Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 3, August 2002, pp. 453–462.

A result concerning the stability of some difference equations and its applications

HARK-MAHN KIM

Department of Mathematics, Chungnam National University, Taejon 305-764, Korea
E-mail: hmkim@math.cnu.ac.kr

MS received 25 March 2002

Abstract. In this paper, we investigate the Hyers–Ulam stability problem for the difference equation $f(x + p, y + q) - \varphi(x, y)f(x, y) - \psi(x, y) = 0$.

Keywords. Hyers–Ulam stability; difference equation.

1. Introduction

Mathematical computations frequently are based on equations, called difference equations or recurrence equations that allow us to compute the value of a function recursively from a given set of values. These equations occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology, and other fields. The study of linear difference equations is important for a number of reactions. Many types of problems are naturally formulated as linear equations [14].

In 1940, Ulam [18] raised a question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If the answer is affirmative, we would call the equation of homomorphism $H(xy) = H(x)H(y)$ stable. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1,3–13,15–17].

Now, we investigate Hyers–Ulam stability problem for the following difference equation:

$$f(x + p, y + q) - \varphi(x, y)f(x, y) - \psi(x, y) = 0, \tag{1.1}$$

$$f(x + p, y + q) - \varphi(x, y)f(x, y) = 0. \tag{1.2}$$

Thus we find situations when the approximate solutions of an equation differing slightly from a given difference equation must be close to the true solution of the given equation.

It is important to provide methods and suitable criterion that describe the nature and behavior of solutions of difference systems, without actually constructing or approximating them. In contrast with differential equations, since the existence and uniqueness of solutions of discrete initial value problems is already guaranteed, one of the problems is the study of asymptotic behavior of solutions of the difference system [2].

Apart from the above point of view, in this paper we examine the situations that the difference equation (1.1) is stable in the sense of Hyers and Ulam, and for a given δ-approximate function we construct a true solution of the difference equation near it. Throughout this paper, let $\delta > 0$ and $p, q \in \mathbb{N}$ be fixed, and $\mathbb{N}$ denote the set of all positive integers.

2. Main results

Before taking up the main subject we point out the following situation which is similar to that of elementary homogeneous linear differential equation. That is, if a particular solution f_p of (1.1) is given, then the general solution f of (1.1) has the form $f = f_h + f_p$, where f_h is a solution of (1.2).

In the next theorem, let two functions $\varphi : \mathbb{N} \times \mathbb{N} \to (0, \infty)$, $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ satisfy

$$\varepsilon(x, y) := \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{\varphi(x + ip, y + iq)} < \infty, \tag{2.1}$$

$$\varepsilon'(x, y) := \sum_{j=0}^{\infty} \frac{\psi(x + jp, y + jq)}{\prod_{i=0}^{j} \varphi(x + ip, y + iq)} < \infty \tag{2.2}$$

for all $x, y \in \mathbb{N}$.

We now investigate the Hyers–Ulam stability problem for eq. (1.1). That is, the difference equation (1.1) is stable in the sense of Hyers and Ulam under the conditions subject to (2.1) and (2.2).

Theorem 2.1. *Suppose that functions $f, \psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ and φ satisfy the inequality*

$$|f(x + p, y + q) - \varphi(x, y) f(x, y) - \psi(x, y)| \le \delta \tag{2.3}$$

for all $x, y \in \mathbb{N}$. Then there exist unique functions $T, T_h, T_p : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that T, T_p satisfy eq. (1.1), *T_h satisfies eq.* (1.2) *and the relations*

$$\begin{aligned}
&|f(x, y) - T(x, y)| \le \delta \varepsilon(x, y), \\
&|f(x, y) - T_h(x, y)| \le \delta \varepsilon(x, y) + |\varepsilon'(x, y)|, \\
&|T_p(x, y)| \le |\varepsilon'(x, y)|, \\
&T(x, y) = T_h(x, y) + T_p(x, y)
\end{aligned} \tag{2.4}$$

hold for all $x, y \in \mathbb{N}$. If the range of f is $(0, \infty)$, then the range of T_h is $(0, \infty)$.

