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The Wegner estimate and the integrated density of states for some random operators

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Abstract. The integrated density of states (IDS) for random operators is an important function describing many physical characteristics of a random system. Properties of the IDS are derived from the Wegner estimate that describes the influence of finite-volume perturbations on a background system. In this paper, we present a simple proof of the Wegner estimate applicable to a wide variety of random perturbations of deterministic background operators. The proof yields the correct volume dependence of the upper bound. This implies the local Hölder continuity of the integrated density of states at energies in the unperturbed spectral gap. The proof depends on the L^p-theory of the spectral shift function (SSF), for $p \geq 1$, applicable to pairs of self-adjoint operators whose difference is in the trace ideal $\mathcal{I}_p$, for $0 < p \leq 1$. We present this and other results on the SSF due to other authors. Under an additional condition of the single-site potential, local Hölder continuity is proved at all energies. Finally, we present extensions of this work to random potentials with nonsign definite single-site potentials.

Keywords. Schrödinger operators; localization; random potentials.

1. Introduction and main results

Much progress has been made in the study of random systems describing the propagation of electrons and classical waves in randomly perturbed media. In this paper, we concentrate on the Wegner estimate and on some recent results concerning the integrated density of states for random operators on $\mathbb{R}^d$, for $d \geq 1$. The Wegner estimate also plays a key role in the proof of localization for random systems, but we will not discuss localization here, and refer the reader to various references [1,10,13,14,18,21,34]. The Wegner estimate is a fine analysis of the effect of finite-volume, random perturbations V_Λ, for a bounded region $\Lambda \subset \mathbb{R}^d$, on the spectrum of a self-adjoint operator H_0, describing the background, unperturbed situation. More specifically, a *Wegner estimate* is an upper bound on the probability that the spectrum of the local Hamiltonian H_Λ lies within an η-neighborhood of a given energy E. A good Wegner estimate is one for which the upper bound depends

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linearly on the volume $|\Lambda|$, and vanishes as the size of the energy neighborhood η shrinks to zero. The linear dependence on the volume is essential for the proof of the regularity properties of the IDS. The rate of vanishing of the upper bound as $\eta \to 0$ determines the continuity of the IDS.

We present a new, simple proof of a good Wegner estimate applicable to random operators with some additional conditions on the single-site potential. This proof uses more directly the ideas of Krein, Birman, and Simon than the proof in [9]. As in [9], the proof employs L^p-estimates on the spectral shift function related to the single-site perturbation. This result allows us to prove exponential localization and the local Hölder continuity of the integrated density of states for more models than previously known.

The models that can be treated by this method are described as follows. We can treat both multiplicative (M) and additive (A) perturbations of a background self-adjoint operator H_0^X, for $X = M$ or $X = A$. Additively perturbed operators describe electron propagation, and multiplicatively perturbed operators describe the propagation of acoustic and electromagnetic waves. We refer to [10] for a further discussion of the physical interpretation of these operators. For the Wegner estimate, we are interested in perturbations V_Λ of a background operator H_0^X, that are local with respect to a bounded region $\Lambda \subset \mathbb{R}^d$. Multiplicatively perturbed operators H_Λ^M are of the form

$$H_\Lambda^M = A_\Lambda^{-1/2} H_0^M A_\Lambda^{-1/2}, \tag{1.1}$$

where $A_\Lambda = 1 + V_\Lambda$ is assumed to be invertible (cf. [10] for a discussion of this condition). Additively perturbed operators H_Λ^A are of the form

$$H_\Lambda^A = H_0^A + V_\Lambda. \tag{1.2}$$

The unperturbed, background medium in the multiplicative case is described by a divergence form operator

$$H_0^M = -C_0 \rho_0^{1/2} \nabla \cdot \rho_0^{-1} \nabla \rho_0^{1/2} C_0, \tag{1.3}$$

where ρ_0 and C_0 are positive functions that describe the unperturbed density and sound velocity. We assume that ρ_0 and C_0 are sufficiently regular so that $C_0^\infty(\mathbb{R}^d)$ is an operator core for H_0^M. The unperturbed, background medium in the additive case is described by a Schrödinger operator H_0 given by

$$H_0^A = (-i\nabla - A)^2 + W, \tag{1.4}$$

where A is a vector potential with $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(\mathbb{R}^d)$ and $W_+ \in K_d^{\mathrm{loc}}(\mathbb{R}^d)$.

