

Truncation method for operators with disconnected essential spectrum

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Abstract. In this short paper, the usage of truncation method to get information about essential spectrum of bounded as well as semi-bounded linear operators on separable Hilbert spaces, is investigated. In addition to this, the problem of predicting the gaps in the essential spectrum of self-adjoint operators, linear algebraically is also considered.

Keywords. Operator; singular values; spectrum.

1. Introduction

Let H be a separable Hilbert space and let $B(H)$ denote the class of all bounded linear operators on H . For $T \in B(H)$, the k th approximation member $S_k(T)$ of T is defined by

$$S_k(T) = \inf\{\|T - F\| : F \in B(H), \text{rank } F \leq k - 1\}.$$

Thus $S_1(T) = \|T\|$ and

$$S_1(T) \geq S_2(T) \geq \dots.$$

The essential norm $\|T\|_{\text{ess}}$ of T is given by

$$\|T\|_{\text{ess}} = \inf\{\|T - K\| : K \in K(H)\},$$

where $K(H)$ denotes the class of all compact operators on H . Goghberg [6] has shown that

$$\lim_{k \rightarrow \infty} S_k(T) = \|T\|_{\text{ess}}.$$

Thus if T is a positive operator, then the sequence of singular values approximates the upperbound of the essential spectrum $\sigma_{\text{ess}}(T)$ of T .

Now let $\{e_1, e_2, \dots\}$ be an orthonormal basis in H and let P_n denote the orthogonal projection of H onto $\text{span}\{e_1, e_2, \dots, e_n\}$. Then $T_n = P_n T P_n$ can be thought of as a finite truncation at the n th level of T and hence can be regarded as an $n \times n$ complex matrix. It has been proved [2] that

$$\lim_{n \rightarrow \infty} (S_k(T_n)) = S_k(T) \text{ for each } k \geq 1.$$

Now let the numbers m , M , v and μ be defined as follows:

$$\begin{aligned} m &= \inf_{\|x\|=1} \langle Tx, x \rangle & M &= \sup_{\|x\|=1} \langle Tx, x \rangle \\ v &= \inf \sigma_{\text{ess}}(T) & \mu &= \sup \sigma_{\text{ess}}(T). \end{aligned}$$

It has been proved [2] that the discrete spectrum as well as the bounds v and μ can be computed by the finite section method.

Now let $\sigma_{\text{ess}}(T)$ be disconnected and assume that it is a finite union of disjoint intervals,

$$\sigma_{\text{ess}}(T) = \cup_1^N [a_i, b_i], \quad b_i < a_{i+1}, \quad i = 1 \dots N-1.$$

The earlier analysis [2] says that a_1 and b_N and the discrete spectrum outside $[a_1, b_N]$ can be computed by the finite section method. In this short paper the possibility of finding the other end values a_i 's and b_i 's as well as the discrete spectral values inside the gaps, is investigated. The case of semibounded, self-adjoint linear operators is also considered. This is given in §2. Another problem investigated is the prediction of gaps in $\sigma_{\text{ess}}(T)$ linear algebraically and is given in a very small in §2.

2. Disconnected essential spectrum

Throughout T is assumed to be a self adjoint operator on H whose essential spectrum has at most finite number of gaps. First a simple observation is made.

Lemma 2.1. *Let $T \in B(H)$ be self-adjoint and $\sigma_{\text{ess}}(T) = [a, b] \cup \{c\}$ where $b < c$ and c is isolated in $\sigma(T)$. Then the bounds a, b and c can be determined linear algebraically by truncation.*

Proof. We may assume that T is positive. Then the number c can be determined by truncation, being the upper bound of $\sigma_{\text{ess}}(T)$. Let P_b be the associated spectral projection of T . Then the infinite truncation $(I - P_b)T(I - P_b)$ has essential spectrum $\{0\} \cup [a, b]$. Here b can be obtained from truncations of $(I - P_b)T(I - P_b)$. This can be again considered as truncations of T , being truncations of truncations. \square

Remark 2.2. The discrete spectral values in $[b, c]$ can be obtained by truncations using the singular value's of truncations of $(I - P_b)T(I - P_b)$ [2].