Proof. Replacing x, y by $x + p, y + q$, respectively, in (2.3), we have

$$\begin{aligned}
|f(x + 2p, y + 2q) - \varphi(x + p, y + q) f(x + p, y + q) \\
- \psi(x + p, y + q)| \le \delta
\end{aligned} \tag{2.5}$$

for all $x,\ y \in \mathbb{N}$. Combining the last inequality with (2.3), we get the relation

$$|f(x+2p, y+2q) - \varphi(x+p, y+q)\varphi(x, y)f(x, y) - \psi(x+p, y+q) - \varphi(x+p, y+q)\psi(x, y)| \le \delta + \delta\varphi(x+p, y+q) \tag{2.6}$$

for all $x,\ y \in \mathbb{N}$. Now we use induction on $m \in \mathbb{N}$ to prove

$$\begin{aligned}&\Bigg| f(x+(m+1)p, y+(m+1)q) - \prod_{i=0}^{m} \varphi(x+ip, y+iq) f(x, y) \\ &\quad - \sum_{j=0}^{m} \psi(x+jp, y+jq) \prod_{i=j+1}^{m} \varphi(x+ip, y+iq) \Bigg| \\ &\quad \le \delta \sum_{j=0}^{m} \prod_{i=j+1}^{m} \varphi(x+ip, y+iq)\end{aligned} \tag{2.7}$$

for all $x,\ y \in \mathbb{N}$, where $\prod_i^j(\cdot) = 1$ conveniently if $i > j$. Thus we obtain the inequality which plays an important role in the sequel,

$$\begin{aligned}&\left| \frac{f(x+(m+1)p, y+(m+1)q)}{\prod_{i=0}^{m} \varphi(x+ip, y+iq)} - f(x, y) - \sum_{j=0}^{m} \frac{\psi(x+jp, y+jq)}{\prod_{i=0}^{j} \varphi(x+ip, y+iq)} \right| \\ &\quad \le \delta \sum_{j=0}^{m} \frac{1}{\prod_{i=0}^{j} \varphi(x+ip, y+iq)}.\end{aligned} \tag{2.8}$$

We claim that the sequence

$$\begin{aligned}&\Bigg\{ T_m(x, y) = \frac{f(x+(m+1)p, y+(m+1)q)}{\prod_{i=0}^{m} \varphi(x+ip, y+iq)} \\ &\qquad - \sum_{j=0}^{m} \frac{\psi(x+jp, y+jq)}{\prod_{i=0}^{j} \varphi(x+ip, y+iq)} \Bigg\}\end{aligned} \tag{2.9}$$

is a Cauchy sequence. Indeed, for $m > n$ we get by (2.8)

$$\begin{aligned}&|T_m(x, y) - T_n(x, y)| = \frac{1}{\prod_{i=0}^{n} \varphi(x+ip, y+iq)} \\ &\quad \cdot \Bigg| \frac{f(x+(m+1)p, y+(m+1)q)}{\prod_{i=(n+1)}^{m} \varphi(x+ip, y+iq)} - f(x+(n+1)p, y+(n+1)q) \\ &\quad - \sum_{j=(n+1)}^{m} \frac{\psi(x+jp, y+jq)}{\prod_{i=(n+1)}^{j} \varphi(x+ip, y+iq)} \Bigg| \\ &\quad \le \frac{\delta}{\prod_{i=0}^{n} \varphi(x+ip, y+iq)} \sum_{j=(n+1)}^{m} \frac{1}{\prod_{i=(n+1)}^{j} \varphi(x+ip, y+iq)} \\ &\quad \to 0 \ \text{ as } \ m > n \to \infty.\end{aligned} \tag{2.10}$$

Therefore, we can now define a function $T : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$T(x, y) = \lim_{m\to\infty} T_m(x, y) \tag{2.11}$$

for any $(x, y) \in \mathbb{N} \times \mathbb{N}$.

Since $T_{m-1}(x + p, y + q) = \varphi(x, y)T_m(x, y) + \psi(x, y)$, we have

$$T(x + p, y + q) = \varphi(x, y)T(x, y) + \psi(x, y) \tag{2.12}$$

for any $x, y \in \mathbb{N}$. That is, T is a solution of eq. (1.1).