In this note, we will limit ourselves to Anderson-type perturbations. These methods can also be used to treat the breather-type perturbations, and we refer the reader to [9] for details. Let $\tilde{\Lambda}$ denote the lattice points in the region Λ, so that $\tilde{\Lambda} \equiv \Lambda \cap \mathbb{Z}^d$. The local perturbation in the Anderson-type model is defined by

$$V_\Lambda(x) = \sum_{i \in \tilde{\Lambda}} \lambda_i(\omega) u_i(x - i - \xi_i(\omega')), \tag{1.5}$$

provided the random variables $\xi_i(\omega')$, modeling thermal vibrations, are small enough so that one of the conditions (H3), (H3a), (H3b), or (H3c) (given below) holds. To simplify the discussion in this paper, however, we take $\xi_i(\omega') = 0$. The functions u_i are nonzero and compactly supported in a neighborhood of the origin. They need not be of the form $u_i(x) = u(x)$, for some fixed u, since ergodicity plays no role in the Wegner estimate. When the sum in (1.5) extends over all the lattice points $\mathbb{Z}^d$, we write V_ω for the potential and H_ω^X, with the operator for $X = A$ given by (1.2) with V_Λ replaced by V_ω, and similarly for (1.1) in the case $X = M$.

The hypotheses for the models are listed here. The first two (H1) and (H1a) concern the spectrum of the unperturbed operator H_0^X. The second is a local compactness condition on H_0^X. There are four different conditions on the single-site potential u_j. Finally, there are two possible conditions on the random variables $\lambda_k(\omega)$. We note that the Wegner estimate is a local estimate so that for a finite region $\Lambda \subset \mathbb{R}^d$, only a finite number of single-site potentials are involved. We denote the ball of radius $R > 0$ about the origin by $B(R)$, and by $\Lambda_r(k) = \{x \in \mathbb{R}^d \mid |k_j - x_j| < r/2,\ j = 1, \dots, d\}$, the cube of side length $r > 0$, centered at k.

(H1) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, for $X = A$ and for $X = M$. The operator H_0^X is semi-bounded and has an open spectral gap. That is, there exist constants $-\infty < M_0 \leq C_0 \leq B_- < B_+ < C_1 \leq \infty$ so that $\sigma(H_0) \subset [M_0, \infty)$, and

$$\sigma(H_0) \cap (C_0, C_1) = (C_0, B_-] \cup [B_+, C_1).$$

(H1a) The self-adjoint operator H_0^X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and H_0^X is semi-bounded with $\sigma(H_0^X) \subset [M_0, \infty)$, for some $M_0 > -\infty$.

(H2) The operator H_0^X is locally compact in the sense that for any $\chi \in L^\infty(\mathbb{R}^d)$ with compact support, the operator $\chi(H_0^X - M_1)^{-1}$ is compact for any $M_1 < M_0$.

(H3) The single-site potentials u_k, $k \in \mathbb{Z}^d$, are nonzero. For the Anderson-type model (1.5), we assume that there exists $R > 0$ so that $u_k \in C_0(B(R))$, and that $u_k \geq 0$ for each $k \in \mathbb{Z}^d$. Furthermore, we assume that the family $\{u_k \mid k \in \mathbb{Z}^d\}$ is equicontinuous.

(H3a) In addition to (H3), we assume that there exists $\epsilon_1 > 0$ so that $u_k \geq \epsilon_1$ on $\Lambda_1(0)$.

(H3b) In addition to (H3), we assume that there is a nonempty subset $B \subset \Lambda_1(0)$ so that supp $u_k \subset B$.

(H3c) The single-site potentials $u_k \in C_0(\mathbb{R}^d)$. For each $k \in \mathbb{Z}^d$, there exists a nonempty open set B_k containing the origin so that the single-site potential $u_k \neq 0$ on B_k. Furthermore, we assume that

$$\sum_{j \in \mathbb{Z}^d} \left\{ \int_{\Lambda_1(0)} |u_j(x - j)|^p \right\}^{1/p} < \infty, \tag{1.6}$$

for $p \geq d$ when $d \geq 2$ and $p = 2$ when $d = 1$.

(H4) The conditional probability distribution of λ_0, conditioned on $\lambda_0{}^\perp \equiv \{\lambda_i \mid i \neq 0\}$, is absolutely continuous with respect to Lebesgue measure. The density h_0 has compact support $[m, M]$, for some constants (m, M) with $-\infty < m < M < \infty$. The density h_0 satisfies $\|h_0\|_\infty < \infty$, where the sup norm is defined with respect to the probability measure $\mathbb{P}$.

