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Sums of two polynomials with each having real zeros symmetric with the other

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Abstract. Consider the polynomial equation

$$\prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i) = 0,$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$. All zeros of this equation lie on the imaginary axis. In this paper, we show that no two of the zeros can be equal and the gaps between the zeros in the upper half-plane strictly increase as one proceeds upward. Also we give some examples of geometric progressions of the zeros in the upper half-plane in cases $n = 6, 8, 10$.

Keywords. Polynomial; zero; geometric progression.

1. Introduction

Throughout this paper, n is an integer ≥ 2. There is an extensive literature concerning zeros of sums of polynomials. A classical text in this subject is Marden's book [2]. The fourth chapter of [2] examines linear combinations of polynomials, and combinations of a polynomial and its derivative. Perhaps the most immediate question of sums of polynomials, $A + B = C$, is "given bounds for the zeros of A and B, what bounds can be given for the zeros of C?" By Fell [1], if all zeros of A and B lie in $[-1, 1]$ with A, B monic and deg $A =$ deg $B = n$, then no zero of C can have modulus exceeding $\cot(\pi/2n)$, the largest zero of $(x + 1)^n + (x - 1)^n$. This suggests to study polynomials having a form something like $A(x) + B(x)$ where all zeros of $A(x)$ are negative and all zeros of $B(x)$ are positive. In this paper, we study zero distributions of the polynomial equation

$$\prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i) = 0, \tag{1.1}$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$. All zeros of (1.1) lie on the imaginary axis. In fact, if z is a zero, then $\prod_{i=1}^{n}(z - r_i) = -\prod_{i=1}^{n}(z + r_i)$. On taking absolute values, one gets that the product of the distances of z from the points $-r_i$ equals the product of the distances of z from the points r_i. Thus, if z is to the left or to the right of the y-axis, one of these distances is bigger.

The main purpose of this paper is to characterize the set S_n of all positive numbers

$$\{s_1, \ldots, s_{\lfloor n/2 \rfloor}\}$$

such that

$$\{\pm i\, s_1, \ldots, \pm i\, s_{\lfloor n/2 \rfloor}\}$$

is the set of nonzero zeros of some

$$P(x) = \prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i),$$

where $0 < r_1 \leq r_2 \leq \cdots \leq r_n$. We remark that an $\{s_i\} \in S$ does not determine an $\{r_i\}$ uniquely, for example,

$$(x-1)(x-2)(x-3) + (x+1)(x+2)(x+3) = (x-\alpha)^3 + (x+\alpha)^3,$$

where $\alpha = \sqrt{22/6} = 1.914\cdots$. We will show in §2 that in general no two of the s_i can be equal. Of special interest is to investigate whether elements of S_n can be arithmetic or geometric progressions. We shall prove in §2 that

$$s_{i+1} - s_i > s_i - s_{i-1}$$

for each i, $2 \leq i \leq \lfloor n/2 \rfloor - 1$, so the gaps between the zeros in the upper half-plane strictly increase as one proceeds upward. We can, however, find some geometric progressions. In fact, we shall give some examples of geometric progressions in S_6, S_8 and S_{10}.

2. Results and proofs

We first show in Proposition 2.1 that no two of the s_i can be equal.

PROPOSITION 2.1

Let

$$P(x) = \prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i),$$

where $0 < r_1 \leq r_2 \leq \cdots \leq r_n$. *Then all zeros of* $P(x)$ *are simple.*

Proof. Suppose, if possible, that $P(z) = P'(z) = 0$. Then

$$\begin{aligned} 0 = P'(z) &= \sum_{i=1}^{n}\left\{\prod_{j\neq i}(z - r_j) + \prod_{j\neq i}(z + r_j)\right\} \\ &= \sum_{i=1}^{n}\left\{\prod_{j\neq i}(z - r_j) - \frac{z - r_i}{z + r_i}\prod_{j\neq i}(z - r_j)\right\} \end{aligned}$$

on using $\prod_{i=1}^{n}(z - r_i) = -\prod_{i=1}^{n}(z + r_i)$. Thus,

$$0 = \sum_{i=1}^{n}\frac{2r_i}{z + r_i}\prod_{j\neq i}(z - r_j) = \left\{\sum_{i=1}^{n}2r_i\prod_{j\neq i}(z^2 - r_j^2)\right\} \Big/ \prod_{i=1}^{n}(z + r_i).$$

In other words, since $z = is$ for some real number s, we have

$$0 = \sum_{i=1}^{n} r_i \prod_{j \neq i} (s^2 + r_j^2)$$

an impossibility, since all terms are positive. $\square$

For main result of this paper, we need the following lemma.

