isomorphism class belongs to $\mathcal{F}(PR_X^S)$, and the geometric quotient morphism $q: PR_X^S \longrightarrow \mathcal{PU}_X^S(4,0)$ is just the morphism $\Phi(PR_X^S)([(\mathbf{V},\mathcal{D})])$. Then the following concludes Step (B).

PROPOSITION 6.8

(1) *Let* $(\mathbf{V}_t,\mathcal{D}_t)$ *denote the base-change of* $(\mathbf{V},\mathcal{D})$ *to* $X\otimes_k K$ *via a point* $t\in PR_X^S(K)$ *where* K *is an algebraically closed extension field of* k. *Then the subset of points of the topological space underlying* PR_X^S *given by*

$$\{t\ |\dim_K(\mathrm{End}(\mathbf{V}_t)) = 4\}$$

is closed and hence inherits the structure of a reduced closed subscheme QR_X^S *of* PR_X^S.

(2) QR_X^S *is saturated with respect to* q *and* q *restricted to* QR_X^S *is a principal* $PGL(n,k)$*-bundle over its (scheme-theoretic) image* $\mathcal{QU}_X^S(4,0)$ *which is a reduced closed subscheme of* $\mathcal{PU}_X^S(4,0)$.

The proof of (1) follows from (A) and upper-semicontinuity of fiber dimensions. As for the proof of (2): As just recalled, $\mathcal{PU}_X^S(4,0)$ is a geometric quotient under a free action and hence by Proposition 0.9 of Mumford *et al* [12], $q : PR_X^S \longrightarrow \mathcal{PU}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle. In particular q is a flat finite-type morphism (in fact, a smooth morphism since $PGL(n,k)$ is smooth) and is hence open. The condition defining QR_X^S is true at a point $t \in QR_X^S(K)$ iff it is true at all points in the $PGL(n,k)$-orbit of t, since $\Phi(K)$ is bijective by (4b) of Theorem 6.5 above. Thus QR_X^S is saturated with respect to q, i.e., $QR_X^S = q^{-1}(q(QR_X^S))$. But as noted above, q is open and surjective, so

$$q(PR_X^S - QR_X^S) = \mathcal{PU}_X^S(4,0) - q(QR_X^S)$$

is open, implying that $q(QR_X^S)$ is closed in $\mathcal{PU}_X^S(4,0)$. Since PR_X^S is reduced, $q(QR_X^S)$, given the reduced induced closed subscheme structure of $\mathcal{PU}_X^S(4,0)$ is the scheme-theoretic image of $q|QR_X^S$. Denote this reduced closed subscheme by $\mathcal{QU}_X^S(4,0)$. Now $q|QR_X^S : QR_X^S \longrightarrow \mathcal{QU}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle since it is the base-change of $q : PR_X^S \longrightarrow \mathcal{PU}_X^S(4,0)$ to $\mathcal{QU}_X^S(4,0)$. This proves (2). We are left with step (C): the determination of $\mathcal{N}_X^S(4,0)$ as an open subscheme of $\mathcal{QU}_X^S(4,0)$.

PROPOSITION 6.9

The subset of points $[(V,\Delta)] \in \mathcal{PU}_X^S(4,0)$ *such that* $\mathrm{End}(V) \cong M(2,k)$ *is the set of closed points of a locally closed subscheme* $\mathcal{N}_X^S(4,0)$ *and the morphism* $\zeta_2^S : \mathcal{U}_X^S(2,0) \longrightarrow \mathcal{PU}_X^S(4,0)$ *of Proposition* 6.6 *factors through* $\mathcal{N}_X^S(4,0)$.

Proof. By the standard theorem on cohomology and base-change, part (1) of Proposition 6.8 implies that the coherent sheaf

$$\mathcal{A} := (p_{QR_X^S})_*(\mathcal{E}nd(\mathbf{V}|QR_X^S \times X))$$

on QR_X^S is locally free of rank 4. It has the natural structure of a sheaf of associative $\mathcal{O}_{QR_X^S}$-algebras with identity. If $W \in \mathcal{U}_X^S(2,0)$, then $[(W \oplus W, \Delta(W))] \in \mathcal{PU}_X^S(4,0)$ (this is a point of the set-theoretic image of ζ_2^S). Further, $\mathrm{End}(W \oplus W) \cong M(2,k)$ since W is stable. So by the relevant analogue of Proposition 3.3, $\exists$ a maximal open subscheme NR_X^S of QR_X^S restricted to which $\mathcal{A}$ is a sheaf of Azumaya $\mathcal{O}_{NR_X^S}$-algebras. By part (2) of Proposition 6.8, the topological image of NR_X^S under $q|QR_X^S$ determines an open subscheme $\mathcal{N}_X^S(4,0)$ of $\mathcal{QU}_X^S(4,0)$ and $q|NR_X^S : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle. By part (2) of Proposition 6.6, it is clear that the topological map underlying ζ_2^S factors through $\mathcal{N}_X^S(4,0)$, and since $\mathcal{U}_X^S(2,0)$ and $\mathcal{QU}_X^S(4,0)$ are reduced, this morphism indeed factors through $\mathcal{N}_X^S(4,0)$. Q.E.D.

Integrality of $\mathcal{N}_X^S(4,0)$

PROPOSITION 6.10

The morphism $\zeta_2^S : \mathcal{U}_X^S(2,0) \longrightarrow \mathcal{N}_X^S(4,0)$ *of Proposition* 6.9 *is bijective on K-valued points for each algebraically closed extension field* $K \supset k$. *Since* $\mathcal{U}_X^S(2,0)$ *is irreducible, it therefore follows that* $\mathcal{N}_X^S(4,0)$ *is an integral scheme of finite type.*

The proof of the above depends on the following crucial result of Seshadri:

Theorem 6.11 (Props. 3–4, [16])

(a) *Let K be any algebraically closed extension of k, and let $(V, \Delta) \in \mathcal{PV}_X^S(4,0)(K)$ such that* $\dim_K(\mathrm{End}(V)) = 4$. *Consider the canonical representation of* $\mathrm{End}(V)$ *on the fiber V_{P_K} of V at $P_K \in X_K(K) = (X \otimes_k K)(K)$, where P_K is the base-change of $P \in X(k)$ to K.* (*As noted earlier, $(V, \Delta) \in \mathcal{PV}_{X_K}^S(4,0) \Rightarrow V \in \mathcal{V}_{X_K}^{SS}(4,0)$. Therefore this canonical representation is faithful.*) *Then we have*: (1) *this representation can be identified with the dual of the right regular representation of* $\mathrm{End}(V)$ *and* (2) *the structure group of the principal bundle* $\mathrm{Pr}(V)$ *of V may be identified with* $\mathrm{Aut}(V_{P_K})$, *and via the above representation of* $\mathrm{End}(V)$, *the structure group of this principal bundle can be reduced to the opposite group of the group of units in* $\mathrm{End}(V)$.

(b) *When properly formulated, all the above results remain true for families parametrized by* $\mathrm{Spec}(R)$ *where R is a complete noetherian local k-algebra with residue field k.*

We briefly indicate why ζ_2^S is surjective on geometric points. Let $[(V, \Delta)] \in \mathcal{N}_X^S(4,0)(K)$ where K is any algebraically closed extension of k. By the definition of $\mathcal{N}_X^S(4,0)$ it follows that $\mathrm{End}(V) \cong M(2,K)$ – the algebra of (2×2)-matrices over K. Applying (a) of Theorem 6.11, we get that the representation of $\mathrm{Aut}(V)$ on V_{P_K} is equivalent to the diagonal representation of $\mathrm{GL}(2,K)$ in $\mathrm{GL}(4,K)$. Further applying (b) of Theorem 6.11, we see that there exists a principal $\mathrm{Aut}(V) \cong \mathrm{GL}(2,K)$-bundle from which the principal bundle obtained by extension of structure group to $\mathrm{Aut}(V_{P_K})$ is $\mathrm{Pr}(V)$. This means that $\exists$ a rank 2 degree zero bundle W on X_K such that $V \cong W \oplus W$. But since V is semi-stable, the same must be true of W. Further, if W is not stable, then it contains a line sub-bundle L of degree zero, which implies that V contains a sub-bundle isomorphic to $L \oplus L$ from which one can prove that (V, Δ) cannot be parabolic semi-stable. Now by the analogue of Proposition 6.4 for X_K, $\exists\ \Delta'$ such that $(W \oplus W, \Delta') \in \mathcal{PV}_{X_K}^S(4,0)$. By part (2) of Proposition 6.6, it follows that $\zeta_2^S([W]) = [(W \oplus W, \Delta')]$. Let $(W \oplus W, \Delta'') \in \mathcal{PV}_{X_K}^S(4,0)$ be the parabolic stable bundle induced by an isomorphism $V \cong W \oplus W$ from (V, Δ). By the analogue of (2) of Proposition 6.4 for X_K, $(W \oplus W, \Delta') \cong (W \oplus W, \Delta'')$. This implies that $[(V, \Delta)]$ is the image of $[W]$ under ζ_2^S. Thus ζ_2^S is indeed surjective.

6.2 *Generic smoothness and birationality*

For a moment, we revert to the notations of §3 of A: taking there the base scheme to be $\mathrm{Spec}(k)$, and W the four-dimensional vector space corresponding to $\mathbf{W}$ and $w \in W$ any non-zero vector, and denoting Id-w-Sp-$\mathrm{Azu}_{\mathbf{W}}$ and Id-w-$\mathrm{Azu}_{\mathbf{W}}$ in this case respectively by Id-w-Sp-Azu_W and Id-w-Azu_W, from (3a) of Theorem 3.8 we get that Id-w-Sp-$\mathrm{Azu}_W \cong A_k^9$ and is exactly the reduced closed subscheme structure on the closure of the $\mathrm{Stab}(w)$-orbit Id-w-Azu_W of (2×2)-matrix algebra structures on W with multiplicative identity w.

Lemma 6.12. *On* Id-w-Sp-Azu_W $\exists$ *a natural sheaf* $\mathcal{B}_k''$ *of associative algebras with identity whose underlying module is free of rank* 4 *and which is universal* (*resp. locally universal*) *for sheaves $\mathcal{F}$ of associative $\mathcal{O}_T$-algebras with identity on integral k-schemes T satisfying the following properties*: (1) *the module underlying $\mathcal{F}$ is free* (*resp. locally free*) *of rank* 4, *and* (2) $\exists$ *a point $t \in T(k)$ such that the fiber $\mathcal{F}_t$ at t of $\mathcal{F}$ is isomorphic to a* (2×2)*-matrix algebra over k.*

To prove the above, we may without loss of generality assume that $\mathcal{F}$ is free, and with the notations of the proof of Theorem 3.4, we see that one may take $\mathcal{B}_k''$ to be $(\mathcal{B}_w|\text{Id-}w\text{-Sp-Azu}_W)$. Continuing with the above notations, let $a \in \text{Id-}w\text{-Azu}_W$ be a closed point and let R_a be the completion of the local ring at that point. Let $\mathcal{B}_{k,a}''$ be the algebra induced by $\mathcal{B}_k''$ over R_a. If $\mathfrak{m}_a$ is the maximal ideal of R_a, then by the definition of $\text{Id-}w\text{-Azu}_W$, one has that $\mathcal{B}_{k,a}''/(\mathfrak{m}_a\mathcal{B}_{k,a}'')$ can be identified with the (2×2)-matrix algebra structure over k corresponding to the point $a \in \text{Id-}w\text{-Azu}_W$. Since R_a is a complete local ring, this implies by property (6) of Proposition 3.2 that $\mathcal{B}_{k,a}''$ can be identified with the (2×2)-matrix algebra structure over R_a corresponding to the natural morphism $\text{Spec}(R_a) \longrightarrow \text{Id-}w\text{-Azu}_W$. So if ρ_a denotes the dual of the right regular representation of $\mathcal{B}_{k,a}''$, then ρ_a may be identified with the dual of the right regular representation of $M(2, R_a)$. But for this latter representation, one observes that if $R_a \longrightarrow K$ is any k-homomorphism into any algebraically closed extension field K of k, then the induced representation is equivalent to the diagonal representation of $M(2, K)$ in $M(4, K)$. Now $\text{Id-}w\text{-Sp-Azu}_W$ is a variety, so its complete local rings are reduced, and hence the above observation implies the following result:

Lemma 6.13. *The above representation ρ_a is equivalent to the diagonal representation of $M(2, R_a)$ in $M(4, R_a)$.*

The above lemma will be applied in the proof of the following proposition.

PROPOSITION 6.14

Let A be a complete noetherian local k-algebra with residue field k. Then the canonical map

$$\mathsf{Mor}_k(\text{Spec}A, \mathcal{U}_X^S(2,0)) \longrightarrow \mathsf{Mor}_k(\text{Spec}A, \mathcal{N}_X^S(4,0)) \ :\ f \longmapsto \zeta_2^S \circ f$$

is surjective, where ζ_2^S is the morphism of Proposition 6.9.

Proof. Special Case: First assume that A is reduced. Start with $g \in \mathsf{Mor}_k(\text{Spec}A, \mathcal{N}_X^S(4,0))$. Since $q|NR_X^S : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle (see proof of Proposition 6.9), g lifts to $g' \in \mathsf{Mor}_k(\text{Spec}A, NR_X^S)$. Let $(\mathbf{V}_A, \mathcal{D}_A)$ be the pull back to $X_A := X \times \text{Spec}A$, of the restriction of the locally universal family $(\mathbf{V}, \mathcal{D})$ (recalled before Proposition 6.8) to $X \times NR_X^S$, via the morphism $(\text{Id}_X \times g')$.

Let $\mathcal{A}_A$ be the algebra corresponding to the pullback via g' to $\text{Spec}A$ of the sheaf of Azumaya algebras $\mathcal{A}|NR_X^S$ introduced in the proof of Proposition 6.9. Since $\mathcal{N}_X^S(4,0)$ is integral, and since $q|NR_X^S$ is a principal $PGL(n,k)$-bundle, NR_X^S is also integral. Hence by Lemma 6.12, $\mathcal{A}|NR_X^S$ is locally isomorphic to a pullback of $\mathcal{B}_k''|\text{Id-}w\text{-Azu}_W$. Therefore, it follows from Lemma 6.13 above that the dual of the right regular representation of $\mathcal{A}_A$ is equivalent to the diagonal representation of $M(2, A)$ in $M(4, A)$.

$$
\begin{tikzcd}
(\mathcal{W}\oplus\mathcal{W}, \mathcal{D}(\mathcal{W})) \arrow[d] & & (\mathbf{V}, \mathcal{D}) \arrow[d] \\
R_X^S \arrow[rr, dashed, "\zeta_r''"] \arrow[dd, "q_2"'] & & NR_X^S \arrow[dd, "q"] \\
& \mathrm{Spec}(A) \arrow[ul, "f'"] \arrow[ur, "g'"'] \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathcal{U}_X^S(2,0) \arrow[rr, "\zeta_2^S"'] & & \mathcal{N}_X^S(4,0)
\end{tikzcd}
$$

Let $\mathrm{Pr}(\mathbf{V}_A)$ denote the principal bundle of $\mathbf{V}_A$. Identify the structure group of this principal bundle with $\mathrm{Aut}_A(\mathcal{A}_A)$. By (b), Theorem 6.11, this structure group may be reduced, via the representation of $\mathcal{A}_A$ of the previous paragraph, to the opposite group of the group of units in $\mathcal{A}_A$. But as seen in the previous paragraph, this is equivalent to the diagonal embedding of $\mathrm{GL}(2, A)$ in $\mathrm{GL}(4, A)$. Thus $\exists$ a rank 2 vector bundle $\mathbf{W}_A$ on X_A such that $\mathbf{W}_A \oplus \mathbf{W}_A \cong \mathbf{V}_A$.