(H4a) In addition to (H4), the density h_0 is assumed to be locally absolutely continuous.

We refer to the review article of Kirsch [19] for a proof of the fact that these hypotheses imply the essential self-adjointness of H_ω^A on $C_0^\infty(\mathbb{R}^d)$ (see [10] for the $X = M$ case). As stated in hypothesis (H4), we will assume that the random variables are independent, and identically distributed, but the results hold in the correlated case, and in the case that the supports of the single-site potentials are not necessarily compact (cf. [8,22]).

Our main results under these hypotheses on the unperturbed operator H_0^X, and the local perturbation V_Λ, concern two cases depending upon whether or not the single-site potentials are sign-definite. For the case of sign-definite single-site potentials, hypotheses (H3), (H3a), or (H3b), our main theorem is the following.

Theorem 1.1. *Assume* (H1), (H2), (H3), *and* (H4). *For any* $E_0 \in G = (B_-, B_+)$, *for any* $q > 1$, *and for any* $\eta < (1/2)\,\mathrm{dist}\,(E_0, \sigma(H_0^X))$, *there exists a finite constant* C_{E_0}, *depending on* $[\mathrm{dist}\,(\sigma(H_0^X), E_0)]^{-1}$, *the dimension* d, *and* $q > 1$, *such that*

$$\mathbb{P}\left\{\mathrm{dist}\,(E_0, \sigma(H_\Lambda^X)) \le \eta\right\} \le C_{E_0}\eta^{1/q}\,|\Lambda|\,. \tag{1.7}$$

If, in addition, the single-site potential satisfies (H3a), *then the result* (1.7) *holds for* $q = 1$ *and for* $H_\omega^X \mid \Lambda$, *with Dirichlet boundary conditions on the boundary of* Λ, *and for any* $E_0 \in \mathbb{R}$.

There are several prior results on the Wegner estimate for multidimensional, continuous Schrödinger operators with Anderson-type potentials constructed from fixed-sign, single-site potentials. Kotani and Simon [26] proved a Wegner estimate with a $|\Lambda|$-dependence for Anderson models with overlapping single-site potentials satisfying (H3a). This condition was removed and extensions were made to the band-edge case in [6] and [1]. An extension to multiplicative perturbations was made in [10,12–14]. These methods require a spectral averaging theorem (cf. [7] and references therein). Wegner's original proof [35] for Anderson models did not require spectral averaging. Following Wegner's argument, Kirsch gave a nice, short proof of the Wegner estimate in [20], but obtained a $|\Lambda|^2$-dependence. Recently, Stollmann [32] presented a short, elementary proof of the Wegner estimate for Anderson-type models with singular single-site probability distributions that are assumed to be simply Hölder continuous. He also obtains a $|\Lambda|^2$-dependence. These proofs, and the proof in this paper, do not require spectral averaging.

An immediate consequence of Theorem 1.1 concerns the IDS. In order to discuss the IDS, we need to assume that the model is ergodic. For example, we can take $u_j = u$, for all $j \in \mathbb{Z}^d$. Let Σ denote the deterministic spectrum of the family H_ω.

Theorem 1.2. *Assume* (H1), (H2), (H3), *and* (H4), *and that the model is ergodic. The integrated density of states is Hölder continuous of order* $1/q$, *for any* $q > 1$, *on the interval* (B_-, B_+). *If, in addition, we assume that the single-site potential satisfies* (H3a), *then the integrated density of states is locally Lipschitz continuous on* Σ.

Concerning the second case of nonsign-definite single-site potentials, hypothesis (H3c), our main results are not quite as general (see [15]). The first results concern energies below the bottom of the spectrum of H_0^A, and are given in Theorem 4.1 and Corollary 4.2. For the case of energies in an unperturbed spectral gap of H_0^X, we must suppose that the disorder is sufficiently small. The main results for this case are given in Theorem 4.3. These are, however, the first general results for nonsign-definite single-site potentials. Some related results concern the IDS for magnetic Schrödinger operators with unbounded

Gaussian random potentials studied by Hupfer, Leschke, Müller, and Warzel [17]. They prove a Wegner estimate for these models and that the IDS is absolutely continuous at all energies. Veselić [33] recently considered the nonsign-definite case for a restricted class of Anderson-type potentials that we discuss at the end of § 4.