Theorem 2.3. *Let $T \in B(H)$ be self-adjoint and $\sigma_{\text{ess}}(T) = [a, b] \cup [c, d]$ where $a < b < c < d$. Assume that b is known and is not an accumulation point of discrete spectrum of T . Then c can be obtained by the truncation method. The discrete spectrum in (b, c) can also be computed using truncations.*

Proof. Consider the function defined by

$$\begin{aligned} \mu(t) &= 0, & t &\leq b \\ &= t - b, & t &\geq b. \end{aligned}$$

Thus $\mu(T)$ has essential spectrum $\{0\} \cup [c - b, d - b]$. Hence c can be obtained from truncations of $\mu(T)$.

Now

$$\begin{aligned} \mu(T) &= \int_b^d (t - b) dE_t \\ &= \int_b^d t dE_t - b(I - E_b) \\ &= (I - E_b)T(I - E_b) - b(I - E_b), \end{aligned}$$

where $\{E_t\}_{t \in \mathbb{R}}$ is the spectral measure of T . Here truncations of $\mu(T)$ can be regarded as perturbations of truncations of T . This completes the proof. \square

The following proposition can also be proved in a similar way.

PROPOSITION 2.4

Let $T \in B(H)$ be self-adjoint. Assume that $\sigma_{\text{ess}}(T) = \bigcup_1^N [a_i, b_i]$, when $N < \infty$, $b_i < a_{i+1}$, for $i = 1 \dots N-1$ and b_l is not a limit point of discrete spectrum of T . If b_l is known for $l = 1, \dots, N-1$, then a_l can be computed by truncation for each l . Also the discrete spectrum can be recovered by truncation method.

Now the so-called semibounded self-adjoint operators are considered. Recall that a self-adjoint linear operator T in H is called semibounded if $I + T$ has a bounded inverse [7]. To make truncation meaningful, it is assumed that the orthonormal set $\{e_1, e_2, \dots\}$ is in domain T .

Theorem 2.5. Let S be a semibounded self-adjoint operator in H . Assume that $P_n + P_n S P_n$ are invertible on range P_n for all n and $(P_n + P_n S P_n)^{-1}$ converges to $(I + S)^{-1}$ strongly as $n \rightarrow \infty$ where P_n is the orthogonal projection onto the space spanned by $\{e_1, e_2, \dots, e_n\}$. Then

$$\lim_{n \rightarrow \infty} S_k(P_n - P_n S P_n)^{-1} = S_k(I + S)^{-1}, \text{ for each } k \geq 1.$$

Proof. It is known that [2]

$$\lim_{n \rightarrow \infty} S_k(P_n(I + S)^{-1}P_n) = S_k(I + S)^{-1} \text{ for each } k \geq 1.$$

Now $(P_m + P_m S P_m)^{-1} \rightarrow (I + S)^{-1}$ strongly. Here $P_n(P_m + P_m S P_m)^{-1}P_n$ converges to $P_n(I + S)^{-1}P_n$ uniformly. Hence

$$\lim_{m \rightarrow \infty} S_k(P_n(P_m + P_m S P_m)^{-1}P_n) = S_k(P_n(I + S)^{-1}P_n) \text{ for each } n.$$

This completes the proof. \square

3. Gaps in the essential spectrum

The problem of detecting gaps in the essential spectrum of self-adjoint operators, linear algebraically is considered here. The reason for considering this problem is mainly due to some results in [2] in connection with spectral approximation. There the crucial assumption is the connectedness of the essential spectrum. So how do we know this if an operator is given? We restrict our attention to bounded self-adjoint operators. Here we provide some simple, near trivial observation in this hard but very interesting problem.

For $T \in B(H)$ and self-adjoint, let $T_n = P_n T P_n$ be the truncation defined in section 2 for each $n \leq 1$. As before let $\{a_{n1} \leq a_{n2} \leq \dots \leq a_{nn}\}$ be the set of eigenvalues of T_n arranged in the increasing order of magnitude and v and μ be the upper and lower bounds of $\sigma_{\text{ess}}(T)$. In the following proposition we assume that $T \in A$, the Arveson's class [1].