From (2.8) we have for any $x, y \in \mathbb{N}$

$$\begin{aligned}&\left|\frac{f(x + (m + 1)p, y + (m + 1)q)}{\prod_{i=0}^{m} \varphi(x + ip, y + iq)} - f(x, y)\right| \\ &\qquad \le \delta \sum_{j=0}^{m} \frac{1}{\prod_{i=0}^{j} \varphi(x + ip, y + iq)} + \left|\sum_{j=0}^{m} \frac{\psi(x + jp, y + jq)}{\prod_{i=0}^{j} \varphi(x + ip, y + iq)}\right|.\end{aligned} \tag{2.13}$$

Using similar argument to that of (2.10), we obtain that the sequence

$$\left\{\frac{f(x + (m + 1)p, y + (m + 1)q)}{\prod_{i=0}^{m} \varphi(x + ip, y + iq)}\right\} \tag{2.14}$$

is a Cauchy sequence and the function T_h given by

$$T_h(x, y) = \lim_{m\to\infty} \left\{\frac{f(x + (m + 1)p, y + (m + 1)q)}{\prod_{i=0}^{m} \varphi(x + ip, y + iq)}\right\} \tag{2.15}$$

is defined for any $(x, y) \in \mathbb{N} \times \mathbb{N}$.

As in the case of (2.12), we have

$$T_h(x + p, y + q) = \varphi(x, y)T_h(x, y)$$

for any $x, y \in \mathbb{N}$. That is, T_h is a solution of eq. (1.2).

As a result,

$$T(x, y) - T_h(x, y) := T_p(x, y) = -\sum_{j=0}^{\infty} \frac{\psi(x + jp, y + jq)}{\prod_{i=0}^{j} \varphi(x + ip, y + iq)} \tag{2.16}$$

is well-defined and a particular solution of (1.1) by the comment preceding the theorem.

We also have the inequality (2.4) by taking the limit on both sides in (2.8) and (2.13).

Now assume that T', T_h', T_p' are the mappings satisfying the conclusions in the theorem. Since T, T' satisfy eq. (1.1), it then follows from (2.4) that

$$\begin{aligned}&|T(x, y) - T'(x, y)| \\ &\quad = \frac{|T(x + (m + 1)p, y + (m + 1)q) - T'(x + (m + 1)p, y + (m + 1)q)|}{\prod_{i=0}^{m} \varphi(x + ip, y + iq)}\end{aligned}$$

$$\leq \frac{2\delta}{\prod_{i=0}^{m}\varphi(x+ip,\, y+iq)} \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{\varphi(x+(m+1+i)p,\, y+(m+1+i)q)}$$

$$\to 0 \text{ as } m \to \infty. \tag{2.17}$$

This implies the uniqueness of T. Similarly we have the uniqueness of T_h by the same method of (2.17). This completes the proof of the theorem. □

We note that if $\psi = 0$, then $T = T_h$, $T_p = 0$ since $\varepsilon'(x, y) = 0$ for any $x, y \in \mathbb{N}$.

In the next corollary, let two functions $\varphi : \mathbb{N} \to (0, \infty)$, $\psi : \mathbb{N} \to \mathbb{R}$ satisfy

$$\varepsilon(x) := \sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{\varphi(x+ip)} < \infty, \tag{2.18}$$

$$\varepsilon'(x) := \sum_{j=0}^{\infty} \frac{\psi(x+jp)}{\prod_{i=0}^{j}\varphi(x+ip)} < \infty \tag{2.19}$$

for all $x \in \mathbb{N}$. Then we obtain the Hyers–Ulam stability problem for a single variable.

COROLLARY 2.2

Suppose that functions $f, \psi : \mathbb{N} \to \mathbb{R}$ *and* φ *satisfy the inequality*

$$|f(x+p) - \varphi(x)f(x) - \psi(x)| \leq \delta$$

for all $x \in \mathbb{N}$. *Then there exist unique functions* $T, T_h, T_p : \mathbb{N} \to \mathbb{R}$ *such that* T, T_p *satisfy the equation* $f(x+p) - \varphi(x)f(x) - \psi(x) = 0$, T_h *satisfies the equation* $f(x+p) - \varphi(x)f(x) = 0$ *and the relations*

$$|f(x) - T(x)| \leq \delta\varepsilon(x),$$

$$|f(x) - T_h(x)| \leq \delta\varepsilon(x) + |\varepsilon'(x)|,$$

$$|T_p(x)| \leq |\varepsilon'(x)|,$$

$$T(x) = T_h(x) + T_p(x)$$

hold for all $x \in \mathbb{N}$.