The existence of the integrated density of states for additively perturbed, infinite-volume, ergodic models like (1.2) is well-known. A textbook account is found in the lecture notes of Kirsch [19]. The same proof applies to the multiplicatively perturbed model (1.1) with minor modifications. Recently, Nakamura [27] showed the uniqueness of the IDS, in the sense that it is independent of Dirichlet or Neumann boundary conditions, in the case of Schrödinger operators with magnetic fields. The same proof applies to the multiplicatively perturbed model. It is interesting to note that the proof uses the L^1-theory of the spectral shift function. Another proof of the uniqueness of the IDS for Schrödinger operators with magnetic fields is given by [11].

The contents of this paper are as follows. The L^p-theory of the spectral shift function (SSF) for $p > 1$ is developed in § 2. We also give a summary of other estimates on the SSF. We give a simple proof of Wegner's estimate in § 3. This proof is different from the proof in [9] and especially transparent. In § 4, we extend the results to Anderson-type potentials with nonsign-definite single-site potentials following [15].

2. The L^p-theory of the spectral shift function, $1 \leq p \leq \infty$

The L^p-theory of the spectral shift function for $p \in [1, \infty]$ can be viewed as an interpolation between the two well-known cases of $p = 1$ and $p = \infty$. Let us recall the L^1 and L^∞-theory, which can be found in the review paper of Birman and Yafaev [4], and the book of Yafaev [36]. Suppose that H_0 and H are two self-adjoint operators on a separable Hilbert space $\mathcal{H}$ having the property that $V \equiv H - H_0$ is in the trace class. We denote by $\|V\|_1$ the trace norm of V. Under these conditions, we can define the Krein spectral shift function (SSF) $\xi(\lambda; H, H_0)$ through the perturbation determinant. Let $R_0(z) = (H_0 - z)^{-1}$, for $\mathrm{Im}\, z \neq 0$. We then have

$$\xi(\lambda; H, H_0) \equiv \frac{1}{\pi} \lim_{\epsilon \to 0^+} \arg \det\, (1 + V R_0(\lambda + i\epsilon)). \tag{2.1}$$

It is well-known that

$$\int_{\mathbb{R}} \xi(\lambda; H, H_0)\, \mathrm{d}\lambda = \mathrm{Tr}\, V, \tag{2.2}$$

and that the SSF satisfies the L^1-estimate:

$$\|\xi(\cdot\,; H, H_0)\|_{L^1} \;\leq\; \|V\|_1. \tag{2.3}$$

At the other extreme, $p = \infty$, we recall that for a perturbation V of rank K, the SSF is essentially bounded and satisfies the bound

$$\|\xi(\cdot; H, H_0)\|_{L^\infty} \;\leq\; K. \tag{2.4}$$

In particular, for rank one perturbations, we have

$$|\xi(\lambda; H, H_0)| \leq 1. \tag{2.5}$$

This implies that $\|\xi(\cdot; H, H_0)\|_{L^\infty} \le K$ for finite-rank perturbations V.

Let us now consider the case $1 < p < \infty$ (cf. [30]). Let A be a compact operator on $\mathcal{H}$ and let $\mu_j(A)$ denote the jth singular value of A. We say that $A \in \mathcal{I}_{1/p}$, for some $p \ge 1$, if

$$\sum_j \mu_j(A)^{1/p} < \infty. \tag{2.6}$$

We define a nonnegative functional on the ideal $\mathcal{I}_{1/p}$ by

$$\|A\|_{1/p} \equiv \left(\sum_j \mu_j(A)^{1/p} \right)^p. \tag{2.7}$$

For $p > 1$, this functional is not a norm but satisfies

$$\|A + B\|_{1/p}^{1/p} \le \|A\|_{1/p}^{1/p} + \|B\|_{1/p}^{1/p}. \tag{2.8}$$

If we define a metric $\rho_{1/p}(A, B) \equiv \|A - B\|_{1/p}^{1/p}$ on $\mathcal{I}_{1/p}$, then the linear space $\mathcal{I}_{1/p}$ is a complete, separable linear metric space. The finite rank operators are dense in $\mathcal{I}_{1/p}$ (cf. [3]).