Since $\mathbf{V}_A$ is a family in $\mathcal{V}_X^{SS}(4,0)$, $\mathbf{W}_A$ is a family in $\mathcal{V}_X^{SS}(2,0)$. It follows from the fact that $(\mathbf{V}_A, \mathcal{D}_A)$ is parabolic stable that $\mathbf{W}_A$ is a family in $\mathcal{V}_X^S(2,0)$. We now use the following facts about $\mathcal{U}_X^S(2,0)$ from [15]: the integral smooth open subscheme $\mathcal{U}_X^S(2,0)$ of $\mathcal{U}_X^{SS}(2,0)$ is the geometric quotient under a free action of $PGL(m,k)$ (for a suitable m) on an integral smooth open subscheme R_X^S of a certain Grothendieck Quot scheme. There is a locally universal family $\mathcal{W}$ of vector bundles on X in $\mathcal{V}_X^S(2,0)$ parametrized by R_X^S which is tautological in the sense that if $q_2 : R_X^S \longrightarrow \mathcal{U}_X^S(2,0)$ is the geometric quotient morphism and if $\mathcal{W}_r$ is defined to be the base-change of $\mathcal{W}$ to a closed point $r : \mathrm{Spec}(k) \longrightarrow R_X^S$, then after identifying $X \times_k \mathrm{Spec}(k)$ with X, $q_2(r) = [\mathcal{W}_r]$. Now $\mathcal{U}_X^S(2,0)$ has the universal mapping property for families of rank 2 degree zero stable bundles, so one gets a morphism $f \in \mathsf{Mor}_k(\mathrm{Spec}A, \mathcal{U}_X^S(2,0))$. The proof of the present proposition will follow if one shows that $\zeta_2^S \circ f = g$. Since q_2 is a smooth morphism, f factors through q_2, i.e., $\exists\ f' \in \mathsf{Mor}_k(\mathrm{Spec}A, R_X^S)$ such that $q_2 \circ f' = f$. Let $r \in R_X^S$ be a closed point above the image of the closed point via f. Then $\mathbf{W}_A \oplus \mathbf{W}_A$ inherits the structure of a family $(\mathbf{W}_A \oplus \mathbf{W}_A, \Delta_A)$ in $\mathcal{PV}_X^S(4,0)$ parametrized by $\mathrm{Spec}A$ via f' from the family $(\mathcal{W}\oplus\mathcal{W}|X \times U_r^S, \mathcal{D}(\mathcal{W}))$ given by the following result (whose proof involves an application of Nakayama's lemma and the fact that parabolic semi-stability is an open condition on the parameter space).

Lemma 6.15. *Let* $W \in \mathcal{V}_X^S(2,0)$ *and let* r *be a closed point of* R_X^S *such that* $q_2(r) = [W] = [\mathcal{W}_r]$. *Let* $\Delta(\mathcal{W}_r)$ *be given by* (1), *Proposition* 6.4 *such that* $(\mathcal{W}_r \oplus \mathcal{W}_r, \Delta(\mathcal{W}_r)) \in \mathcal{PV}_X^S(4,0)$. *Then* $\Delta(\mathcal{W}_r)$ *can be extended to give a family* $\mathcal{D}(\mathcal{W})$ *of parabolic structures in* $\mathcal{PV}_X^S(4,0)$ *parametrized by a suitable open neighborhood* U_r^S *of* r *such that the underlying family in* $\mathcal{V}_X^{SS}(4,0)$ *is* $(\mathcal{W} \oplus \mathcal{W})|X \times U_r^S$.

Because of the local universality of the family $(\mathbf{V}, \mathcal{D})$ on PR_X^S, and because $\mathcal{U}_X^S(2,0)$ is reduced, the composition $\zeta_2^S \circ q_2$ factors locally through $q : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$, i.e., one has a morphism ζ_r'' from a suitable neighborhood of r into NR_X^S such that $q \circ \zeta_r'' = \zeta_2^S \circ q_2$.

The pullbacks of the family $(\mathbf{V}, \mathcal{D})$ via $\zeta_r'' \circ f'$ and g' are isomorphic as families in $\mathcal{PV}_X^S(4,0)$ because of the isomorphism $\mathbf{W}_A \oplus \mathbf{W}_A \cong \mathbf{V}_A$ and (3) of Proposition 6.4. So the morphisms $\zeta_r'' \circ f'$ and g' differ by an A-valued point of $PGL(n,k)$ – here the fact that $q|NR_X^S : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle is used. This means that $q \circ \zeta_r'' \circ f' = q \circ g' \;\; \Rightarrow \;\; \zeta_2^S \circ q_2 \circ f' = g \;\; \Rightarrow \;\; \zeta_2^S \circ f = g$. This finishes off the proof of the present proposition for the case when A is reduced.

Proof of Proposition 6.14 *for arbitrary* A. Again start with $g \in \mathrm{Mor}_k(\mathrm{Spec}A, \mathcal{N}_X^S(4,0))$. Let $n \in \mathcal{N}_X^S(4,0)$ be the image of the closed point under g. Let A_n denote the completion of the local ring of $\mathcal{N}_X^S(4,0)$ at n. Then g factors through $\mathrm{Spec}A_n$ by a morphism g_n. Note that A_n is reduced, so by the Special Case considered for Proposition 6.14, one gets $\phi_n \in \mathrm{Mor}_k(\mathrm{Spec}A_n, \mathcal{U}_X^S(2,0))$ such that $\zeta_2^S \circ \phi_n$ is the canonical morphism from $\mathrm{Spec}A_n$ into $\mathcal{N}_X^S(4,0)$. Now one needs to just take $f := \phi_n \circ g_n$. Q.E.D. for Prop. 6.14

PROPOSITION 6.16

Let $\mathcal{N}_X^S(4,0)'$ *be the non-empty dense open subscheme of smooth points of* $\mathcal{N}_X^S(4,0)$ *and let* $\mathcal{U}_X^S(2,0)'$ *be the open subscheme of* $\mathcal{U}_X^S(2,0)$ *given by the inverse image of* $\mathcal{N}_X^S(4,0)'$ *under the morphism* ζ_2^S *of Proposition* 6.9. *Then* ζ_2^S *restricted to* $\mathcal{U}_X^S(2,0)'$ *is a smooth morphism.*

Proof. Since $\mathcal{U}_X^S(2,0)$ is smooth, by Proposition 10.4 of Chap. III of Hartshorne's book 'Algebraic Geometry', it is enough to prove that the differential of ζ_2^S at each closed point of $\mathcal{U}_X^S(2,0)'$ is surjective. But this follows by applying the previous proposition to the case $A = k[\varepsilon]/(\varepsilon^2)$ and remembering that ζ_2^S is topologically an injective map. Q.E.D.

Theorem 6.17. *The bijective morphism* $\zeta_2^S : \mathcal{U}_X^S(2,0) \longrightarrow \mathcal{N}_X^S(4,0)$ *is an isomorphism over the smooth locus* $\mathcal{N}_X^S(4,0)'$ *of* $\mathcal{N}_X^S(4,0)$.

The proof essentially follows from the generic smoothness of ζ_2^S just seen and the fact that a bijective etale morphism is an isomorphism.

6.3 *Smoothness of the limiting scheme*

Recall from (2), Proposition 6.8, that $q : QR_X^S \longrightarrow \mathcal{QU}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle and hence so is $q|NR_X^S : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$ (see the proof of Proposition 6.9). Let NR_X denote the canonical reduced induced closed subscheme structure on the closure of NR_X^S in QR_X^S (or in PR_X^S). Since $q : NR_X^S \longrightarrow \mathcal{N}_X^S(4,0)$ is a principal

$PGL(n,k)$-bundle and since $\mathcal{N}_X^S(4,0)$ is integral, NR_X^S and hence NR_X are also integral. Let $\mathcal{N}_X(4,0)$ denote the canonical reduced induced closed subscheme structure on the closure of $\mathcal{N}_X^S(4,0)$ in $\mathcal{QU}_X^S(4,0)$ (or in $\mathcal{PU}_X^S(4,0)$). Since $\mathcal{N}_X^S(4,0)$ is integral, so is $\mathcal{N}_X(4,0)$.

Theorem 6.18. *The local ring of NR_X at each closed point is regular.*

The proof will be divided into several steps.

Step 1: Lifting Criterion. Let A_n be the completion of the local ring of NR_X at a closed point n. It is enough to show that A_n is regular. For this, it is enough to prove that the canonical map

$$\mathsf{Mor}_k(\mathrm{Spec}A, \mathrm{Spec}A_n) \longrightarrow \mathsf{Mor}_k(\mathrm{Spec}A_0, \mathrm{Spec}A_n)\ :\ g \longmapsto g \circ i_0$$

is surjective for any closed immersion $\mathrm{Spec}A_0 \overset{i_0}{\hookrightarrow} \mathrm{Spec}A$ where $A_0,\ A$ are finite dimensional local k-algebras with residue field k.

Step 2: The family $(\mathbf{V}_0, \mathcal{D}_0)$. Start with $g_0 \in \mathsf{Mor}_k(\mathrm{Spec}A_0, \mathrm{Spec}A_n)$ and let the composition of g_0 with the canonical map $\mathrm{Spec}A_n \longrightarrow NR_X$ be denoted $g_0' \in \mathsf{Mor}_k(\mathrm{Spec}A_0, NR_X)$. Recall that $\exists$ a locally universal family $(\mathbf{V}, \mathcal{D})$ parametrized by PR_X^S. Let $(\mathbf{V}_0, \mathcal{D}_0)$ be the induced family parametrized by $\mathrm{Spec}A_0$ via the composition of g_0' with the closed immersion $NR_X \subset PR_X^S$.

Step 3: The morphism $f_0 : \mathrm{Spec}A_0 \longrightarrow \mathrm{Spec}R_a$. By the definition of QR_X^S (see part (1) of Proposition 6.8), one gets a locally free rank 4 sheaf $\mathcal{A}$ of associative algebras with identity on QR_X^S by the descent of the sheaf $\mathcal{E}nd(\mathbf{V}|QR_X^S \times X)$. Let B_0 be the A_0-algebra corresponding to the sheaf induced by the sheaf $\mathcal{A}|NR_X$ via g_0'. Since NR_X is integral and is the closure of NR_X^S, by Lemma 6.12, $\mathcal{A}|NR_X$ is locally isomorphic to the base-change of $\mathcal{B}_k''$ by a morphism from a neighborhood of n (in NR_X) into Id-w-Sp-Azu$_W$. Hence $\exists$ a morphism $f_0 : \mathrm{Spec}A_0 \longrightarrow \mathrm{Spec}R_a$, where $f_0(\text{closed point}) = a$ is a closed point of Id-w-Sp-Azu$_W$ and R_a is the completion of the local ring of Id-w-Sp-Azu$_W$ at a, such that the algebra $B_0 = B_a \otimes_{R_a} A_0$ where B_a is the R_a-algebra induced by $\mathcal{B}_k''$.

$$\begin{tikzcd}
\mathrm{Spec}(A_n) \arrow[r] & NR_x \arrow[r] & PR_x^s \\
\mathrm{Spec}(A_0) \arrow[u, "g_0"] \arrow[r, "i_0"] \arrow[ur, bend left=60, "g_0'"] \arrow[dr, "f_0"] & \mathrm{Spec}(A) \arrow[ul, dashed, "g"] \arrow[r, "f'"] \arrow[d, "f"] & \mathrm{Spec}(R_a') \arrow[ul, "g'"] \arrow[u, "g''"] \arrow[dl] \\
& \mathrm{Spec}(R_a) &
\end{tikzcd}$$

Step 4: Extension of f_0 *to* $f : \mathrm{Spec} A \longrightarrow \mathrm{Spec} R_a$. Since Id-$w$-Sp-Azu$_{\mathbf{W}} \cong A_k^9$ (see the beginning of §6.2), R_a is a regular local ring. Therefore f_0 lifts to an f as required. If $B := B_a \otimes_{R_a} A$, then clearly $B_0 = B \otimes_A A_0$.

Step 5: Factorization of f via a closed immersion. f itself may not be a closed immersion, but it lifts to a closed immersion $f' : \mathrm{Spec} A \longrightarrow \mathrm{Spec} R'_a$, $R'_a := R_a[[Y_1, \ldots Y_m]]$ for a suitable $m < \dim_k(A)$. Note that since R_a is regular, so is R'_a. Let $B'_a := B_a \otimes_{R_a} R'_a$. Note that $B'_a \otimes_{R'_a} A = B$ and $B'_a \otimes_{R'_a} A_0 = B_0$.

Step 6: The family $(\mathbf{V}_a, \mathcal{D}_a)$. One chooses an R_a-basis for the free rank 4 algebra B_a and thus gets bases for B'_a, B and B_0 respectively over R'_a, A and A_0. Therefore, the algebras of endomorphisms of the underlying modules for B'_a, B and B_0 respectively over R'_a, A and A_0 are identified with the matrix algebras $M(4, R'_a)$, $M(4, A)$ and $M(4, A_0)$. Consider the duals of the right regular representations of B'_a, B and B_0. Now if H'_a, H and H_0 respectively denote the opposite groups to the groups of units $(B'_a)^\times$, $B^\times$ and ${B_0}^\times$, then the images of these groups are naturally identified as subgroups of $\mathrm{GL}(4, R'_a)$, $\mathrm{GL}(4, A)$ and $\mathrm{GL}(4, A_0)$.