3. Applications

We list some examples of difference equations which are stable by Theorem 2.1 in the sense of Hyers–Ulam.

For some a $(0 < a < 1)$ the function $f(x) = \int_0^\infty t^{x-1}a^t \mathrm{d}t$ $(x \in \mathbb{N})$ is a solution of the homogeneous linear difference equation

$$f(x+1) + \frac{x}{\ln a} f(x) = 0. \tag{3.1}$$

In particular, in the case $a = 1/e$ it is well-known that the functional equation (3.1) on an interval $(0, \infty)$ is the gamma functional equation and Jung obtained its Hyers–Ulam stability [13]. We obtain from Theorem 2.1 the Hyers–Ulam stability for a single variable as follows:

COROLLARY 3.1

Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ *satisfies the inequality*

$$\left| f(x+1) + \frac{x}{\ln a} f(x) \right| \leq \delta \tag{3.2}$$

for all $x \in \mathbb{N}$. *Then there exists a unique function* $T : \mathbb{N} \to \mathbb{R}$ *such that* T *satisfies eq.* (3.1) *and the relation*

$$|f(x) - T(x)| \leq \left(\frac{1}{a} - 1 \right) \delta \tag{3.3}$$

holds for all $x \in \mathbb{N}$.

Proof. We apply Theorem 2.1 with $\varphi(x) = x/-\ln a$ and $\psi(x) = 0$. For any $x \in \mathbb{N}$

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{-\ln a}{x+i} = \left(\frac{-\ln a}{x} + \frac{(-\ln a)^2}{x(x+1)} + \frac{(-\ln a)^3}{x(x+1)(x+2)} + \cdots \right)$$
$$\leq \frac{1}{a} - 1. \tag{3.4}$$

Thus we lead to the conclusion. □

The function $f(x) = \int_0^1 t^x/(5-t)\mathrm{d}t$ $(x \in \mathbb{N})$ is a solution of the nonhomogeneous linear difference equation

$$f(x+1) - 5f(x) + \frac{1}{x+1} = 0. \tag{3.5}$$

We obtain from Theorem 2.1 the Hyers–Ulam stability as follows:

COROLLARY 3.2

Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ *satisfies the inequality*

$$\left| f(x+1) - 5f(x) + \frac{1}{x+1} \right| \leq \delta \tag{3.6}$$

for all $x \in \mathbb{N}$. *Then there exist unique functions* $T, T_h, T_p : \mathbb{N} \to \mathbb{R}$ *such that* T, T_p *satisfy eq.* (3.5), T_h *satisfies the equation* $f(x+1) - 5f(x) = 0$ *and the relations*

$$|f(x) - T(x)| \leq \frac{\delta}{4}, \quad |f(x) - T_h(x)| \leq \frac{\delta}{4} + \sum_{i=1}^{\infty} \frac{1}{5^i \cdot (i+1)},$$
$$|T_p(x)| \leq \sum_{i=1}^{\infty} \frac{1}{5^i \cdot (i+1)}, \quad T(x) = T_h(x) + T_p(x) \tag{3.7}$$

hold for all $x \in \mathbb{N}$.