Since $\mathcal{I}_{1/p} \subset \mathcal{I}_1$, for all $p \ge 1$, we refer to $A \in \mathcal{I}_{1/p}$ as being super-trace class. Consequently, we can define the SSF for a pair of self-adjoint operators H_0 and H for which $V = H - H_0 \in \mathcal{I}_{1/p}$. Our main theorem is the following:

Theorem 2.1. *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_{1/p}$, for some $p \ge 1$. Then, the SSF $\xi(\lambda; H, H_0) \in L^p(\mathbb{R})$, and satisfies the bound*

$$\|\xi(\cdot\,; H, H_0)\|_{L^p} \le \|V\|_{1/p}^{1/p}. \tag{2.9}$$

Notice that this theorem provides the correct estimates for the endpoints $p = 1$ and $p = \infty$, where we take $1/\infty = 0$, and that the bound on the right side of (2.9) in this case is a constant depending only on the rank of V. In this sense, Theorem 2.1 is an interpolation theorem for the SSF in L^p-spaces for $p \in [0, \infty]$. The proof of Theorem 2.1 follows the same lines as the proof for the trace class case as found in, for example, Yafaev [36]. This bound was recently improved by Hundertmark and Simon [16].

Theorem 2.2 [16]. *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_1$. Let $F : [0, \infty) \to \mathbb{R}^+$ be a nonnegative, convex function with $F(0) = 0$. Then, the SSF $\xi(\lambda; H, H_0)$ satisfies the bound*

$$\int_{\mathbb{R}} F(|\xi(\lambda; H, H_0)|)\, \mathrm{d}\lambda \le \sum_{j=1}^{\infty} [F(j) - F(j-1)]\, \mu_j(V). \tag{2.10}$$

If one takes $F(t) = t^p$, $p \ge 1$ in Theorem 2.2, one obtains

$$\int_{\mathbb{R}} |\xi(\lambda; H, H_0)|^p\, \mathrm{d}\lambda = \sum_j (j^p - (j-1)^p)\mu_j(V). \tag{2.11}$$

The bound is better than the bound in Theorem 2.1, and provides an optimal upper bound for the L^p-norm of the SSF.

Other integral bounds on the SSF were obtained by Pushnitski [28]. Among them, we mention the following result concerning Schrödinger operators. We recall that for unbounded operators, such as Schrödinger operators, the SSF is defined through the invariance principle. Suppose that H_0 and H are two self-adjoint operators and $g : \mathbb{R} \to \mathbb{R}$ is a function so that $[g(H) - g(H_0)] \equiv V_{\text{eff}} \in \mathcal{I}_1$. Then, we define the SSF for the pair (H_0, H) by

$$\xi(\lambda; H, H_0) \equiv \operatorname{sgn}(g') \, \xi(g(\lambda); g(H), g(H_0)). \tag{2.12}$$

Theorem 2.3 [28]. *Let $d \geq 3$. Suppose that $H_0 = -\Delta$ and $H = H_0 + V$, where the potential $V \geq 0$ and satisfies the bound*

$$V(x) \leq C_0(1 + \|x\|)^{-\rho}, \quad \text{for} \ \ \rho > d. \tag{2.13}$$

Then, there exists a finite constant $C_1 \geq 0$, such that for any nonnegative, monotone decreasing function f, we have

$$\int_0^\infty \xi(\lambda; H, H_0) f(\lambda) \, \mathrm{d}\lambda \ \leq \ C_1 \int_0^\infty \lambda^{(d/2)-1} f(\lambda) \, \mathrm{d}\lambda \ \int_{\mathbb{R}^d} V(x) \, \mathrm{d}x. \tag{2.14}$$

In addition to these integral bounds on the SSF, we would like to mention the pointwise bound of Sobolev [31].

Theorem 2.4 [31]. *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_1$. Also suppose that*

$$\lim_{\epsilon \to 0^+} \| |V|^{1/2} (H_0 - \lambda - i\epsilon)^{-1} |V|^{1/2} \|_{1/p} \ < \infty, \tag{2.15}$$

for some $p \geq 1$. Then, there exists a finite constant $C_p > 0$, so that for all $\lambda > 0$, the SSF $\xi(\lambda; H, H_0)$ satisfies the bound

$$|\xi(\lambda; H, H_0)| \ \leq \ C_p \| |V|^{1/2} (H_0 - \lambda - i0)^{-1} |V|^{1/2} \|_{1/p}^{1/p}. \tag{2.16}$$

For one-dimensional Schrödinger operators, Kostrykin and Schrader [23] proved the following pointwise bound on the SSF.