By part (b), Theorem 6.11, the structure group $\mathrm{GL}(4, A_0)$ of the principal bundle $\mathrm{Pr}(\mathbf{V}_0)$ of $\mathbf{V}_0$ can be reduced to H_0. Now since A_0 is an artinian quotient of the complete noetherian local k-algebra R'_a, by Lemma 1 of §5 of [16], the principal H_0-bundle $\mathrm{Pr}(\mathbf{V}_0)$ extends to a principal H'_a-bundle. Let $\mathbf{V}_a$ be the vector bundle on $X \times \mathrm{Spec} R'_a$ of rank 4 gotten from this principal H'_a-bundle by the canonical representation of H'_a via the dual of the right regular representation of B'_a. Then clearly $\mathbf{V}_a \otimes_{R'_a} A_0 = \mathbf{V}_0$. The parabolic structure

$$\mathcal{D}_0 \in \Gamma(\mathrm{Spec} A_0,\ \mathbb{P}((\mathbf{V}_0)_{P_0}))$$

on $\mathbf{V}_0$ extends to a parabolic structure

$$\mathcal{D}_a \in \Gamma(\mathrm{Spec} R'_a,\ \mathbb{P}((\mathbf{V}_a)_{P_a}))$$

on $\mathbf{V}_a$ because R'_a is local and because projective spaces are smooth. Here $P_0 := \{P\} \times \mathrm{Spec} A_0 \cong \mathrm{Spec} A_0$ and a similar definition holds for P_a. Now at the closed point, the corresponding member of the family $(\mathbf{V}_a, \mathcal{D}_a)$ is parabolic stable, and so the base being local, the family itself is parabolic stable.

Step 7: The morphism $g' : \mathrm{Spec} R'_a \longrightarrow NR_X$. Since $(\mathbf{V}, \mathcal{D})$ is locally universal, one gets a morphism $g'' : \mathrm{Spec} R'_a \longrightarrow PR_X^S$ such that the family induced via g'' from $(\mathbf{V}, \mathcal{D})$ is isomorphic to $(\mathbf{V}_a, \mathcal{D}_a)$. One sees as follows that the morphism g'' factors through a morphism g' into NR_X. Now a in Id-w-Sp-Azu$_W$ is a specialization of (2×2)-matrix algebras. So if $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec} R'_a$ is the generic point then $B'_a \otimes_{R'_a} K \cong M(2, K)$ and hence $H'_a \otimes_{R'_a} K \cong \mathrm{GL}(2, K)$. But this means that the base-change $((\mathbf{V})_a)_K$ of $(\mathbf{V})_a$ to $\mathrm{Spec}(K)$ splits as a direct sum $(\mathbf{W})_K \oplus (\mathbf{W})_K$ of stable rank 2 degree zero vector bundles on X_K. Hence $\mathrm{End}((\mathbf{V})_K) \cong M(2, K)$ which implies by the definition of NR_X^S that the composite map

$$\mathrm{Spec}(K) \longrightarrow \mathrm{Spec} R'_a \xrightarrow{g''} PR_X^S$$

factors through NR_X^S. But $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(R'_a)$ being the generic point, this means that the topological image of g'' lands inside $NR_X = \overline{NR_X^S}$. Now since R'_a is reduced, g'' factors through a morphism $g' : \mathrm{Spec} R'_a \longrightarrow NR_X$.

Step 8: The lifting of g_0 to a morphism $g : \mathrm{Spec}A \longrightarrow \mathrm{Spec}A_n$. Let i_0 denote the closed immersion of $\mathrm{Spec}A_0$ into $\mathrm{Spec}A$. Then $g' \circ f' \circ i_0$ and g'_0 both induce isomorphic families on $X \times \mathrm{Spec}A_0$ from $(\mathbf{V}, \mathcal{D})$ since $\mathbf{V}_a$ and $\mathcal{D}_a$ were extended from $\mathbf{V}_0$ and $\mathcal{D}_0$. Hence these two morphisms differ by an A_0-valued point λ_0 of $PGL(n,k)$:

$$\lambda_0 \cdot (g' \circ f' \circ i_0) = g'_0.$$

Now since $PGL(n,k)$ is smooth, λ_0 lifts to an R'_a-valued point λ_a. Let λ_A be the A-valued point of $PGL(n,k)$ induced by λ_a and let $\widehat{g} := \lambda_A \cdot (g' \circ f')$. Then one has by the very definition of an action (the action of $PGL(n,k)$ on NR_X) that

$$\begin{aligned} f' \circ (\lambda_a \cdot g') = \lambda_A \cdot (g' \circ f') \ \Rightarrow\ \widehat{g} \circ i_0 &= (\lambda_A \cdot (g' \circ f')) \circ i_0 \\ &= \lambda_0 \cdot (g' \circ f' \circ i_0) = g'_0. \end{aligned}$$

The images of the closed point of $\mathrm{Spec}A_0$ under g'_0 and $\widehat{g} \circ i_0$ are one and the same point $n \in NR_X(k)$. But since A is complete, $\widehat{g}$ factors through a morphism $g : \mathrm{Spec}A \longrightarrow \mathrm{Spec}A_n$. Now it is easy to check that g lifts g_0 using the following simple result:

Lemma 6.19. *Let B be a noetherian domain, $\mathfrak{p} \subset B$ a prime ideal, and $\widehat{f}$, $\widehat{g}$ ring homomorphisms from $\widehat{B_{\mathfrak{p}}}$ into a complete noetherian local ring $\widehat{R}$. Then $\widehat{f}|B = \widehat{g}|B \Rightarrow \widehat{f} = \widehat{g}$.*

Thus the proof of Theorem 6.18 is established.

COROLLARY 6.20

NR_X and $\mathcal{N}_X(4,0)$ are smooth, and the morphism $\zeta_2^S : \mathcal{U}_X^S(2,0) \longrightarrow \mathcal{N}_X^S(4,0)$ is an isomorphism.

Proof. NR_X is an integral scheme of finite type and hence is smooth by Theorem 6.18. Now $q : QR_X^S \longrightarrow \mathcal{QU}_X^S(4,0)$ is a principal $PGL(n,k)$-bundle and hence q is a smooth surjective morphism. Hence

$$q^{-1}(\mathcal{N}_X(4,0)) = q^{-1}(\overline{\mathcal{N}_X^S(4,0)}) = \overline{q^{-1}(\mathcal{N}_X^S(4,0))} = \overline{NR_X^S} = NR_X.$$

This implies that $q : NR_X \longrightarrow \mathcal{N}_X(4,0)$ is also a principal $PGL(n,k)$-bundle. Therefore, NR_X is smooth iff $\mathcal{N}_X(4,0)$ is smooth. But as we just saw, NR_X is smooth. From these and Theorem 6.17, it follows that ζ_2^S is an isomorphism. Q.E.D.

6.4 *The Seshadri Desingularization*

In the following, the isomorphism $(\zeta_2^S)^{-1} : \mathcal{N}_X^S(4,0) \cong \mathcal{U}_X^S(2,0)$ is extended to a desingularization $\pi_2 : \mathcal{N}_X(4,0) \longrightarrow \mathcal{U}_X^{SS}(2,0)$. We first show the existence of a natural surjective map $\pi_2(k) : \mathcal{N}_X(4,0)(k) \longrightarrow \mathcal{U}_X^{SS}(2,0)(k)$. This will be done in 4 steps.

Step 1: The morphism $\pi_1 : \mathcal{N}_X(4,0) \longrightarrow \mathcal{U}_X^{SS}(4,0)$. For the locally universal family $(\mathbf{V}, \mathcal{D})$ in $\mathcal{PV}_X^S(4,0)$ parametrized by PR_X^S, the underlying family of vector bundles $\mathbf{V}$ is a family of semi-stable vector bundles, i.e., a family in $\mathcal{V}_X^{SS}(4,0)$. Since $\mathcal{U}_X^{SS}(4,0)$ has the universal mapping property for families of rank 4 degree zero semi-stable vector bundles, one gets a morphism $\pi_1'' : PR_X^S \longrightarrow \mathcal{U}_X^{SS}(4,0)$. Since this morphism is $PGL(n,k)$-invariant and since $\mathcal{PU}_X^S(4,0)$ is a geometric quotient of PR_X^S under $PGL(n,k)$, π_1'' goes down to a morphism $\pi_1' : \mathcal{PU}_X^S(4,0) \longrightarrow \mathcal{U}_X^{SS}(4,0)$ which is given on closed points by $[(V,D)] \longmapsto [V]$. Let π_1 denote the composition of π_1' with the canonical closed immersion $\mathcal{N}_X(4,0) \subset \mathcal{PU}_X^S(4,0)$.

Since $\mathcal{N}_X(4,0)$ is projective (because $\mathcal{PU}_X^S(4,0)$ is projective), π_1 is proper, hence closed. Further $\mathcal{N}_X(4,0)$ is reduced, so π_1 is surjective onto its scheme-theoretic image which is the same as its topological image (a closed set) given the canonical reduced induced closed subscheme structure. As $\mathcal{N}_X^S(4,0)$ is an open dense subset of $\mathcal{N}_X(4,0)$, $\pi_1(\mathcal{N}_X^S(4,0))$ is a dense subset in $\pi_1(\mathcal{N}_X(4,0))$.

Step 2: The morphism $g_1 : \mathcal{U}_X^{SS}(2,0) \longrightarrow \mathcal{U}_X^{SS}(4,0)$. The association

$$\mathcal{U}_X^{SS}(2,0)(k) \longrightarrow \mathcal{U}_X^{SS}(4,0)(k) \ : \ [W] \longmapsto [W \oplus W]$$

is the underlying map on closed points of a morphism $g_1 : \mathcal{U}_X^{SS}(2,0) \longrightarrow \mathcal{U}_X^{SS}(4,0)$. The topological map underlying this morphism is injective because of the Jordan–Hölder theorem for the category of semi-stable vector bundles of degree zero on X (see [15]).

Since $\mathcal{U}_X^{SS}(2,0)$ is projective, g_1 is proper, hence closed. Further $\mathcal{U}_X^{SS}(2,0)$ is reduced, so g_1 is surjective onto its scheme-theoretic image which is the same as its topological image (a closed set) given the canonical reduced induced closed subscheme structure. As $\mathcal{U}_X^S(2,0)$ is an open dense subset of $\mathcal{U}_X^{SS}(2,0)$, $g_1(\mathcal{U}_X^S(2,0))$ is a dense subset in $g_1(\mathcal{U}_X^{SS}(2,0))$.

Step 3: The map $\pi_2 : \mathcal{N}_X(4,0) \longrightarrow \mathcal{U}_X^{SS}(2,0)$. By the definition of the isomorphism ζ_2^S (Prop. 6.6), one sees that

$$\pi_1 \circ \zeta_2^S \ = \ g_1 \,|\, \mathcal{U}_X^S(2,0).$$

Therefore $g_1(\mathcal{U}_X^S(2,0)) = \pi_1(\mathcal{N}_X^S(4,0))$. Therefore by Steps 1 and 2 above

$$g_1(\mathcal{U}_X^{SS}(2,0)) = \overline{g_1(\mathcal{U}_X^S(2,0))} = \overline{\pi_1(\mathcal{N}_X^S(4,0))} = \pi_1(\mathcal{N}_X(4,0)).$$

Now since g_1 is injective as noted in Step 2 above, there is a well-defined set-theoretic map $\pi_2(k) = (g_1(k))^{-1} \circ \pi_1(k)$ where for a morphism f, $f(k)$ is used to denote the underlying map on closed points.

Note that $\pi_2(k)$ is surjective by construction. This implies in particular that if $[W] \in \mathcal{U}_X^{SS}(2,0)$ then there is a representative V of $[W \oplus W]$ such that $\exists$ a parabolic stable structure Δ on V.

Step 4: The isomorphism $\pi_2^S : \mathcal{N}_X^S(4,0) \longrightarrow \mathcal{U}_X^S(2,0)$. Since $\pi_1 \circ \zeta_2^S = g_1 \,|\, \mathcal{U}_X^S(2,0)$, g_1 is injective and ζ_2^S is an isomorphism, one has by the very definition of $\pi_2(k)$ that its restriction to $\mathcal{N}_X^S(4,0)(k)$ is the inverse of $\zeta_2^S(k)$. Therefore, $\pi_2(k) \,|\, \mathcal{N}_X^S(4,0)(k)$ is the underlying map on closed points of the isomorphism $\pi_2^S := (\zeta_2^S)^{-1}$. Thus we get the following

PROPOSITION 6.21

There exists a surjective set-theoretic map

$$\pi_2(k) : \mathcal{N}_X(4,0)(k) \longrightarrow \mathcal{U}_X^{SS}(2,0)(k)$$

such that its restriction to $\mathcal{N}_X^S(4,0)(k)$ *is the underlying map on closed points of the inverse* (*denoted by* π_2^S) *of the isomorphism* ζ_2^S *of Corollary* 6.20.

Theorem 6.22. *The isomorphism* π_2^S *of Proposition* 6.21 *extends to a surjective morphism* $\pi_2 : \mathcal{N}_X(4,0) \longrightarrow \mathcal{U}_X^{SS}(2,0)$ *whose underlying map on closed points is* $\pi_2(k)$. *Further* π_2 *is a desingularization of* $\mathcal{U}_X^{SS}(2,0)$ *and is an isomorphism over the smooth locus below.*

Proof. Assume that a surjective morphism π_2 exists as in the first statment of the theorem. By Corollary 6.20, $\mathcal{N}_X(4,0)$ is smooth. Further it is already projective, since it is a closed subscheme of the projective scheme $\mathcal{PU}_X^S(4,0)$. Therefore π_2 is a proper morphism. Since its restriction to $\mathcal{N}_X^S(4,0)$ is the isomorphism π_2^S, it is indeed a desingularization. The fact about π_2 being an isomorphism over the smooth points below is a consequence of the 'connectedness-version' of Zariski's Main Theorem. So we only have to prove the first statement of the theorem.

We note that it is enough to show that π_2^S extends to a morphism π_2 whose underlying map on closed points is $\pi_2(k)$; for then since $\mathcal{N}_X(4,0)$ is projective, π_2 will be proper, and since $\pi_2(k)$ is surjective and $\mathcal{U}_X^S(2,0)$ is a variety, this will imply that π_2 is surjective.