Lemma 2.2. *Let the function $G(y)$ be twice differentible on $y \geq 0$. Suppose that $G(0) = 0$, $G'(y) > 0$ and $G''(y) < 0$ for $y \geq 0$. If*

$$G(y_1) = m, \quad G(y_2) = m + d, \quad G(y_3) = m + 2d,$$

with $d > 0$, then

$$y_3 - y_2 > y_2 - y_1.$$

Proof. It is clear that the inverse function G^{-1} exists and is strictly convex. This implies that

$$G^{-1}\left(\frac{x+y}{2}\right) < \frac{G^{-1}(x) + G^{-1}(y)}{2} \quad \text{for } 0 < x < y.$$

Upon applying G^{-1} to both sides of

$$G(y_2) = \frac{G(y_1) + G(y_3)}{2},$$

we find that

$$y_2 < \frac{y_1 + y_3}{2},$$

and the result follows. $\square$

Now we have

Theorem 2.3. *Let the zeros of polynomial $P(x)$ above the real axis be*

$$is_1, \quad is_2, \; \ldots, \; is_{\lfloor n/2 \rfloor}.$$

Then we have

$$s_{i+1} - s_i > s_i - s_{i-1}$$

for each i, $2 \leq i \leq \lfloor n/2 \rfloor - 1$, so the gaps between the zeros in the upper half-plane strictly increase as one proceeds upward.

Proof. Suppose that s is a zero of $P(x)$. Then s lies on the imaginary axis. Let, for each i, $\alpha_i = \angle s(-r_i)o$ and $\beta_i = \pi - \angle s r_i o$, where o denotes the origin. Then we have $0 < \alpha_i < \pi/2$ and $\beta_i = \pi - \alpha_i$ for each i. Since

$$\exp\left(i \sum_{i=1}^{n} \alpha_i\right) + \exp\left(i \sum_{i=1}^{n} \beta_i\right) = 0,$$

we have

$$\sum_{i=1}^{n} \alpha_i \equiv \sum_{i=1}^{n} (\pi - \alpha_i) + \pi \pmod{2\pi}$$

and hence

$$2 \sum_{i=1}^{n} \frac{\alpha_i}{\pi} \equiv n + 1 \pmod{2}.$$

On the other hand,

$$0 < 2 \sum_{i=1}^{n} \frac{\alpha_i}{\pi} < \frac{2}{\pi} \cdot n \cdot \frac{\pi}{2} = n.$$

So

$$2 \sum_{i=1}^{n} \frac{\alpha_i}{\pi} = n - 1, n - 3, n - 5, \ldots, c, \tag{2.1}$$

where $c = 0$ if n is odd and $c = 1$ if n is even. Here $\alpha_i = \tan^{-1} s/r_i$. Define a function in y, $y \geq 0$,

$$G(y) := \frac{2}{\pi} \sum_{i=1}^{n} \tan^{-1} \frac{y}{r_i}.$$

Then $G(0) = 0$ and as y increases from 0 the graph of $G(y)$ increases and is concave downward, since

$$G'(y) = \frac{2}{\pi} \sum_{i=1}^{n} \frac{r_i}{r_i^2 + y^2} \quad \text{and} \quad G''(y) = -\frac{2}{\pi} \sum_{i=1}^{n} \frac{2r_i y}{(r_i^2 + y^2)^2}.$$

Also, by (2.1), the s_j are the solutions to

$$G(y) = n - 1, n - 3, n - 5, \ldots, c, \tag{2.2}$$

where $c = 0$ if n is odd and $c = 1$ if n is even. The numbers on the right side of (2.2) are in the arithmetic progression. So the result follows from Lemma 2.2. $\square$

Finally, we draw our attention to the principal idea (see (2.1)) which makes the proof of Theorem 2.3 work and which is independently interesting. This is the following fact.