Theorem 2.5 [23]. *Let $H_0 = -\mathrm{d}^2/\mathrm{d}x^2$ be the self-adjoint Laplacian on $L^2(\mathbb{R})$, and let $H = H_0 + V$, with the potential V satisfying*

$$\int_{\mathbb{R}} (1 + |x|^2) |V(x)| \, \mathrm{d}x \ < \infty. \tag{2.17}$$

Then, there exists a constant $0 \leq C_V < \infty$, depending on V and independent of λ, so that for all $\lambda \in \mathbb{R}$, the SSF $\xi(\lambda; H, H_0)$ satisfies

$$|\xi(\lambda; H, H_0)| \ \leq \ C_V. \tag{2.18}$$

Moreover, there is a constant $0 \leq C_0 < \infty$, independent of V and $\lambda > 0$, so that for all $\lambda > 0$, the SSF $\xi(\lambda; H, H_0)$ satisfies

$$|\xi(\lambda; H, H_0)| \leq C_0 \left\{ \frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} |V(x)| \, \mathrm{d}x + \frac{1}{4\lambda} \left[\int_{\mathbb{R}} |V(x)| \, \mathrm{d}x \right]^2 \right\}. \tag{2.19}$$

2.1 *Various identities for the SSF*

In this subsection, we study various identities for the SSF. In this setting, we consider a one-parameter family of self-adjoint operators H_λ, $\lambda \in J \equiv [\lambda^-, \lambda^+] \subset \mathbb{R}$.

1. The family $\lambda \in J \to H_\lambda$ is self-adjoint on the same domain D_0. The family is weakly differentiable on J with the derivative $\dot{H}_\lambda \equiv (\mathrm{d}H_\lambda/\mathrm{d}\lambda) \in \mathcal{I}_1$.
2. The map $\lambda \in J \to \|\dot{H}_\lambda\|_1$ is continuous.

The first fundamental result is the Birman–Krein trace formula (cf. [4,36]).

PROPOSITION 2.6

For any $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$\mathrm{Tr}\,\{f(H_{\lambda^+}) - f(H_{\lambda^-})\} = \int_{\mathbb{R}} f'(E)\xi(E; H_{\lambda^+}, H_{\lambda^-})\,\mathrm{d}E. \tag{2.20}$$

We also have a form of the spectral averaging theorem [2,7,29].

PROPOSITION 2.7

Under the conditions stated above, we have

$$\int_J \mathrm{Tr}\,\{E_\lambda(I)\dot{H}_\lambda\}\,\mathrm{d}\lambda = \int_I \xi(E; H_{\lambda^+}, H_{\lambda^-})\,\mathrm{d}E. \tag{2.21}$$

Sketch of the Proof. We will sketch the proof of this identity by working formally. First, for any $f \in C_0^\infty(\mathbb{R})$, we note the basic identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Tr}\{f(H(s))\} = \mathrm{Tr}\,\{f'(H(s))\ \dot{H}(s)\}. \tag{2.22}$$

We now integrate this equation over the interval J,

$$\begin{aligned}\int_J \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{Tr}\{f(H(s))\}\,\mathrm{d}s &= \mathrm{Tr}\{f(H(\lambda^+)) - f(H(\lambda^-))\}\\ &= \int_J \mathrm{Tr}\,\{f'(H(s))\ \dot{H}(s)\}\,\mathrm{d}s\\ &= \int_{\mathbb{R}} f'(E)\xi(E; H(\lambda^+), H(\lambda^-))\,\mathrm{d}E.\end{aligned} \tag{2.23}$$

We used the Birman–Krein trace formula (2.20). We now use the spectral theorem for $H(s)$ to write the integrand on the second line of (2.23) as

$$\mathrm{Tr}\{f'(H(s))\ \dot{H}(s)\} = \int_{\mathbb{R}} f'(E)\,\mathrm{d}\mu_s(E), \tag{2.24}$$

where $\mathrm{d}\mu_s(E)$ is the measure on $\mathbb{R}$ with the formal density given by $\mathrm{Tr}\{E_s(E)\dot{H}(s)\}$, with $E_s(\cdot)$ the spectral family of $H(s)$. We integrate the identity (2.24) over J to obtain

$$\int_J \mathrm{Tr}\,\{f'(H(s))\ \dot{H}(s)\}\,\mathrm{d}s = \int_{\mathbb{R}} f'(E) \int_J \mathrm{d}s\ \ \mathrm{d}\mu_s(E). \tag{2.25}$$

Comparing the formula on the right in (2.25) with the one on the third line of (2.23), we obtain

$$\xi(E; H(\lambda^+), H(\lambda^-))\,\mathrm{d}E = \int_J \mathrm{d}s \;\; \mathrm{d}\mu_s(E). \tag{2.26}$$

Integrating this identity over an interval $I \subset \mathbb{R}$, we obtain

$$\int_I \xi(E; H(\lambda^+), H(\lambda^-))\,\mathrm{d}E = \int_J Tr\{E_s(I)\dot{H}(s)\}\,\mathrm{d}s, \tag{2.27}$$

proving the proposition.