One continues to use the notations introduced in the discussion preceding Proposition 6.21. To begin with, consider the set-theoretic graph of $\pi_2(k)$:

$$\Gamma_{\pi_2(k)} := \{(n,m) \in \mathcal{N}_X(4,0) \times \mathcal{U}_X^{SS}(2,0) \mid \pi_2(k)(n) = m\}.$$

Then one sees clearly that

$$(\mathrm{Id}_{\mathcal{N}_X(4,0)} \times g_1)^{-1}(\Gamma_{\pi_1(k)}) = \Gamma_{\pi_2(k)}.$$

Thus $\Gamma_{\pi_2(k)}$ is the set of closed points of the reduced closed subscheme

$$\Gamma_{\pi_2} := \left(\mathrm{Id}_{\mathcal{N}_X(4,0)} \times g_1\right)^{-1}(\Gamma_{\pi_1})$$

of $\mathcal{N}_X(4,0) \times \mathcal{U}_X^{SS}(2,0)$. Let p_1 and p_2 from $\mathcal{N}_X(4,0) \times \mathcal{U}_X^{SS}(2,0)$ into $\mathcal{N}_X(4,0)$ and $\mathcal{U}_X^{SS}(2,0)$ respectively denote the canonical projections. Since $\mathcal{U}_X^{SS}(2,0)$ is projective, the morphism p_1 is proper. Hence its restriction to the closed subscheme Γ_{π_2} is also proper. Further, this morphism is bijective on closed points and is birational since $\pi_2(k)|\mathcal{N}_X^S(4,0)$ is the underlying map of the isomorphism $\pi_2^S = (\zeta_2^S)^{-1}$. But by Corollary 6.20, $\mathcal{N}_X(4,0)$ is smooth, in particular normal, and hence by Zariski's Main Theorem, the morphism

$$p_1|\Gamma_{\pi_2} : \Gamma_{\pi_2} \longrightarrow \mathcal{N}_X(4,0)$$

is an isomorphism, showing that the morphism

$$\pi_2 := p_2 \circ (p_1|\Gamma_{\pi_2})^{-1} : \mathcal{N}_X(4,0) \longrightarrow \mathcal{U}_X^{SS}(2,0)$$

extends π_2^S, as required. Q.E.D.

6.5 *On the Seshadri desingularization over a general base*

For this part, let R be a normal integral domain which is a universally Japanese (Nagata) ring. For details on such rings see §7.2. Let $\mathcal{X}$ be a complete smooth curve over R, i.e., one is given a proper, smooth, finite-type morphism $\mathcal{X} \longrightarrow \mathrm{Spec}(R)$ such that for every geometric point $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(R)$, the K-scheme $\mathcal{X}_K := \mathcal{X} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(K)$ is an integral, separated scheme of dimension one.

By a *semi-stable* (resp. *stable*) vector bundle on $\mathcal{X}$ of rank n and degree d is meant a locally-free sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{V}$ on $\mathcal{X}$ such that for every geometric point $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(R)$, $\mathcal{V} \otimes_R K$ is a semi-stable (resp. stable) locally-free sheaf on $\mathcal{X}_K$ of rank n and degree d. Using Seshadri's Geometric Invariant Theory over general base [18], one can construct a moduli space $\mathcal{U}^{SS}_{\mathcal{X}}(n, d)$ for semi-stable vector bundles on $\mathcal{X}$ of rank n and degree d. This $\mathcal{U}^{SS}_{\mathcal{X}}(n, d)$ turns out to be a proper R-scheme. Again by using [18], one can construct a proper moduli scheme over R for parabolic semi-stable vector bundles on $\mathcal{X}$, of fixed rank and degree and fixed types of parabolic structures given at a finite number of R-valued points of $\mathcal{X}$ over R, generalizing the construction of Mehta–Seshadri [11]. Using the smoothness of Id-w-Sp-Azu$_{\mathbf{W}}$ over $\mathrm{Spec}(R)$ (Theorem 3.8), and the techniques of Seshadri [18], one can construct a proper R-scheme $\mathcal{N}_{\mathcal{X}}(4, 0)$ which is smooth over $\mathrm{Spec}(R)$, along with a birational, surjective, proper R-morphism

$$\Pi_2 : \mathcal{N}_{\mathcal{X}}(4, 0) \longrightarrow \mathcal{U}^{SS}_{\mathcal{X}}(2, 0)$$

i.e., a desingularization of $\mathcal{U}^{SS}_{\mathcal{X}}(2, 0)$ over $\mathrm{Spec}(R)$. Further, one can show that, when the moduli space $\mathcal{U}^{SS}_{\mathcal{X}}(2, 0)$ has geometrically reduced fibers over R, the above desingularization specializes well. The proofs of all these assertions shall appear in a forthcoming paper [21].

7. Application 2: Existence of the Nori desingularization over a general base

Introduction. This section uses Seshadri's Geometric Invariant Theory over a general base [18] and a theorem of Donkin [4] to extend the construction of a $\mathbb{Z}$-scheme $\mathsf{V}_{(2,\mathbb{Z})}$ of Nori (Appendix, [16]) to a normal domain R which is a universally Japanese (Nagata) ring. The existence and smoothness of the scheme of limits of Azumaya algebra structures on a fixed module W free of rank 4 over R (Theorem 3.8) is used to show that the construction $\mathsf{V}_{(2,R)}$ is a desingularization of the Artin moduli space $\mathsf{Z}_{(2,R)}$ of $R\{X_1, \ldots, X_g\}$-modules of rank 2 over R for $g \geq 2$. It is also shown that this desingularization specializes well to the analogous desingularization over any algebraically closed field which is also an R-algebra, provided the Artin moduli space has geometrically reduced fibers. This happens for example when $R = \mathbb{Z}$ by the work of Donkin [4]. In particular, one gets desingularizations over fields of characteristic 2 (for algebraically closed fields of char.$\neq$2 the existence of such a desingularization follows from [16]).

Nori's method is based on that of Seshadri's which was used for desingularizing $\mathcal{U}^{SS}_{X}(2, 0)$. The latter was shown in the previous section to extend to characteristic 2 and in fact in a characteristic-free manner. Nori's $\mathbb{Z}$-scheme $\mathsf{V}_{(2,\mathbb{Z})}$ comes along with a canonical morphism $\mathsf{V}_{(2,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(2,\mathbb{Z})}$ which is a desingularization provided one shows the existence of a canonical $\mathbb{Z}$-smooth $\mathbb{Z}$-scheme structure on the space of those limits of Azumaya algebra structures on a fixed free $\mathbb{Z}$-module of rank 4, for which multiplicative identities exist. This follows as a special case of the more general result proved in A for a vector bundle of rank 4 over any base scheme.

In §7.1, preliminaries on Artin moduli spaces, Nagata rings and on Seshadri's Geometric Invariant Theory over such rings are recalled. In §7.2, the construction of Nori over $\mathbb{Z}$ is extended and the candidate for the desingularization is defined. In §7.3, the birationality of this candidate with the smooth locus of the relevant Artin moduli scheme is shown. Finally in §7.4, the desingularization is established and its specialization properties are studied.

7.1 *Artin moduli schemes, Nagata rings and Seshadri's GIT over general base*

To begin with, one extends the definition of the Artin moduli space $\mathsf{Z}_{(n,\mathbb{Z})}$ (cf. [1]).

DEFINITION 7.1

Let n, g be integers ≥ 2. Let R be a noetherian commutative ring with 1. Let $\mathsf{M}_{(n,R)}{}^g$ be the g-fold product of the R-affine R-scheme $\mathsf{M}_{(n,R)}$ of $(n \times n)$-matrices and consider the action of the general linear groupscheme $\mathsf{GL}_{(n,R)}$ on $\mathsf{M}_{(n,R)}{}^g$ given by 'simultaneous conjugation'. Let $\mathsf{B}_{(n,R)}$ be the ring of invariants; then $\mathsf{Z}_{(n,R)} := \mathrm{Spec}(\mathsf{B}_{(n,R)})$.

Recall 7.2 (*Facts about Nagata rings*). The standard reference is Chap. 12 of Matsumura's book [10]. An integral domain A is said to satisfy condition N-1 if its integral closure A_K in its quotient field K is a finite A-module. It is said to satisfy condition N-2 if for every finite extension field L/K, the integral closure A_L of A in L is a finite A-module. The properties N-1 and N-2 are preserved under localization and N-2 $\implies$ N-1 whereas noetherianness with N-1 $\implies$ N-2 only in char. 0; there exists an example of Akizuki of a noetherian domain of positive char. which is not N-1. A commutative ring B is called a *Nagata* ring (*pseudo-geometric* ring in Nagata's own terminology and *universally Japanese* ring in Grothendieck's) if it is noetherian and $B/\mathfrak{p}$ is N-2 for each prime $\mathfrak{p}$ of B. Every localization of B and every finitely generated (commutative) B-algebra are then also Nagata, and complete noetherian local rings are Nagata as well. Dedekinds domains of char. 0 such as $\mathbb{Z}$ are Nagata.

The next theorem which gives the basic facts about $\mathsf{Z}_{(n,R)}$ is a direct application of Seshadri's Geometric Invariant Theory over a general base (see Theorem 3, [18]).

Theorem 7.3. *The canonical morphism $\mathsf{M}_{(n,R)}{}^g \longrightarrow \mathsf{Z}_{(n,R)}$ is submersive and surjective. In fact, for every algebraically closed field K which is also an R-algebra, the induced map on K-valued points is just the set-theoretic quotient map by the 'orbit closure intersection equivalence' on $(\mathsf{M}_{(n,R)}{}^g \otimes_R K)(K)$. Further this morphism is a uniform categorical quotient* (*which means it base-changes well under flat base extensions*). *Moreover, if R is a universally Japanese* (*Nagata*) *ring, then $\mathsf{Z}_{(n,R)}$ is a scheme of finite type over R.*

Remark 7.4

(1) The categorical quotient property of the above theorem implies that if R is an S-algebra, then one has a unique morphism $\alpha_{(n,R,S)}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{M}_{(n,R)}{}^g \arrow[r, "\cong"] \arrow[d] & (\mathsf{M}_{(n,S)}{}^g) \otimes_S R \arrow[d, "\text{(base-chg from } S)"] \\ \mathsf{Z}_{(n,R)} \arrow[r, "\alpha_{(n,R,S)}"] & \mathsf{Z}_{(n,S)} \otimes_S R \end{tikzcd}$$

If further R' is an R-algebra, then one gets the following commutative diagram

$$\begin{tikzcd} \mathsf{M}_{(n,R')}{}^g \arrow[r, "\cong"] \arrow[d] & (\mathsf{M}_{(n,R)}{}^g) \otimes_R R' \arrow[r, "\cong"] \arrow[d, "\text{(base-chg from } R)"] & {} \\ \mathsf{Z}_{(n,R')} \arrow[r, "\alpha_{(n,R',R)}"] & (\mathsf{Z}_{(n,R)}) \otimes_R R' \arrow[r, "\alpha_{(n,R,S)} \otimes_R R'"] & {} \end{tikzcd}$$

$$\begin{array}{ccc} \mathsf{M}_{(n,R')}{}^{g} & \xrightarrow{\cong} & ((\mathsf{M}_{(n,S)}{}^{g}) \otimes_S R) \otimes_R R' \cong (\mathsf{M}_{(n,S)}{}^{g}) \otimes_S R' \\ \downarrow & & \downarrow \text{(base-chg from } S) \\ \mathsf{Z}_{(n,R')} & \xrightarrow{\alpha_{(n,R',R)}} & (\mathsf{Z}_{(n,S)} \otimes_S R) \otimes_R R' \cong \mathsf{Z}_{(n,S)} \otimes_S R' \end{array}$$

and the uniqueness of the morphism $\alpha_{(n,R',S)}$ implies the equality

$$\alpha_{(n,R',S)} = (\alpha_{(n,R,S)} \otimes_R R') \circ \alpha_{(n,R',R)}.$$

(2) From Theorem 7.3, it follows that the topological map underlying $\alpha_{(n,R,S)}$ of (1) above is bijective – in fact even bijective on L-valued points for each algebraically closed field L which is also an R-algebra.

(3) The uniformity of the categorical quotient of the above theorem implies that when R is a flat S-algebra, then the base-change of the categorical quotient over S is also a categorical quotient over R, and so the morphism $\alpha_{(n,R,S)}$ of (1) above must be an isomorphism.

Theorem 7.5 ([4], §3). *Let K be an algebraically closed field. Then the uniform categorical quotient $\mathsf{M}_{(n,\mathbb{Z})}{}^{g} \longrightarrow \mathsf{Z}_{(n,\mathbb{Z})}$ of Theorem* 7.3 *specializes well at geometric points, that is, the morphism $\alpha_{(n,K,\mathbb{Z})}$ of* (1) *of Remarks* 7.4 *is an isomorphism.*

Note: *From now on, the value of the integer $g \geq 2$ is fixed.*

Remark 7.6. For any commutative R-algebra S with 1 let $A_S := S\{X_1, \dots, X_g\}$ be the non-commuting polynomial algebra over S in g indeterminates. Consider an A_S-module M which is free of rank n over S. If an S-basis $\{e_1, \dots, e_n\}$ is chosen for M, so that one has an identification $M \cong S^{\oplus n}$, then the A_S-module structure on M defines an A_S-module structure on S^n, which is equivalent to prescribing a g-tuple of $(n \times n)$-matrices with entries in S, i.e., an S-valued point of $\mathsf{M}_{(n,R)}{}^{g}$. If another identification $M \cong S^n$ is chosen, then the corresponding new S-valued point of $\mathsf{M}_{(n,R)}{}^{g}$ is in the $\mathrm{GL}(n, S) = \mathsf{GL}_{(n,R)}(S)$-orbit of the previous one, where the action of $\mathrm{GL}(n, S)$ on $M(n, S)^g = \mathsf{M}_{(n,R)}{}^{g}(S)$ is by simultaneous conjugation. Therefore upto this action, $M(n, S)^g$ parametrizes pairs $(M, \{e_1, \dots, e_n\})$ where M is an A_S-module with S-basis $\{e_1, \dots, e_n\}$. Hence the moduli for such modules is given by the categorical quotient $\mathsf{Z}_{(n,R)}$ as defined above.

Before proceeding, one needs to know about what happens to the geometric points of $\mathsf{M}_{(n,R)}{}^{g}$ corresponding to simple modules in the light of the above remark. To this end, one has the following elementary lemma:

Lemma 7.7

(1) *Let x be a point of the topological space $|\mathsf{M}_{(n,R)}{}^{g}|$ underlying $\mathsf{M}_{(n,R)}{}^{g}$ with the property that if K is an algebraic closure of the residue field $\kappa(x)$ of $\mathsf{M}_{(n,R)}{}^{g}$ at x, then the g-tuple of matrices in $M(n, K)$ to which x corresponds makes K^n into a simple $(K \otimes_R A_R)$-module. Then this property of x is independent of the choice of K. In particular the subset $|(\mathsf{M}_{(n,R)}{}^{g})^s| \subset |\mathsf{M}_{(n,R)}{}^{g}|$ consisting of such points x is well-defined.*

(2) *In the above definition, the phrase 'if K is an algebraic closure of' may be replaced by 'if K is some algebraically closed extension field of' or by 'if K is any algebraically closed extension field of' or further by 'if K is any extension field of'.*

(3) *The property required of x in property* (1) *of Lemma* 7.7 *is also equivalent to the following one*: *'for every extension field K of $\kappa(x)$, the canonical map $(K \otimes_R A_R) \longrightarrow M(n, K) = \mathrm{End}_K(K^n)$ is surjective.*

The proofs of properties (2) and (3) of Lemma 7.7 require Burnside's theorem.