PROPOSITION 2.4

Let the zeros of polynomial $P(x)$ above the real axis be

$$is_1,\ is_2,\ \ldots,\ is_{\lfloor n/2 \rfloor}.$$

If $\alpha_j(s_k)$ denotes the angle formed at the real number r_j by the triangle joining r_j, is_k and the origin, then the sums

$$\theta_k = \sum_{j=1}^{n} \alpha_j(s_k)$$

for $k = \lfloor n/2 \rfloor, \ldots, 2, 1$ *are, respectively, the numbers*

$$\frac{\pi(n-1)}{2}, \quad \frac{\pi(n-3)}{2}, \quad \frac{\pi(n-5)}{2}, \ldots, \frac{\pi c}{2},$$

where $c = 0$ *if* n *is odd and* $c = 1$ *if* n *is even. In particular, these are independent of the* r_j*'s.*

From Theorem 2.3, a further natural question of interest is whether the s_i can be in geometric progression. For the rest of this paper, we give some examples of geometric progressions in S_6, S_8 and S_{10}.

Some simple examples of geometric progressions in S_6 arise from the identity

$$\begin{aligned}
&(x-a)^3(x-b)^3 + (x+a)^3(x+b)^3 \\
&\quad = 2(u+ab)(u^2 + (3a^2 + 8ab + 3b^2)u + (ab)^2) \\
&\quad = 2\,Q_1(u)Q_2(u), \quad u = x^2, \quad \text{say,}
\end{aligned}$$

since the product of the zeros of Q_2 is the square of the zero of Q_1. For another point of view on geometric progressions contained in S_6, observe the identity

$$\begin{aligned}
&(x-h)(x-hq^2)(x-hq^4)(x-hq^6)(x-hq^8)(x-hq^{10}) \\
&\quad + (x+h)(x+hq^2)(x+hq^4)(x+hq^6)(x+hq^8)(x+hq^{10}) \\
&= 2(x^2+1)(x^2+h^2q^{10})(x^2+h^4q^{20}) \\
&\quad - 2x^2(x^2+h^2q^{10})\left(1 - q^2(q^4+1)\frac{q^7-1}{q-1}\frac{q^7+1}{q+1}h^2 + h^4q^{20}\right).
\end{aligned}$$

If we take $q = 2$ and $h = 0.5950\ldots$, a zero of

$$2^{20}h^4 - 371348h^2 + 1,$$

the last factor on the right side of the identity becomes zero and we have a case in which the r_i and the s_i are *both* in geometric progression, and the r_i are all distinct.

For S_8 and S_{10}, we have examples below. The following can be verified by computer algebra. First, for S_8, if t is the largest real zero $(= 14.415\ldots)$ of

$$t^4 - 12t^3 - 34t^2 - 12t + 1 = 0,$$

then the zeros of

$$(x-1)^4(x-t)^4 + (x+1)^4(x+t)^4 = 0$$

in the upper half-plane are in geometric progression with ratio equal to the larger real zero $(= 4.611\ldots)$ of

$$z^4 - 4z^3 - 2z^2 - 4z + 1 = 0.$$

Finally, for S_{10}, if t is the largest real zero $(= 24.375\cdots)$ of

$$t^4 - 22t^3 - 57t^2 - 22t + 1 = 0$$

then the zeros of

$$(x-1)^5(x-t)^5+(x+1)^5(x+t)^5$$

in the upper half-plane are in geometric progression with ratio equal to the larger real zero $(= 3.985\cdots)$ of

$$z^4-4z^3+z^2-4z+1=0.$$

In fact, for every positive integer m, it seems that

$$\prod_{i=1}^{2m}(x-g_i)+\prod_{i=1}^{2m}(x+g_i)=2\prod_{i=1}^{m}(x^2+r_i),$$

for some real positive geometric progressions $\{g_1,\ldots,g_{2m}\}$ and $\{r_1,\ldots,r_m\}$. But it remains an open problem.

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