Let us note that if we formally take f so that $f'(x) = \chi_I(x)$, then the result (2.27) follows from the second and third lines of (2.23). □

2.2 *The integrated density of states*

The integrated density of states (IDS) is defined as follows: We consider the Hamiltonian H_ω restricted to a cube Λ with Dirichlet boundary conditions on $\partial\Lambda$, the boundary of the cube. This operator, denoted by H_Λ^D, has discrete spectrum. Let $N_\Lambda^D(\lambda)$ be the number of eigenvalues of H_Λ^D, including multiplicity, less than or equal to λ. If the following limit exists

$$\lim_{|\Lambda|\to\infty} \frac{N_\Lambda^D(\lambda)}{|\Lambda|} \equiv N(\lambda), \tag{2.28}$$

and it is called the IDS. It is known for the models discussed here that $N(\lambda)$ exists, is nonrandom, and a monotone increasing function of λ. We refer to [19] for a proof of this result.

There is an interesting connection between the IDS $N(\lambda)$ and the SSF for the pair (H_Λ, H_0), with $H_\Lambda = H_0 + V_\Lambda$, that involves the *spectral shift density* introduced by Kostrykin and Schrader [23,24]. For any $g \in C_0^1(\mathbb{R})$, they prove that the following limit

$$\lim_{|\Lambda|\to\infty} \int g(\lambda)\, \frac{\xi(\lambda; H_0 + V_\Lambda, H_0)}{|\Lambda|}\, \mathrm{d}\lambda \tag{2.29}$$

exists and is nonrandom.

Theorem 2.8 [24]. *For the models discussed here, the integrated density of states $N(E)$ exists, and belongs to $L^q_{\mathrm{loc}}(\mathbb{R})$, for any $q \geq 1$. Furthermore, if $N_0(\lambda)$ is the IDS for H_0, we have the following identity, for any $g \in C_0^1(\mathbb{R})$,*

$$\lim_{|\Lambda|\to\infty} \int g(\lambda)\, \frac{\xi(\lambda; H_0 + V_\Lambda, H_0)}{|\Lambda|}\mathrm{d}\lambda = \int g(\lambda)(N_0(\lambda) - N(\lambda))\,\mathrm{d}\lambda. \tag{2.30}$$

We remark that the proof of $N(\lambda) \in L^q_{\mathrm{loc}}(\mathbb{R})$, for any $q > 1$, uses the estimate (2.9).

3. Proof of Wegner's estimate

We first formulate Wegner's estimate in general terms for a family of random operators satisfying some assumptions. We then show that these assumptions are verified for some Anderson-type models.

3.1 *An abstract Wegner's estimate*

We give a rather general proof of a Wegner estimate under the following assumptions:

(A1) The operator H_Λ depends on $N = \mathcal{O}(|\Lambda|)$ random variables $\{\lambda_1, \dots, \lambda_N\}$, distributed according to the distribution $h_0(\lambda)\mathrm{d}\lambda$ with $h_0 \in L^\infty((\lambda^-, \lambda^+))$, for finite $\lambda^-, \lambda^+ \in \mathbb{R}$.

(A2) For a bounded interval $I \subset \mathbb{R}$, the following identity holds for some finite $C_0 > 0$:

$$\mathrm{Tr}\,\{E_\Lambda(I_\eta)\} \;\le\; C_0\,\mathrm{Tr}\left\{\sum_{j=1}^{N}\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right\}. \tag{3.1}$$

(A3) Let $\omega_j^\pm \equiv \{(\lambda_1, \dots, \lambda_j = \lambda^\pm, \dots, \lambda_n)\}$ be the set of all configurations for which the random variable λ_j is fixed at the minimum, respectively, maximum value. For any $p > 1$, let $g(x) = (x + M_0)^{-k}$, for some $k > (pd/2) + 2$. Then, there is a finite constant $C = C(p, d, M_0) > 0$ so that

$$\sup_{j=1,\dots,N}\left(\sup_{k\neq j}\;\|g(H_{\omega_j^+}) - g(H_{\omega_j^-})\|_{1/p}\right) \;\le C < \infty. \tag{3.2}$$

These assumptions can be modified for the multiplicatively perturbed model (1.1), but we do not do this here, and concentrate on the additively perturbed model (1.2).