DEFINITION 7.8

For each R-algebra S (which is commutative with 1), let $(\mathsf{M}_{(n,R)}{}^g)^s(S) \subset (\mathsf{M}_{(n,R)}{}^g)(S)$ consist of those elements of $M(n, S)^g$ for which the canonical map of S-algebras $A_S \longrightarrow M(n, S)$ is surjective.

The above definition gives a subfunctor of (the functor of points of) $\mathsf{M}_{(n,R)}{}^g$ in view of the right-exactness of tensor product. In fact one has the following elementary result as an application of Nakayama's lemma.

Lemma 7.9. *Let S be a noetherian commutative ring with* 1, *and $\psi : S\{X_1, \dots, X_g\} \longrightarrow M(n, S)$ an S-algebra homomorphism. Let $U_S \subset |\mathrm{Spec}(S)|$ be the subset of the topological space underlying* $\mathrm{Spec}(S)$ *consisting of prime ideals $\mathfrak{p}$ such that $\psi \otimes_S \kappa(\mathfrak{p})$ is surjective. Then the subset U_S is open and thus acquires the canonical structure of an open subscheme. The subfunctor $(\mathsf{M}_{(n,R)}{}^g)^s$ of the above definition is open, i.e., it is represented by an open subscheme of $\mathsf{M}_{(n,R)}{}^g$. This open subscheme will also be denoted by $(\mathsf{M}_{(n,R)}{}^g)^s$. The subset $|(\mathsf{M}_{(n,R)}{}^g)^s|$ of* (1) *of the previous lemma is indeed the topological space underlying this open subscheme and therefore the canonical open immersion $(\mathsf{M}_{(n,R)}{}^g)^s \hookrightarrow \mathsf{M}_{(n,R)}{}^g$ base-changes well.*

Theorem 7.10 ([1]). *When K is an algebraically closed field, the action of $\mathsf{PGL}_{(n,K)}$ on $(\mathsf{M}_{(n,K)}{}^g)^s$ is scheme-theoretically free, so that $\mathsf{Z}_{(n,K)}{}^s$ is a geometric quotient under a free action. This geometric quotient $\mathsf{Z}_{(n,K)}{}^s$ is the smooth open subscheme of $\mathsf{Z}_{(n,K)}$ and its set of L-valued points corresponds to the set of isomorphism classes of simple $A_L = L\{X_1, \dots, X_g\}$-modules of dimension n over L, for each algebraically closed extension field L of K. Further, the set of L-valued points of $\mathsf{Z}_{(n,K)}$ can be identified canonically with the set of equivalence classes of A_L-modules of dimension n over L under the equivalence $M \sim M'$ iff $\mathsf{gr}(M) \cong \mathsf{gr}(M')$ where $\mathsf{gr}(M)$ denotes the associated graded module $\oplus_{i=0}^{r-1} (M_{i+1}/M_i)$ with $M_0 \subset M_1 \subset \cdots \subset M_r = M$ a Jordan–Hölder series for M.*

The above results combined with Seshadri's GIT over general base imply the following theorem.

Theorem 7.11. *The open subscheme $(\mathsf{M}_{(n,R)}{}^g)^s$ of Lemma* 7.9 *is $\mathsf{PGL}_{(n,R)}$-invariant and if one denotes its quotient by $\mathsf{Z}_{(n,R)}{}^s := (\mathsf{M}_{(n,R)}{}^g)^s/\mathsf{PGL}_{(n,R)}$, then the canonical quotient morphism $(\mathsf{M}_{(n,R)}{}^g)^s \longrightarrow \mathsf{Z}_{(n,R)}{}^s$ is also a quotient of the type mentioned in Theorem* 7.3. *Further the open immersion $(\mathsf{M}_{(n,R)}{}^g)^s \hookrightarrow \mathsf{M}_{(n,R)}{}^g$ descends to give an open immersion $\mathsf{Z}_{(n,R)}{}^s \hookrightarrow \mathsf{Z}_{(n,R)}$.*

Remark 7.12

(1) When R is normal and integral, since $\mathsf{M}_{(n,R)}{}^g$ (resp. $(\mathsf{M}_{(n,R)}{}^g)^s$) is also normal and integral, it follows from geometric invariant theory that $\mathsf{Z}_{(n,R)}$ (resp. $\mathsf{Z}_{(n,R)}{}^s$) is normal and integral.

(2) Assertions analogous to those in (1), Remark 7.4 are valid for ${\mathsf{Z}_{(n,R)}}^s$. In particular, the diagrams we got from (1) of Remark 7.4 by replacing ${\mathsf{M}_{(n,R)}}^g$, $\mathsf{Z}_{(n,R)}$, $\alpha_{(n,R,S)}$ etc by $({\mathsf{M}_{(n,R)}}^g)^s$, ${\mathsf{Z}_{(n,R)}}^s$, $\alpha^s_{(n,R,S)}$ respectively are also commutative and when R is a flat S-algebra, $\alpha^s_{(n,R,S)}$ is an isomorphism. One also has

$$\alpha^s_{(n,R',S)} = (\alpha^s_{(n,R,S)} \otimes_R R') \circ \alpha^s_{(n,R',R)}.$$

Further the topological map underlying $\alpha^s_{(n,R,S)}$ is bijective – in fact even bijective on L-valued points for each algebraically closed field L which is also an R-algebra. Finally, the categorical quotient property of ${\mathsf{Z}_{(n,R)}}^s$ implies that the following diagram is cartesian, showing that $\alpha^s_{(n,R,S)}$ is an affine morphism.

$$\begin{tikzcd}
{\mathsf{Z}_{(n,R)}}^s \arrow[r, "\alpha^s_{(n,R,S)}"] \arrow[d, "\text{open immersion}"'] & {\mathsf{Z}_{(n,S)}}^s \otimes_S R \arrow[d, "\text{(open imm: base-chg from } S)"] \\
\mathsf{Z}_{(n,R)} \arrow[r, "\alpha_{(n,R,S)}"] & \mathsf{Z}_{(n,S)} \otimes_S R
\end{tikzcd}$$

(3) By Seshadri's GIT over general base, it can be seen that the canonical quotient morphism $({\mathsf{M}_{(n,\mathbb{Z})}}^g)^s \longrightarrow {\mathsf{Z}_{(n,\mathbb{Z})}}^s$ specializes well – in fact, one has more, as we will see in Theorem 7.37. Note that the geometric quotient ${\mathsf{Z}_{(n,K)}}^s$ is $\cong {\mathsf{Z}_{(n,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} K$ via $\alpha^s_{(n,K,\mathbb{Z})}$.

(4) Thus the singularities (if any) of the normal variety $\mathsf{Z}_{(n,K)}$ lie outside the open set ${\mathsf{Z}_{(n,K)}}^s$. In fact, even when $\mathrm{char}(K) = 0$, there are singularities, so that $\mathsf{Z}_{(n,\mathbb{Z})}$ is not smooth over $\mathbb{Z}$. To see this, take $K = \mathbb{C}$ and $M := \mathcal{U}_X^{SS}(n, 0)$, the normal projective variety of equivalence classes of semi-stable vector bundles of a fixed rank $n \geq 2$ and degree zero on a smooth projective curve X over K of fixed genus $g \geq 2$ (with $g > n$ when $n = 2$). Let $m_0 \in M$ be the point corresponding to the trivial vector bundle of rank n. Then m_0 is a singular point of M (see the beginning of §6). An application of Luna's Etale Slice Theorem shows that the completion of the local ring of M at m_0 is isomorphic to the completion of the local ring of $\mathsf{Z}_{(n,K)}$ at the point corresponding to the g-tuple of identity matrices. The aim of the present section is to show the existence of the Nori desingularization of $\mathsf{Z}_{(2,R)}$ when R is a normal Nagata domain and that it specializes well whenever $\mathsf{Z}_{(2,R)}$ is geometrically reduced, and hence in particular when $R = \mathbb{Z}$.

7.2 *Extension of Nori's Construction*

Nori in the Appendix to [16] constructs a scheme $\mathsf{Hilb}_{(n,\mathbb{Z})}$ which is a moduli for 'monogenic $A_{\mathbb{Z}}$-modules', and his candidate for desingularizing $\mathsf{Z}_{(2,\mathbb{Z})}$ is caught as a closed subscheme of $\mathsf{Hilb}_{(4,\mathbb{Z})}$. The following shows that the analogue $\mathsf{Hilb}_{(n,R)}$ of $\mathsf{Hilb}_{(n,\mathbb{Z})}$ may also be constructed. The role of monogenic modules here is analogous to that of parabolic vector bundles in the previous section.

DEFINITION 7.13

Let R be a noetherian commutative ring with 1 and as before let the non-commuting polynomial algebra over a ring S in g indeterminates be denoted by $A_S := S\{X_1, \dots, X_g\}$.

For any commutative R-algebra S with 1, let $\mathsf{Hilb}_{(n,R)}(S)$ denote the set of isomorphism classes of pairs (M, m) where (1) M is an A_S-module, which is locally free of rank n as an S-module, and (2) $m \in M$ generates M as A_S-module. Equivalently $\mathsf{Hilb}_{(n,R)}(R)$ is the set of left ideals $I \subset A_S$ such that A_S/I is locally free of rank n as an S-module.

Nori's method of showing the representability of the functor $\mathsf{Hilb}_{(n,\mathbb{Z})}$ also works to give the representability of $\mathsf{Hilb}_{(n,R)}$. This may be shown by the construction of a quotient of the following functor.

DEFINITION 7.14

For each commutative R-algebra S, let $\mathsf{U}_{(n,R)}(S)$ denote the set of triples $(M, \{e_1, \ldots, e_n\}, m)$ where (1) M is an A_S-module, (2) $\{e_1, \ldots, e_n\}$ is a basis for M as an S-module, and (3) M is an m-monogenic A_S-module, i.e., $m \in M$ generates M as A_S-module.

Remark 7.15

(1) Using a reasoning similar to the one in Remark 7.6, one sees that for any commutative R-algebra S, the set of pointed free modules, i.e., triples $(M, \{e_1, \ldots, e_n\}, m)$ where M is an A_S-module which is free of rank n as an S-module with basis $\{e_1, \ldots, e_n\}$ and $m \in M$, may be canonically identified with the set of S-valued points of the product $\mathsf{T}_{(n,R)} := {\mathsf{M}_{(n,R)}}^g \times_R \mathsf{A}^n_R$ where A^n_R is affine n-space over R. Moreover two such triples are isomorphic (as pointed modules) iff the corresponding S-valued points are in the same orbit of $\mathrm{GL}(n, S)$. Here the action on the first factor of $\mathsf{T}_{(n,R)}$ is the one described in Definition 7.1 while the action on the second factor is the canonical one.
(2) Note that $\mathsf{U}_{(n,R)}(S)$ may also be canonically identified (functorially in S) with the set of pairs $(I, \{e_1, \ldots, e_n\})$ where $I \subset A_S$ is a left ideal such that A_S/I is free of rank n as an S-module with basis $\{e_1, \ldots, e_n\}$.
(3) For each commutative R-algebra S, one has a canonical identification of $\mathsf{U}_{(n,R)}(S)$ with a $\mathsf{GL}_{(n,R)}(S)$-invariant subset of $\mathsf{T}_{(n,R)}(S)$. Moreover, this identification is functorial in S. Thus one gets a subfunctor $\mathsf{U}_{(n,R)} \hookrightarrow \mathsf{T}_{(n,R)}$. It can be checked that this is an open subfunctor, and henceforth $\mathsf{U}_{(n,R)}$ shall also denote the open $\mathsf{GL}_{(n,R)}$-invariant subscheme of $\mathsf{T}_{(n,R)}$ which represents it.

When the base $R = \mathbb{Z}$, the relationships between the open subscheme $\mathsf{U}_{(n,\mathbb{Z})}$ mentioned above, the functor $\mathsf{Hilb}_{(n,\mathbb{Z})}$ of Nori and the Artin moduli space $\mathsf{Z}_{(n,\mathbb{Z})}$ are given in the following result:

Theorem 7.16. (Nori, Proposition 1, [16]). *For the action of* $\mathsf{GL}_{(n,\mathbb{Z})}$ *on* $\mathsf{U}_{(n,\mathbb{Z})}$ *described above,* $\mathsf{U}_{(n,\mathbb{Z})} \longrightarrow \mathsf{U}_{(n,\mathbb{Z})}/\mathsf{GL}_{(n,\mathbb{Z})}$ *is a locally-trivial principal* $\mathsf{GL}_{(n,\mathbb{Z})}$*-bundle. Further* $\mathsf{U}_{(n,\mathbb{Z})}/\mathsf{GL}_{(n,\mathbb{Z})}$ *represents* $\mathsf{Hilb}_{(n,\mathbb{Z})}$ *and the first projection* $\mathsf{U}_{(n,\mathbb{Z})} \hookrightarrow \mathsf{T}_{(n,\mathbb{Z})} = {\mathsf{M}_{(n,\mathbb{Z})}}^g \times \mathsf{A}^n_{\mathbb{Z}} \longrightarrow {\mathsf{M}_{(n,\mathbb{Z})}}^g$ *goes down to a projective morphism* $\mathsf{Hilb}_{(n,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(n,\mathbb{Z})}$.

Remark 7.17. It is immediate from the above and the definition of $\mathsf{Hilb}_{(n,R)}$ that $\mathsf{Hilb}_{(n,R)}$ is representable over any base R and that the construction $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{Hilb}_{(n,R)}$ base-changes well. In particular $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{Hilb}_{(n,R)}$ is a universal categorical quotient.