PROPOSITION 3.1

For each positive integer n let $\{W_{nk} : k = 1 \dots n\}$ be a set of numbers such that $0 \leq W_{nk} \leq 1$, and $\sum_{k=1}^n W_{nk} = 1$. If there is a $\delta > 0$ such that

$$\max_{1 \leq j \leq n} \left| \sum_{k=1}^n W_{nk} a_{nk} - a_{nk} \right| \geq \delta \text{ for each } n.$$

Then there is a gap of width $\geq \delta$ in $\sigma_{\text{ess}}(T)$.

Proof. It is clear that $\lim_{n \rightarrow \infty} \sum^n W_{nk}$ is in $[v, \mu]$ since $\lim_{n \rightarrow \infty} a_{n1} = \mu$ and $\lim_{n \rightarrow \infty} a_{nn} = v$. This is a consequence of Arveson's theorem. Again using Arveson's theorem we get the required result.

For the sake of completeness we give a very brief description of Arveson's class. Let A be the class of bounded linear operators on H defined as follows:

With respect to the given $\{P_n\}$ of orthogonal projections considered above, let

$$\deg(T) = \sup_n \text{dimension } P_n T - T P_n, T \in B(H).$$

Then

$$A = \{T \in B(H) \mid T = \sum_k A_k, \deg(A_k) < \infty$$

$$\text{and } \sum_1^\alpha (1 + \deg^2 A_k)^{1/2} \quad \|A_k\| < \infty.$$

□

PROPOSITION 3.2

Let T be a bounded self-adjoint operator on H and let μ be a smooth, real function on the real line R such that $\text{support } (\mu) = [a, b] \subset [m, M]$. For integers m and n , let

$$\mu_{m,n}(t) = \mu(2^m t - n), \quad p \in R.$$

Then σ_{ess} has a gap if and only if $\mu_{m,n}(T)$ is a compact operator on H for large enough m and n . Moreover $\mu_{m,n}(T)$ is 0 if support of $\mu_{m,n}$ does not contain discrete spectral values of T .

Proof. Suppose $\mu_{m,n}(T)$ is compact of some m and n . By functional calculus we have

$$\sigma_{\text{ess}}(\mu_{m,n}(T)) = \mu_{m,n}(\sigma_{\text{ess}}(T)).$$

But $\sigma_{\text{ess}}(\mu_{m,n}(T)) = 0$

So here

$$\text{support } \mu_{m,n} \subset C - \sigma_{\text{ess}}(T).$$

Therefore support $\mu_{m,n}$ is a gap in $\sigma_{\text{ess}}(T)$. Conversely, let $[c, d]$ be a gap in $\sigma_{\text{ess}}(T)$. We can find m and n large enough so that $\text{support } \mu_{m,n} \subset [c, d]$. Hence by spectral mapping theorem in Calkin algebra, $\mu_{m,n}(T)$ is compact.

We conclude this section with the following remarks.

Remark 3.3.

- (1) The most difficult problem here is the functional calculus. Perhaps some suitable functional calculus like Helffer–Sjöstrand functional calculus may be helpful [5]. Another difficult problem that arise here is the estimation of the values of m and n . It is clear that bigger the gap smaller is the value of m .
- (2) Estimation of the number of gaps and the width of gap is another problem. The translation n might give some information about this.

- (3) Finally the question of finding discrete spectral values (if any) trapped inside the gap is another useful and interesting problem. For example assume that $[a, b]$ is a gap in $\sigma_{\text{ess}}(T)$ and let $\alpha \in (a, b)$ a discrete spectral value of T . Then there will exist m and n such that $\text{support } \mu_{m,n} \subset [a, b]$. Then $\mu_{m,n}(T)$ is a compact self-adjoint operator. Let $\beta = \mu(\alpha)$. Now β can be calculated by conventional methods. If μ is simple enough, one can hope to find α by numerical methods.

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