Proof. We apply Theorem 2.1 with $\varphi(x) = 5$ and $\psi(x) = -1/(x+1)$. For any $x \in \mathbb{N}$

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{1}{5} = \left(\frac{1}{5} + \frac{1}{5^2} + \cdots\right) \leq \frac{1}{4} \tag{3.8}$$

and

$$\left|\sum_{j=0}^{\infty} \frac{-1/(x+j+1)}{\prod_{i=0}^{j} 5}\right| = \frac{1}{5}\left(\frac{1}{x+1} + \frac{1}{5(x+2)} + \frac{1}{5^2(x+3)} + \cdots\right) \leq \left(\frac{1}{5\cdot 2} + \frac{1}{5^2\cdot 3} + \frac{1}{5^3 \cdot 4} + \cdots\right). \tag{3.9}$$

This leads to the conclusion. □

The beta function $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ is a solution of the functional equation

$$f(x+1, y+1) - \frac{xy}{(x+y)(x+y+1)} f(x, y) = 0, \quad x, y \in (0, \infty). \tag{3.10}$$

Jun, Kim and Lee [11] obtained the stability theorem of eq. (3.10).

Consider the following homogeneous linear difference equation

$$f(x+2, y+2) - \frac{(x+1)(y+1)}{(x+y+4)^2} f(x, y) = 0, \quad x, y \in \mathbb{N}. \tag{3.11}$$

The function

$$f(x, y) = \int_0^{\pi/2} \sin^x t \cos^y t \,\mathrm{d}t, \ (x, y \in \mathbb{N}) \tag{3.12}$$

is a solution of the functional equation (3.11).

COROLLARY 3.3

Suppose that a function $g : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ *satisfies the inequality*

$$\left|g(x+2, y+2)^{-1} - \frac{(x+y+4)^2}{(x+1)(y+1)} g(x, y)^{-1}\right| \leq \delta \tag{3.13}$$

for all $x, y \in \mathbb{N}$. *Then there exists a unique function* $T : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ *such that* T *satisfies eq.* (3.11) *and the inequality*

$$|g(x, y)^{-1} - T(x, y)^{-1}| \leq \kappa\delta \leq \frac{\delta}{12} \tag{3.14}$$

holds for all $x, y \in \mathbb{N}$, *where* $\kappa = \sum_{j=0}^{\infty} \prod_{i=0}^{j} (2i+1)^2/(4i+4)^2$.

Proof. We apply Theorem 2.1 with $f(x, y) = g(x, y)^{-1}$ and $\varphi(x, y) = (x + y + 4)^2/(x + 1)(y + 1)$. For any $x, y \in \mathbb{N}$, we get

$$\begin{aligned}
\sum_{j=0}^{\infty} \prod_{i=0}^{j} & \frac{(x+2i+1)(y+2i+1)}{(x+y+4i+4)^2} \\
&= \frac{(x+1)(y+1)}{(x+y+4)^2}\left(1 + \frac{(x+3)(y+3)}{(x+y+8)^2}\right. \\
&\quad \left. + \frac{(x+3)(y+3)(x+5)(y+5)}{(x+y+8)^2(x+y+12)^2} + \cdots\right) \\
&\le \frac{1}{4^2}\left(1 + \frac{3^2}{8^2} + \frac{3^2 5^2}{8^2 12^2} + \frac{3^2 5^2 7^2}{8^2 12^2 16^2} \cdots\right) \\
&\le \frac{1}{12}. \qquad (3.15)
\end{aligned}$$

By Theorem 2.1, there exists a unique function $F : \mathbb{N} \times \mathbb{N} \to (0, \infty)$ such that F satisfies the equation

$$F(x + 2, y + 2) - \frac{(x + y + 4)^2}{(x + 1)(y + 1)} F(x, y) = 0, \quad x, y \in \mathbb{N} \qquad (3.16)$$

and the inequality

$$|f(x, y) - F(x, y)| \le \frac{\delta}{12} \qquad (3.17)$$

holds for all $x, y \in \mathbb{N}$. If we define $T(x, y) = F(x, y)^{-1}$, then T satisfies (3.11) and inequality (3.14). Thus we lead to the conclusion. $\square$

For some a $(0 < a < 1)$, the function $f(x) = \int_0^\infty \mathrm{d}t/(t^2 + a^2)^x$ $(x \in \mathbb{N})$ is a solution of the homogeneous linear difference equation

$$f(x + 1) - \frac{2x - 1}{2xa^2} f(x) = 0. \qquad (3.18)$$

(Using the integration by substitution $t = a \tan u$ and then the relation $\int \cos^n u \mathrm{d}u = \dfrac{\sin u \cos^{n-1} u}{n} + \dfrac{n-1}{n} \int \cos^{n-1} u \mathrm{d}u$, we obtain the eq.(3.18).)