Remark 7.18. Now it will be shown that there is a projective morphism $\mathsf{Hilb}_{(n,R)} \longrightarrow \mathsf{Z}_{(n,R)}$ (generalizing the case of $R = \mathbb{Z}$ in Theorem 7.16) such that the following diagram commutes

$$\begin{array}{ccc} \mathsf{U}_{(n,R)} & \longrightarrow & {\mathsf{M}_{(n,R)}}^g \\ \downarrow & & \downarrow \\ \mathsf{Hilb}_{(n,R)} & \longrightarrow & \mathsf{Z}_{(n,R)} \end{array}$$

where the top horizontal arrow is the $\mathsf{GL}_{(n,R)}$-equivariant morphism given by the restriction, to the $\mathsf{GL}_{(n,R)}$-invariant open subscheme $\mathsf{U}_{(n,R)}$, of the canonical first projection $\mathsf{T}_{(n,R)} = ({\mathsf{M}_{(n,R)}}^g \times \mathsf{A}^n_R) \longrightarrow {\mathsf{M}_{(n,R)}}^g$ which is also $\mathsf{GL}_{(n,R)}$-equivariant. Since $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{Hilb}_{(n,R)}$ is a categorical quotient (7.17), it is clear that $\exists$ a unique morphism $\mathsf{Hilb}_{(n,R)} \longrightarrow \mathsf{Z}_{(n,R)}$ such that the above diagram commutes, so only its projectivity has to be shown. For this, observe that the above diagram can be expanded to give the following commutative diagram

$$\begin{array}{ccccc} \mathsf{U}_{(n,R)} & \longrightarrow & {\mathsf{M}_{(n,R)}}^g & = & ({\mathsf{M}_{(n,\mathbb{Z})}}^g) \otimes_{\mathbb{Z}} R \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle(\text{base-chg from } \mathbb{Z})} \\ \mathsf{Hilb}_{(n,R)} & \longrightarrow & \mathsf{Z}_{(n,R)} & \xrightarrow{\alpha_{(n,R,\mathbb{Z})}} & \mathsf{Z}_{(n,\mathbb{Z})} \otimes_{\mathbb{Z}} R \end{array}$$

where $\alpha_{(n,R,\mathbb{Z})}$ exists by case (1), Remark 7.4, and is the unique morphism that makes the right square commute. Again by the categorial quotient property of $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{Hilb}_{(n,R)}$, it follows that the composition of the lower horizontal arrows must be the same as the base-change from $\mathbb{Z}$ of the morphism $\mathsf{Hilb}_{(n,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(n,\mathbb{Z})}$ of Theorem 7.16 (where Remark 7.17 has been used to identify the base-change to R of $\mathsf{U}_{(n,\mathbb{Z})} \longrightarrow \mathsf{Hilb}_{(n,\mathbb{Z})}$ with $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{Hilb}_{(n,R)}$). But this last morphism is projective, and $\alpha_{(n,R,\mathbb{Z})}$ is separated. So the first lower horizontal arrow is projective as claimed.

DEFINITION 7.19

With the notations of Definition 7.13, let $\mathsf{Hilb}'_{(n,R)}(S) \subset \mathsf{Hilb}_{(n,R)}(S)$ denote the subset corresponding to two-sided ideals I.

Remark 7.20

(1) For an ideal $I \in \mathsf{Hilb}'_{(n,R)}(S)$, note that A_S/I is not only a monogenic S-module locally-free of rank n, but also an S-algebra which is associative and has an identity for multiplication.
(2) $\mathsf{Hilb}'_{(n,R)}$ is a closed subfunctor of $\mathsf{Hilb}_{(n,R)}$. So by Nori's theorem, $\mathsf{Hilb}'_{(n,R)}$ is represented by a closed subscheme of $\mathsf{U}_{(n,R)}/\mathsf{GL}_{(n,R)}$. In the following, $\mathsf{Hilb}_{(n,R)}$ (respectively $\mathsf{Hilb}'_{(n,R)}$) will denote both the functor as well as its representing scheme.

DEFINITION 7.21

Let $\mathsf{P}_{(n,R)}$ denote the restriction of the locally-trivial principal $\mathsf{GL}_{(n,R)}$-bundle $\mathsf{U}_{(n,R)} \longrightarrow \mathsf{U}_{(n,R)}/\mathsf{GL}_{(n,R)} = \mathsf{Hilb}_{(n,R)}$ to the closed subscheme $\mathsf{Hilb}'_{(n,R)} \subset \mathsf{Hilb}_{(n,R)}$ defined above.

Remark 7.22. $\mathsf{P}_{(n,R)}$ is a closed subscheme of $\mathsf{U}_{(n,R)}$ Further, by part 2 of Remark 7.15, it is easy to see that $\mathsf{P}_{(n,R)}(S)$ can be identified (functorially in the R-algebra S) with the set of pairs $(I, \{e_1, \ldots, e_n\})$ where $I \subset A_S$ is a two-sided ideal such that A_S/I is free of rank n as an S-module with basis $\{e_1, \ldots, e_n\}$.

Recall from §2, the X-scheme of associative algebra structures on a fixed X-vector bundle $\mathbf{W}$ of rank 4 denoted by Id-w-Assoc$_{\mathbf{W}}$. We take $X = \mathrm{Spec}(R)$ and $\mathbf{W}$ to correspond to the free R-module $W := R^{\oplus n}$ of rank $n = m^2$ with the standard basis and in this case we denote $\mathrm{Alg}_{\mathbf{W}}$, Id-Assoc$_{\mathbf{W}}$, Azu$_{\mathbf{W}}$, Sp-Azu$_{\mathbf{W}}$ respectively by Alg_W, Id-Assoc$_W$, Azu$_W$, Sp-Azu$_W$. The smoothness of the R-scheme which will eventually desingularize $\mathsf{Z}_{(2,R)}$ will be deduced from the smoothness of Sp-Azu$_W$ (Theorem 3.8). As a first step, the following relates Id-Assoc$_W$ to $\mathsf{P}_{(n,R)}$.

DEFINITION 7.23

Let S be a commutative R-algebra. Let $(I, \{e_1, \ldots, e_n\}) \in \mathsf{P}_{(n,R)}(S)$ be as in Remark 7.22. The associative S-algebra with identity A_S/I defines an associative S-algebra structure with identity on $W \otimes_R S$ via the S-module isomorphism $(A_S/I) \cong W \otimes_R S$ defined by mapping the S-basis $\{e_1, \ldots, e_n\}$ onto the standard S-basis on $W \otimes_R S = R^{\oplus n} \otimes_R S = S^{\oplus n}$. In this way one gets a mapping

$$g_{(n,R)}(S) : \mathsf{P}_{(n,R)}(S) \longrightarrow \text{Id-Assoc}_W(S).$$

Remark 7.24

(1) It is clear from the above definition that $g_{(n,R)}(S)$ is functorial in S, i.e., one has a morphism of R-schemes

$$g_{(n,R)} : \mathsf{P}_{(n,R)} \longrightarrow \text{Id-Assoc}_W.$$

(2) Recall that $\mathsf{P}_{(n,R)}$ is a locally-closed $\mathsf{GL}_{(n,R)}$-invariant subscheme of the scheme $\mathsf{T}_{(n,R)}$ defined in part 1 of Remark 7.15 above. Further also recall from §2 that Id-Assoc$_W$ is a $\mathsf{GL}_{(n,R)} = \mathsf{GL}_W$-invariant subscheme of Alg_W. Now with respect to these actions of $\mathsf{GL}_{(n,R)}$, the morphism $g_{(n,R)}$ above is equivariant.

Nori shows in Lemma 1, Appendix, [16], that the morphism $g_{(n,\mathbb{Z})}$ is a smooth morphism. Since $\mathsf{P}_{(n,R)}$ base-changes well (by construction) and Id-Assoc$_W$ base-changes well (§2), and further the definition of $g_{(n,R)}$ shows it also base-changes well, one gets the following proposition.

PROPOSITION 7.25

$g_{(n,R)}$ is a smooth morphism.

Note: *From now on, W will denote the free module $R^{\oplus 4}$ of rank 4 over R given the standard basis.*

DEFINITION 7.26

Define the schemes $\mathsf{L}_{(2,R)}$, $\mathsf{H}_{(2,R)}$, $\mathsf{V}_{(2,R)}$, and $\mathsf{V}^s_{(2,R)}$ as per the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Azu}_W \arrow[r, "\text{open}"] & \text{Sp-Azu}_W \arrow[r, "\text{closed}"] & \text{Id-Assoc}_W \\
\mathsf{H}_{(2,R)} \arrow[u] \arrow[r, "\text{open}"] \arrow[d] & \mathsf{L}_{(2,R)} \arrow[u] \arrow[r, "\text{closed}"] \arrow[d] & \mathsf{P}_{(4,R)} \arrow[u, "g_{(4,R)}"'] \arrow[d] \\
\mathsf{V}^s_{(2,R)} \arrow[r, "\text{open}"] & \mathsf{V}_{(2,R)} \arrow[r, "\text{closed}"] & \mathsf{Hilb}'_{(4,R)}
\end{tikzcd}$$

The top row of the above diagram is naturally a $\mathsf{GL}_{(4,R)} = \mathsf{GL}_W$-equivariant sequence by §3. $\mathsf{L}_{(2,R)}$ and $\mathsf{H}_{(2,R)}$ are respectively the base-changes of Sp-Azu$_W$ and Azu$_W$. Since $g_{(4,R)}$ is $\mathsf{GL}_{(4,R)}$-equivariant as remarked above, it follows that the middle row of the above diagram is also $\mathsf{GL}_{(4,R)}$-equivariant. An application of Theorem 3.9 shows that the $\mathrm{GL}_4 R$-equivariant closed subscheme $\mathsf{L}_{(2,R)}$ descends to give a closed subscheme $\mathsf{V}_{(2,R)}$. Let $\mathsf{V}^s_{(2,R)}$ denote the canonical open subscheme structure on the topological image of $\mathsf{H}_{(2,R)}$ in $\mathsf{V}_{(2,R)}$ – this image is open since $\mathsf{L}_{(2,R)} \longrightarrow \mathsf{V}_{(2,R)}$ is also a locally-trivial principal $\mathsf{GL}_{(4,R)}$-bundle and hence is a flat morphism of finite type of noetherian schemes which is open. The canonical morphism $\mathsf{H}_{(2,R)} \longrightarrow \mathsf{V}_{(2,R)}$ factors through $\mathsf{V}^s_{(2,R)}$. The scheme $\mathsf{V}_{(2,\mathbb{Z})}$ is Nori's candidate for a birational model for $\mathsf{Z}_{(2,\mathbb{Z})}$.

Before proceeding, we need the following result connected with Theorem 3.9.

Theorem 7.27. *In addition to the hypotheses of Theorem* 3.9*, further assume that G, $\iota : Q \hookrightarrow B$, and $f : B \longrightarrow T$ base-change well. Then $\iota' : Z \hookrightarrow T$ also base-changes well.*

Proof. Let $S = \mathrm{Spec}(R)$, R' a commutative R-algebra with 1, and $S' := \mathrm{Spec}(R')$. Then because $Z_{S'}$ is the scheme-theoretic image of $Q_{S'} = Q_S \times_S S'$ under $f_{S'} \circ \iota_{S'} = (f \circ \iota) \times_S S'$ in $T_{S'} = T_S \times_S S'$, $\exists$ an S'-morphism $\zeta : Z_{S'} \longrightarrow Z \times_S S'$ such that $\zeta \circ f'_{S'} = f'_S \times_S S'$. Thus ζ is a surjective closed immersion. Note that $Z_{S'}$ and $Z \times_S S'$ are both smooth$/S'$. It follows that if S' were reduced, then ζ would have to be an isomorphism. Thus the morphisms induced by ζ at the geometric points of S' are all isomorphisms. Hence ζ is flat and hence faithfully flat (since it is surjective). But then ζ being a closed immersion implies that it must be an isomorphism. Q.E.D.

Theorem 7.28. *The scheme $\mathsf{V}_{(2,R)}$ of Definition* 7.26 *is smooth$/R$ and also base-changes well.*

Proof. Remembering that $\mathrm{rank}_R(W) = 4$, since Sp-Azu$_W$ is smooth$/R$ (Theorem 3.8) and since $\mathsf{L}_{(2,R)} \longrightarrow$ Sp-Azu$_W$ is a smooth morphism being the base-change of $g_{(4,R)}$ (cf. Proposition 7.25), it follows that $\mathsf{L}_{(2,R)}$ is also smooth$/R$. The smoothness$/R$ of $\mathsf{L}_{(2,R)}$ and $\mathsf{V}_{(2,R)}$ are equivalent by Theorem 3.9.

From the description of $\mathsf{P}_{(4,R)}$ in Remark 7.22, and the definition of $g_{(n,R)}$ (cf. Remark 7.24), it is immediate that the portion consisting of the top two rows of the commutative diagram of Definition 7.26 base-changes well. That the whole diagram including the bottom row also base-changes well is now a consequence of Definition 7.26 and Theorem 7.27. Q.E.D.

7.3 Birationality over the locus of simple modules

R continues to be a normal Nagata domain, and as before $A_S := S\{X_1, \ldots, X_g\}$ for any ring S. The aim is to establish an isomorphism $\gamma^s_{(2,R)} : \mathsf{V}^s_{(2,R)} \longrightarrow \mathsf{Z}_{(2,R)}{}^s$ which will capture the 'birational part' of the desingularizing morphism $\gamma_{(2,R)}$ to be constructed later.

Lemma 7.29. *Let K be an algebraically closed field which is also an R-algebra. Let $I \subset A_K$ be a two-sided ideal such that there exists a K-algebra isomorphism $\phi_I : A_K/I \cong M_n(K)$, where $M_n(K)$ is the algebra of $(n \times n)$-matrices over K. Such an isomorphism defines an A_K-module structure M_{ϕ_I} on the K-vector space K^n (given the standard basis so that* $\mathrm{End}_K(K^n) = M_n(K)$*). Then one has*:

(1) *M_{ϕ_I} is simple and its isomorphism class does not depend on ϕ_I. As a consequence, one writes simply M_I for M_{ϕ_I}.*
(2) *If $M_I \cong M_{I'}$ (as A_K-modules), then $I = I'$.*
(3) *Given a simple A_K-module structure M on K^n, $\exists\ I$ such that $M \cong M_I$.*

The proofs of (1) and (2) are elementary. The proof of (3) uses Burnside's theorem.

Lemma 7.30. *The construction $\mathsf{V}^s_{(2,R)}$ base-changes well.*

Proof. This was already seen implicitly in the proof of Theorem 7.28. Another way of seeing this is from the description of the functor of points of $\mathsf{V}^s_{(2,R)}$: for each commutative R-algebra S with 1, $\mathsf{V}^s_{(2,R)}(S)$ may be identified functorially in S (cf. (1) of Remark 7.20 and Definition 7.26) with the set of two-sided ideals $I \subset A_S$ such that the quotient A_S/I is locally-free of rank 4 as an S-module and is also an Azumaya S-algebra. Q.E.D.

By the above lemma, if $I \in \mathsf{V}^s_{(2,R)}(L)$ where L is an algebraically closed field which is also an R-algebra, then A_L/I is a four-dimensional Azumaya L-algebra. But by part (2) of Proposition 3.2, this algebra is isomorphic to $M_2(L)$. Therefore it defines a A_L-module structure on L^2 and following the notations of Lemma 7.29 above, the isomorphism class of this simple module is denoted $[M_I]$. Next let K be an algebraically closed subfield of L which is also an R-algebra. Observe that $[M_I] \in \mathsf{Z}_{(2,K)}{}^s(L)$ by Artin's description of $\mathsf{Z}_{(n,K)}{}^s$ (Theorem 7.10). Note also that by Lemma 7.30,

$$\begin{aligned}\mathsf{V}^s_{(2,R)}(L) :&= (\mathsf{V}^s_{(2,R)} \otimes_R L)(L) = ((\mathsf{V}^s_{(2,R)} \otimes_R K) \otimes_K L)(L)\\ &= (\mathsf{V}^s_{(2,K)} \otimes_K L)(L) =: \mathsf{V}^s_{(2,K)}(L).\end{aligned}$$

Now parts 2 and 3 of Lemma 7.29 clearly imply the following lemma.

Lemma 7.31. *With the above notations, the association*

$$\gamma^s_{(2,K)}(L) : \mathsf{V}^s_{(2,K)}(L) \longrightarrow \mathsf{Z}_{(2,K)}{}^s(L)\ ,\ I \longmapsto [M_I]$$

is a well-defined bijective map.

DEFINITION 7.32

Continuing with the above notations, let

$$\begin{aligned}\gamma^s_{(2,R)}(L) : \mathsf{V}^s_{(2,R)}(L) \longrightarrow \mathsf{Z}_{(2,R)}{}^s(L) &= (\mathsf{Z}_{(2,R)}{}^s \otimes_R L)(L)\\ &= (\mathsf{Z}_{(2,R)}{}^s \otimes_R K)(L)\end{aligned}$$

denote the composition $\alpha^s_{(2,K,R)}(L) \circ \gamma^s_{(2,K)}(L)$ where $\alpha^s_{(2,K,R)}(L) : \mathsf{Z}_{(2,K)}{}^s(L) \longrightarrow (\mathsf{Z}_{(2,R)}{}^s \otimes_R K)(L)$ is the bijective map of (2), of Remark 7.12. Note that by Lemma 7.31, $\gamma^s_{(2,R)}(L)$ is bijective.

Before showing that the above maps are maps underlying morphisms, one needs the following definition.

DEFINITION 7.33

The diagonal embedding $\mathsf{M}_{(n,R)} \hookrightarrow \mathsf{M}_{(n^2,R)}$ commutes with the conjugation actions of $\mathsf{PGL}_{(n,R)}$ and $\mathsf{PGL}_{(n^2,R)}$ (for the diagonal embedding of $\mathsf{PGL}_{(n,R)}$ in $\mathsf{PGL}_{(n^2,R)}$). Therefore by Theorem 7.3 there is an induced morphism $\Delta_{(n,R)} : \mathsf{Z}_{(n,R)} \longrightarrow \mathsf{Z}_{(n^2,R)}$.

Remark 7.34

(1) Let $K \subset L$ be an extension of algebraically closed fields. On L-valued points, it is easy to see that the morphism $\Delta_{(n,K)}$ sends the equivalence class of a A_L-module M to the equivalence class of the A_L-module $M \oplus \cdots \oplus M$ (n summands), with the equivalence described in Theorem 7.10. The uniqueness of the summands upto an ordering in the associated graded module for a Jordan–Hölder Series implies that $\Delta_{(n,K)}(L)$ is injective.

(2) If S is a commutative R-algebra with 1, then one has the following diagram

$$\begin{array}{ccc}
\mathsf{Z}_{(n,S)} & \xrightarrow{\alpha_{(n,S,R)}} & \mathsf{Z}_{(n,R)} \otimes_R S \\
{\scriptstyle \Delta_{(n,S)}}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_{(n,R)} \otimes_R S} \\
\mathsf{Z}_{(n^2,S)} & \xrightarrow[\alpha_{(n^2,S,R)}]{} & \mathsf{Z}_{(n^2,R)} \otimes_R S
\end{array}$$

which commutes because of the categorical quotient property of $\mathsf{Z}_{(n,S)}$.

(3) Taking $S = K$ an algebraically closed field in (2) above, one sees from part 1 of Remark 7.34 above and (2) of 7.4 that $\Delta_{(n,R)}$ is topologically injective–even injective on L-valued points for each algebraically closed field L which is an R-algebra.

(4) It can be seen that $\Delta_{(n,\mathbb{Z})}$ and $\Delta_{(n,K)}$ are closed immersions.

By Theorem 7.16 there is a projective morphism $\mathsf{Hilb}_{(4,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(4,\mathbb{Z})}$. Recall from Definition 7.26 that $\mathsf{V}_{(2,\mathbb{Z})}$ is a closed subscheme of $\mathsf{Hilb}'_{(4,\mathbb{Z})} \subset \mathsf{Hilb}_{(4,\mathbb{Z})}$. Let $\mu_{(2,\mathbb{Z})} : \mathsf{V}_{(2,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(4,\mathbb{Z})}$ be the induced morphism, which is clearly projective.

Theorem 7.35. (Nori, Appendix, [16])

(1) *There exists a projective morphism $\gamma_{(2,\mathbb{Z})} : \mathsf{V}_{(2,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(2,\mathbb{Z})}$ whose restriction to the open subscheme $\mathsf{V}^s_{(2,\mathbb{Z})}$ factors through the open subscheme $\mathsf{Z}_{(2,\mathbb{Z})}{}^s$ by an isomorphism $\gamma^s_{(2,\mathbb{Z})}$ for which the map on K-valued points is precisely the map $\gamma^s_{(2,\mathbb{Z})}(K)$ of Definition* 7.32, *for each algebraically closed field K.*

(2) *Let K be an algebraically closed field. There exists a projective morphism $\gamma_{(2,K)} : \mathsf{V}_{(2,K)} \longrightarrow \mathsf{Z}_{(2,K)}$ whose restriction to the open subscheme $\mathsf{V}^s_{(2,K)}$ factors through the open subscheme $\mathsf{Z}_{(2,K)}{}^s$ by an isomorphism $\gamma^s_{(2,K)}$ for which the map on L-valued points is precisely the map $\gamma^s_{(2,K)}(L)$ of Lemma* 7.31, *for each algebraically closed extension field L of K.*

Theorem 7.36. *The morphisms* $\gamma_{(2,\mathbb{Z})} : \mathsf{V}_{(2,\mathbb{Z})} \longrightarrow \mathsf{Z}_{(2,\mathbb{Z})}$ *and* $\gamma_{(2,K)} : \mathsf{V}_{(2,K)} \longrightarrow \mathsf{Z}_{(2,K)}$ *are desingularizations, and in fact, the base-change of* $\gamma_{(2,\mathbb{Z})}$ *to* K *may be canonically identified with* $\gamma_{(2,K)}$, *i.e., the desingularization* $\gamma_{(2,\mathbb{Z})}$ *has a good specialization property.*

Proof. The morphism $\gamma_{(2,\mathbb{Z})}$ is projective (Theorem 7.35) and is birational as asserted above. The smoothness over $\mathbb{Z}$ of $\mathsf{V}_{(2,\mathbb{Z})}$ follows from the case $R = \mathbb{Z}$ of Theorem 7.28. Similar arguments hold when $\mathbb{Z}$ is replaced by K. The good specialization property is a consequence of the fact that $\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K = \alpha_{(2,K,\mathbb{Z})} \circ \gamma_{(2,K)}$, where the identification $\mathsf{V}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K \cong \mathsf{V}_{(2,K)}$ has as usual been made in view of Theorem 7.28, and the fact that $\alpha_{(2,K,\mathbb{Z})}$ is an isomorphism (Donkin's result, Theorem 7.5). Q.E.D.

Now the discussion proceeds to construct the isomorphism $\gamma^s_{(2,R)}$ when R is a normal Nagata domain (7.2).

Theorem 7.37. *Let* R *be a normal Nagata domain. Then the morphism* $(\mathsf{M}_{(2,R)})^s \longrightarrow {\mathsf{Z}_{(2,R)}}^s$ (*Theorem* 7.11) *base-changes well to extensions* R' *of* R *which is also a normal Nagata domain. In other words, the canonical morphism* $\alpha^s_{(2,R',R)}$ *of part* 2 *of Remark* 7.12 *is an isomorphism such that the following diagram commutes*:

$$\begin{tikzcd}
({\mathsf{M}_{(2,R')}}^g)^s \arrow[r, equal] \arrow[d] & ({\mathsf{M}_{(2,R)}}^g)^s \otimes_R R' \arrow[d] \\
{\mathsf{Z}_{(2,R')}}^s \arrow[r, "\cong", "\alpha^s_{(2,R',R)}"'] & {\mathsf{Z}_{(2,R)}}^s \otimes_R R'
\end{tikzcd}$$

Proof. Recall from part 2 of Remark 7.12 that $\alpha^s_{(2,R',R)}$ is bijective (in fact it is bijective on L-valued points for every algebraically closed field L which is also an R'-algebra). Further ${\mathsf{Z}_{(2,R')}}^s$ is of finite type over R' by Theorem 7.3, and so it follows that $\alpha^s_{(2,R',R)}$ is a morphism of finite type. Therefore in the sense of EGA I [5], §6.11.3, it is a quasi-finite morphism. Note also that it is an affine morphism (cf. part 2, Remark 7.12) and hence it is separated.

Next, note the following properties of ${\mathsf{Z}_{(2,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} R'$, which, as seen above, is of finite type over R'. Since R' is an integral domain, it follows (by GIT) that ${\mathsf{Z}_{(2,R')}}^s$ is integral, and so it is immediate that ${\mathsf{Z}_{(n,R)}}^s \otimes_R R'$ is irreducible. Now by Theorems 7.28 and 7.35, ${\mathsf{Z}_{(2,\mathbb{Z})}}^s$ is smooth$/\mathbb{Z}$ and hence its base-change ${\mathsf{Z}_{(2,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} R'$ is also smooth$/R'$. Therefore ${\mathsf{Z}_{(2,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} R'$ is integral. Further, since R' is normal, it follows that ${\mathsf{Z}_{(2,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} R'$ is also normal.

Finally, let $Q(R')$ denote the quotient field of R'. By part 2 of Remark 7.12, one has the following commutative diagram

$$\begin{tikzcd}
({\mathsf{M}_{(2,Q(R'))}}^g)^s \arrow[r, equal] \arrow[d] & ({\mathsf{M}_{(2,R')}}^g)^s \otimes_{R'} Q(R') \arrow[r] \arrow[d] & ({\mathsf{M}_{(2,R')}}^g)^s \arrow[d] \\
{\mathsf{Z}_{(2,Q(R'))}}^s \arrow[r, "\alpha^s_{(2,Q(R'),R')}"] \arrow[d] & {\mathsf{Z}_{(2,R')}}^s \otimes_{R'} Q(R') \arrow[r] \arrow[d] & {\mathsf{Z}_{(2,R')}}^s \arrow[d] \\
\mathrm{Spec}(Q(R')) \arrow[r, equal] & \mathrm{Spec}(Q(R')) \arrow[r] & \mathrm{Spec}(R')
\end{tikzcd}$$

where $\alpha^s_{(2,Q(R'),R')}$ is an isomorphism. Also by part 2, Remark 7.12 one has that

$$(\alpha^s_{(2,R',\mathbb{Z})} \otimes_{R'} Q(R')) \circ \alpha^s_{(2,Q(R'),R')} = \alpha^s_{(2,Q(R'),\mathbb{Z})}.$$

Therefore, $\alpha^s_{(2,R',\mathbb{Z})}$ is an isomorphism by Zariski's Main Theorem. The same is true of $\alpha^s_{(2,R,\mathbb{Z})}$. But again by the uniqueness of the morphism $\alpha^s_{(2,R',R)}$ (part 2, Remark 7.12), one has that

$$(\alpha^s_{(2,R,\mathbb{Z})} \otimes_R R') \circ (\alpha^s_{(2,R',R)}) = \alpha^s_{(2,R',\mathbb{Z})}$$

where the canonical identification $(\mathsf{Z}_{(2,\mathbb{Z})}{}^s \otimes_{\mathbb{Z}} R) \otimes_R R' \cong \mathsf{Z}_{(2,\mathbb{Z})}{}^s \otimes_{\mathbb{Z}} R'$ has been made. From this it follows that $\alpha^s_{(2,R',R)}$ is indeed an isomorphism. Q.E.D.

DEFINITION 7.38

By base-changing the isomorphism $\gamma^s_{(2,\mathbb{Z})}$ of Theorem 7.35 to R and using Lemma 7.30, one gets an isomorphism $\gamma^s_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R : \mathsf{V}^s_{(2,R)} \cong \mathsf{Z}_{(2,\mathbb{Z})}{}^s \otimes_{\mathbb{Z}} R$. Let the composition of this isomorphism with the inverse of the isomorphism $\alpha^s_{(2,R,\mathbb{Z})}$ of Theorem 7.37 be denoted by $\gamma^s_{(2,R)} : \mathsf{V}^s_{(2,R)} \xrightarrow{\cong} \mathsf{Z}_{(2,R)}{}^s$.

7.4 *Construction and specialization of the desingularization*

Theorem 7.39. *Let R be a normal Nagata domain. Then there exists a unique projective morphism $\gamma_{(2,R)} : \mathsf{V}_{(2,R)} \longrightarrow \mathsf{Z}_{(2,R)}$ whose restriction to the open subscheme $\mathsf{V}^s_{(2,R)}$ factors through the isomorphism $\gamma^s_{(2,R)}$ constructed earlier – in other words, $\gamma_{(2,R)}$ is a desingularization. If it is further assumed that the geometric fibers of $\mathsf{Z}_{(2,R)}$ are reduced, then this desingularization specializes well – in particular, this is indeed the case when $R = \mathbb{Z}$.*

The proof of the above theorem is divided into several steps.

Step 1: Defining the underlying set-theoretic map. By Theorem 7.35 and the good base-change property of $\mathsf{V}_{(2,R)}$ of Theorem 7.28, one gets a morphism

$$\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R : \mathsf{V}_{(2,R)} \longrightarrow \mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R.$$

Let $|\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R| : |\mathsf{V}_{(2,R)}| \longrightarrow |\mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R|$ be the underlying map of topological spaces. Note that this map is surjective. Next let $|\alpha_{(2,R,\mathbb{Z})}| : |\mathsf{Z}_{(2,R)}| \longrightarrow |\mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R|$ denote the bijective map of topological spaces underlying the morphism $\alpha_{(2,R,\mathbb{Z})}$ ($\alpha_{(2,R,\mathbb{Z})}$ is bijective on L-valued points for each algebraically closed field L which is a $\mathbb{Z}$-algebra – see (1) and (2) of Remark 7.4). It follows that the map of sets

$$|\gamma_{(2,R)}| := |\alpha_{(2,R,\mathbb{Z})}|^{-1} \circ |\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R| : |\mathsf{V}_{(2,R)}| \longrightarrow |\mathsf{Z}_{(2,R)}|$$

is surjective. Further note that this map restricted to the open subset $\mathsf{V}^s_{(2,R)} \hookrightarrow \mathsf{V}_{(2,R)}$ is a morphism, i.e., $|\gamma_{(2,R)}|$ restricted to $|\mathsf{V}^s_{(2,R)}|$ factors through $|\mathsf{Z}_{(2,R)}{}^s|$ by a set-theoretic map denoted $|\gamma^s_{(2,R)}|$ which is none other than the map underlying the isomorphism $\gamma^s_{(2,R)}$ of Definition 7.38.