Here we denote $\omega = \sum_{j=0}^{\infty} \frac{2^j a^{2j} (j+1)!}{\prod_{i=0}^{j} 2i+1}$ for abbreviation. We obtain from Theorem 2.1 the Hyers–Ulam stability as follows:

COROLLARY 3.4

Suppose that a function $f : \mathbb{N} \to \mathbb{R}$ *satisfies the inequality*

$$\left| f(x + 1) - \frac{2x - 1}{2xa^2} f(x) \right| \le \delta \qquad (3.19)$$

for all $x \in \mathbb{N}$. *Then there exists a unique function* $T : \mathbb{N} \to \mathbb{R}$ *such that* T *satisfies eq.* (3.18), *and the relation*

$$|f(x) - T(x)| \leq 2a^2\omega \tag{3.20}$$

holds for all $x \in \mathbb{N}$.

Proof. We apply Theorem 2.1 with $\varphi(x) = (2x-1)/2xa^2$ and $\psi(x) = 0$. For any $x \in \mathbb{N}$

$$\begin{aligned}\sum_{j=0}^{\infty} \prod_{i=0}^{j} \frac{a^2(2x+2i)}{2x+2i-1} &= \frac{2xa^2}{2x-1}\left(1 + \frac{a^2(2x+2)}{2x+1}\right.\\ &\quad \left. + \frac{a^4(2x+2)(2x+4)}{(2x+1)(2x+3)} + \cdots\right)\\ &\leq 2a^2\left(1 + \frac{4a^2}{3} + \frac{4\cdot 6a^4}{3\cdot 5} + \cdots\right)\\ &\leq 2a^2\omega. \end{aligned} \tag{3.21}$$

Thus we lead to the conclusion. □

References

[1] Aczél J and Dhombres J, Functional Equations in Several Variables (Cambridge Univ. Press) (1989)

[2] Agarwal R P, Difference Equations and Inequalities: Theory, Methods, and Applications (Marcel Dekker, Inc.) (1992)

[3] Baker J, The stability of the cosine equation, *Proc. Am. Math. Soc.* **80** (1980) 411–416

[4] Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994) 431–436

[5] Grabiec A, The generalized Hyers–Ulam stability of a class of functional equations, *Publ. Math. Debrecen* **48** (1996) 217–235

[6] Hyers D H, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.* **27** (1941) 222–224

[7] Hyers D H, Isac G and Rassias Th M, Stability of Functional Equations in Several Variables (Birkhäuser, Basel) (1998)

[8] Hyers D H, Isac G and Rassias Th M, On the asymptoticity aspect of Hyers–Ulam stability of mappings, *Proc. Am. Math. Soc.* **126** (1998) 425–430

[9] Hyers D H and Rassias Th M, Approximate homomorphisms, *Aequationes Math.* **44** (1992) 125–153

[10] Jun K W and Lee Y H, On the Hyers–Ulam–Rassias stability of a pexiderized quadratic inequality, *Math. Ineq. Appl.* **4(1)** (2001) 93–118

[11] Jun K W, Kim G H and Lee Y W, Stability of generalized gamma and beta functional equations, *Aequationes Math.* **60** (2000) 15–24

[12] Jung S-M, On the Hyers–Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222** (1998) 126–137

[13] Jung S-M, On the stability of gamma functional equation, *Result. Math.* **33** (1998) 306–309

[14] Kelley W G and Peterson A C, Difference Equations, An Introduction with Applications (Academic Press, Inc.) (1991)

[15] Páles J, On a Pexider-type functional equation for quasideviation means, *Acta Math. Hungar.* **51** (1988) 205–224

[16] Rassias Th M, On the stability of the linear mapping in Banach spaces, *Proc. Am. Math. Soc.* **72** (1978) 297–300
[17] Széklyhidi L, Regularity properties of polynomials on groups, *Acta Math. Hungar.* **45** (1985) 15–19
[18] Ulam S M, Problems in Modern Mathematics (New York: Wiley) (1964) Science edition, Chap. VI