Step 2: Showing $\mathsf{V}_{(2,R)}$ *is integral, normal and separated of finite type over* R. Since R is reduced and normal, and $\mathsf{V}_{(2,R)}$ is smooth/R (Theorem 7.28), $\mathsf{V}_{(2,R)}$ is certainly reduced, normal and separated. To see that it is irreducible, recall the diagram of Definition 7.26. Since $g_{(4,R)}$ is a smooth morphism (Proposition 7.25) and hence open, and Azu_W is an open dense subscheme of Sp-Azu_W (by the definition of Sp-Azu_W), it follows that $\mathsf{H}_{(2,R)}$ is an open dense subscheme of $\mathsf{L}_{(2,R)}$. Further as noted in Definition 7.26, $\mathsf{L}_{(2,R)} \longrightarrow \mathsf{V}_{(2,R)}$ is open and surjective and hence $\mathsf{V}^s_{(2,R)}$ is open and dense in $\mathsf{V}_{(2,R)}$. But $\mathsf{V}^s_{(2,R)} \cong {\mathsf{Z}_{(2,R)}}^s$ via the isomorphism $\gamma^s_{(2,R)}$ (Definition 7.38), and since $\mathsf{Z}_{(2,R)}$ is irreducible, it follows that $\mathsf{V}_{(2,R)}$ is also irreducible. By Remark 7.18, one has a projective morphism $\mathsf{Hilb}_{(4,R)} \longrightarrow \mathsf{Z}_{(4,R)}$, and so the composite morphism $\mathsf{V}_{(2,R)} \hookrightarrow \mathsf{Hilb}_{(4,R)} \longrightarrow \mathsf{Z}_{(4,R)}$ is also projective since the first one is a closed immersion. This put together with the fact that $\mathsf{Z}_{(4,R)}$ is of finite type over R (from Theorem 7.3 since R is a Nagata ring) implies that $\mathsf{V}_{(2,R)}$ is also of finite type over R.

Step 3: Construction of the reduced graph. Let the set-theoretic graph of $|\gamma_{(2,R)}|$ be denoted by

$$\Gamma_{|\gamma_{(2,R)}|} \subset |\mathsf{V}_{(2,R)}| \times |\mathsf{Z}_{(2,R)}|.$$

That this set is closed follows from the fact that it is the topological space underlying the inverse-image of $\Gamma_{(\gamma_{(2,\mathbb{Z})}\otimes_{\mathbb{Z}} R)}$ – the graph of the morphism $\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R$ induced by the base-change of $\gamma_{(2,\mathbb{Z})}$ of Theorem 7.35 – under the morphism $(\mathrm{Id}_{\mathsf{V}_{(2,R)}} \times \alpha_{(2,R,\mathbb{Z})})$. Let the canonical reduced induced closed subscheme structure on $\Gamma_{|\gamma_{(2,R)}|}$ be denoted

$$\Gamma_{\gamma_{(2,R)}} \hookrightarrow (\mathsf{V}_{(2,R)} \times_R \mathsf{Z}_{(2,R)})$$

in spite of the fact that the morphism $\gamma_{(2,R)}$ has yet to be shown to exist. Let the base-change of this closed subscheme by the canonical open immersion $\mathsf{V}^s_{(2,R)} \times_R {\mathsf{Z}_{(2,R)}}^s \hookrightarrow \mathsf{V}_{(2,R)} \times_R \mathsf{Z}_{(2,R)}$ give the closed subscheme

$$\Gamma'_{\gamma^s_{(2,R)}} \hookrightarrow \mathsf{V}^s_{(2,R)} \times_R {\mathsf{Z}_{(2,R)}}^s.$$

Since this subscheme is also an open subscheme of $\Gamma_{\gamma_{(2,R)}}$, it follows that it is also reduced. One also has another reduced closed subscheme $\Gamma_{\gamma^s_{(2,R)}} \hookrightarrow \mathsf{V}^s_{(2,R)} \times_R {\mathsf{Z}_{(2,R)}}^s$ corresponding to the graph of the (iso)morphism $\gamma^s_{(2,R)}$. In fact one has the equality of closed subschemes $\Gamma_{\gamma^s_{(2,R)}} = \Gamma'_{\gamma^s_{(2,R)}}$ which follows from the easy check that their underlying topological spaces are the same, since both are reduced. Let $p_1 : \mathsf{V}_{(2,R)} \times_R \mathsf{Z}_{(2,R)} \longrightarrow \mathsf{V}_{(2,R)}$ denote the canonical first projection. It is now straightforward that

$$(p_1|\Gamma_{\gamma_{(2,R)}}) : \Gamma_{\gamma_{(2,R)}} \longrightarrow \mathsf{V}_{(2,R)}$$

when further restricted to the open subscheme $\Gamma_{\gamma^s_{(2,R)}} = \Gamma'_{\gamma^s_{(2,R)}}$ factors through the open subscheme $\mathsf{V}^s_{(2,R)}$ by an isomorphism (onto $\mathsf{V}^s_{(2,R)}$). Hence $(p_1|\Gamma_{\gamma_{(2,R)}})$ is birational. It is easy to check that this map is also set-theoretically bijective. Since R is a Nagata ring, $\mathsf{Z}_{(2,R)}$ is of finite type over R due to Theorem 7.3. Therefore it follows that the morphism $(p_1|\Gamma_{\gamma_{(2,R)}})$ is also a morphism of finite type and hence also quasi-finite in the sense of EGA I, §6.11.3. That this morphism is also affine (and hence separated) follows from the fact that $\mathsf{Z}_{(2,R)}$ is an affine R-scheme. From these observations it follows that $\Gamma_{\gamma_{(2,R)}}$ is a reduced separated scheme of finite-type over R.

Step 4: Irreducibility of the graph. Pick a point $\psi : \mathrm{Spec}(K) \longrightarrow \Gamma_{\gamma_{(2,R)}}$ where K is an algebraically closed field which is also an R-algebra. By projecting onto $\mathsf{V}_{(2,R)}$ and $\mathsf{Z}_{(2,R)}$ one gets points $n \in \mathsf{V}_{(2,R)}(K) = \mathsf{V}_{(2,K)}(K)$ and $m \in \mathsf{Z}_{(2,R)}(K) = (\mathsf{Z}_{(2,R)} \otimes_R K)(K)$ such that $(\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K)(n) = (\alpha_{(2,R,\mathbb{Z})} \otimes_R K)(m)$ because of the definition of $|\Gamma_{\gamma_{(2,R)}}|$. One has the following commutative diagram:

$$\begin{tikzcd}
\mathsf{V}_{(2,K)} \arrow[r, equal] \arrow[d, "\gamma_{(2,K)}"'] & \mathsf{V}_{(2,R)} \otimes_R K \arrow[r, equal] \arrow[d] & \mathsf{V}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K \arrow[d, "\gamma_{(2,\mathbb{Z})}\otimes_{\mathbb{Z}} K"] \\
\mathsf{Z}_{(2,K)} \arrow[r, "\alpha_{(2,K,R)}"] & \mathsf{Z}_{(2,R)} \otimes_R K \arrow[r, "\alpha_{(2,R,\mathbb{Z})}\otimes_R K"] & \mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K
\end{tikzcd}$$

where the outermost arrows commute by Theorem 7.36 and the central vertical downward arrow has been defined so that the diagram commutes. By parts 1 and 2 of Remark 7.4, the composition of the lower horizontal arrows must be the bijective morphism $\alpha_{(2,K,\mathbb{Z})}$ (in fact, this last morphism is an isomorphism, which implies that $\alpha_{(2,K,R)}$ is a surjective closed immersion). So the second of the lower horizontal arrows of the above commutative diagram is bijective. Therefore one has that $(\alpha_{(2,K,R)} \circ \gamma_{(2,K)})(K)(n) = m$, i.e., (n, m) is a K-point of the graph Γ_K of $(\alpha_{(2,K,R)} \circ \gamma_{(2,K)})$. Γ_K is an integral closed subscheme of

$$(\mathsf{V}_{(2,R)} \otimes_R K) \otimes_K (\mathsf{Z}_{(2,R)} \otimes_R K) = (\mathsf{V}_{(2,R)} \otimes_R \mathsf{Z}_{(2,R)}) \otimes_R K$$

since it is isomorphic to $\mathsf{V}_{(2,K)}$. By part 2 of Remark 7.12, there is also another commutative diagram

$$\begin{tikzcd}
{\mathsf{Z}_{(2,K)}}^s \arrow[r, "\alpha^s_{(2,K,R)}"] \arrow[d, "\text{open}"'] & {\mathsf{Z}_{(2,R)}}^s \otimes_R K \arrow[r, "\alpha^s_{(2,R,\mathbb{Z})}\otimes_R K"] \arrow[d, "\text{open}"'] & {\mathsf{Z}_{(2,\mathbb{Z})}}^s \otimes_{\mathbb{Z}} K \arrow[d, "\text{open}"] \\
\mathsf{Z}_{(2,K)} \arrow[r, "\alpha_{(2,K,R)}"] & \mathsf{Z}_{(2,R)} \otimes_R K \arrow[r, "\alpha_{(2,R,\mathbb{Z})}\otimes_R K"] & \mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} K
\end{tikzcd}$$

where the top row consists of isomorphisms given by Theorem 7.37, and the composition of the upper horizontal arrows is the same as the isomorphism $\alpha^s_{(2,K,\mathbb{Z})}$. This put together with the definition of $\gamma^s_{(2,R)}$ (Definition 7.38) implies that $(\alpha_{(2,K,R)} \circ \gamma_{(2,K)})|\mathsf{V}^s_{(2,K)}$ is the same as the isomorphism $\gamma^s_{(2,R)} \otimes_R K$ followed by the canonical open immersion ${\mathsf{Z}_{(2,R)}}^s \otimes_R K \hookrightarrow \mathsf{Z}_{(2,R)} \otimes_R K$. Hence the graph Γ^s_K of $\gamma^s_{(2,R)} \otimes_R K$ is an open, and therefore dense, subset of the graph Γ_K of $(\alpha_{(2,K,R)} \circ \gamma_{(2,K)})$. Hence if U_ψ is an R-open neighborhood of the point represented by ψ, then its base-change to K contains the K-point (n, m) and therefore must intersect Γ^s_K, i.e., it contains a K-point of $\Gamma_{\gamma^s_{(2,R)}}$ (since the base-change of the graph of a morphism may be canonically identified with the graph of the base-change of that morphism). Since every point of $|\Gamma_{\gamma_{(2,R)}}|$ is a limit point of $|\Gamma_{\gamma^s_{(2,R)}}|$, one has $|\Gamma_{\gamma_{(2,R)}}| \subset \overline{|\Gamma_{\gamma^s_{(2,R)}}|}$ (closure in $|\mathsf{V}_{(2,R)} \times_R \mathsf{Z}_{(2,R)}|$). But on the other hand $\Gamma_{\gamma^s_{(2,R)}} \hookrightarrow \Gamma_{\gamma_{(2,R)}}$ is an open subscheme and $\Gamma_{\gamma_{(2,R)}}$ is closed which implies that $\overline{|\Gamma_{\gamma^s_{(2,R)}}|} \subset$ (and hence =) $|\Gamma_{\gamma_{(2,R)}}|$. Now $\Gamma_{\gamma^s_{(2,R)}}$ is irreducible since it is isomorphic to $\mathsf{V}^s_{(2,R)}$ and hence $\Gamma_{\gamma_{(2,R)}}$ is also irreducible.

Step 5: The desingularization and its specializations. It now follows from Zariski's Main Theorem that $(p_1|\Gamma_{\gamma_{(2,R)}}) : \Gamma_{\gamma_{(2,R)}} \longrightarrow \mathsf{V}_{(2,R)}$ is an isomorphism. Thus one gets a morphism

$$\gamma_{(2,R)} := (p_2|\Gamma_{\gamma_{(2,R)}}) \circ (p_1|\Gamma_{\gamma_{(2,R)}})^{-1} : \mathsf{V}_{(2,R)} \longrightarrow \mathsf{Z}_{(2,R)}$$

for which the underlying map is $|\gamma_{(2,R)}|$ and whose graph is indeed $\Gamma_{\gamma_{(2,R)}}$. Here $p_2 : \mathsf{V}_{(2,R)} \times_R \mathsf{Z}_{(2,R)} \longrightarrow \mathsf{Z}_{(2,R)}$ denotes the canonical second projection. Note that by construction, $\gamma_{(2,R)}|\mathsf{V}^s_{(2,R)}$ factors through $\mathsf{Z}_{(2,R)}{}^S$ by the isomorphism $\gamma^s_{(2,R)}$.

Projectivity of the morphism $\gamma_{(2,R)}$. Now $\alpha_{(2,R,\mathbb{Z})} \circ \gamma_{(2,R)}$ and $\gamma_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R$ are two morphisms from the reduced scheme $\mathsf{V}_{(2,R)}$ into the separated scheme $\mathsf{Z}_{(2,\mathbb{Z})} \otimes_{\mathbb{Z}} R$ that agree on the open dense subscheme $\mathsf{V}^s_{(2,R)}$. Therefore they are equal. But then $\gamma_{(2,\mathbb{Z})}$ is projective, and so the same is true of $\gamma_{(2,R)}$, since $\alpha_{(2,R,\mathbb{Z})}$ is separated.

Specialization properties of $\gamma_{(2,R)}$. Observe that the central downward arrow of the first of the two commutative diagrams of Step 4 when restricted to $\mathsf{V}^s_{(2,R)} \otimes_R K$ factors through the isomorphism $\gamma^s_{(2,R)} \otimes_R K$. Hence this morphism is precisely the same as $\gamma_{(2,R)} \otimes_R K$. As noted in Step 4, $\alpha_{(2,K,R)}$ is already a surjective closed immersion. So under the additional hypothesis that the geometric fibers of $\mathsf{Z}_{(2,R)}$ over R are reduced, $\mathsf{Z}_{(2,R)} \otimes_R K$ is reduced, and so $\alpha_{(2,K,R)}$ becomes an isomorphism. This implies that $\alpha_{(2,R,\mathbb{Z})} \otimes_R K$ is also an isomorphism. End of Proof of Theorem 7.39.

The last part of Step 5 of the above proof shows that Theorem 7.5 generalizes as follows:

Theorem 7.40. *Let R be a normal Nagata domain and suppose that the geometric fibers of $\mathsf{Z}_{(n,R)}$ are reduced. Then these geometric fibers are in fact normal. If K is an algebraically closed field which is also an R algebra, then the uniform categorical quotient $\mathsf{M}_{(n,R)}{}^g \longrightarrow \mathsf{Z}_{(n,R)}$ of Theorem* 7.3 *specializes well, i.e., its base-change to K* (*over R*) *may be functorially identified with the Mumford good quotient $\mathsf{M}_{(n,K)}{}^g \longrightarrow \mathsf{Z}_{(n,K)}$.*

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