Theorem 3.1. *Assume that the random family of Hamiltonians satisfy assumptions* (A1)–(A3). *Then, for any $q > 1$, there exists a finite constant $C_W = C_W(q, d, C_0, C_1, k,$ dist $(I, M_0)) > 0$, so that*

$$\mathbb{E}\{\mathrm{Tr}\,(E_\Lambda(I))\} \;\le\; C_W\|h_0\|_\infty |I|^{1/q}|\Lambda|. \tag{3.3}$$

Proof. 1. Due to hypothesis (A2), we have

$$\begin{aligned}\mathbb{E}\{\mathrm{Tr}\,(E_\Lambda(I))\} \le \quad & C_0\mathbb{E}\left[\mathrm{Tr}\;\left\{\sum_{j=1}^{N}\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right\}\right] \\ \le \quad & C_0\sum_{j=1}^{N}\;\mathbb{E}\left\{\mathrm{Tr}\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right\}. \end{aligned} \tag{3.4}$$

As usual, we select one random variable, say λ_j, and integrate with respect to it, using positivity,

$$\begin{aligned}&\mathbb{E}\{\mathrm{Tr}\,(E_\Lambda(I))\} \\ &\quad\le\quad C_0\sum_{j=1}^{N}\int_I \Pi_{k\neq j}h_0(\lambda_k)\mathrm{d}\lambda_k \int_{[\lambda_j^-,\lambda_j^+]} h_0(\lambda_j)\mathrm{d}\lambda_j\;\mathrm{Tr}\left\{\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right\} \\ &\quad\le\quad C_0\|h_0\|_\infty\mathbb{E}'\left\{\int_{[\lambda_j^-,\lambda_j^+]} \mathrm{d}\lambda_j\;\mathrm{Tr}\;\left[\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right]\right\}, \end{aligned} \tag{3.5}$$

where $\mathbb{E}'$ denotes the expectation with respect to the other random variables λ_k, for $k \neq j = \{1, \ldots, N\}$.

2. We use the spectral averaging formula, Proposition 2.6, to evaluate the integral on the right side of (3.5). This gives

$$\int_{[\lambda_j^-, \lambda_j^+]} \mathrm{d}\lambda_j \ \mathrm{Tr}\left[\left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right) E_\Lambda(I)\right] = \int_I \mathrm{d}E \ \xi(E; H_{\lambda_j^+}, H_{\lambda_j^-}). \tag{3.6}$$

At this stage, we use the L^p-estimate on the SSF and Hölder's inequality. Let χ_I be the characteristic function on the energy interval I. For any $q > 1$, let $p > 1$ be the conjugate index so that $(1/p) + (1/q) = 1$. We then have

$$\int \chi_I(E)\xi(E; H_{\lambda_j^+}, H_{\lambda_j^-}) \, \mathrm{d}E \ \leq \ |I|^{1/q} \|\xi(\cdot; H_{\lambda_j^+}, H_{\lambda_j^-})\|_{L^p(I)}. \tag{3.7}$$

3. The SSF appearing in (3.7) is defined through the invariance principle due to the fact that the Hamiltonians are unbounded. Let $g(E) = (E + M_0)^{-k}$, for some $M_0 >> 0$, the existence of which is guaranteed by (H1), and for some $k > (2d/p) + 2$, where $p > 1$. Note that $\mathrm{sgn} g' = -1$, for $E > -M_0$. We recall, as in (2.12), that the SSF is defined by

$$\xi(E; H_{\lambda_j^+}, H_{\lambda_j^-}) = -\xi(E; g(H_{\lambda_j^+}), g(H_{\lambda_j^-})). \tag{3.8}$$

Using Theorem 2.1, after changing variables in the integral, we find

$$\|\xi(\cdot; H_{\lambda_j^+}, H_{\lambda_j^-})\|_{L^p(I)} \ \leq \ C_1(E_0 - |I|/2 + M_0)^{-(k+1)/pk} \|g(H_{\lambda_j^+}) - g(H_{\lambda_j^-})\|_{1/p}^{1/p}. \tag{3.9}$$

By Proposition 3.2 ahead, the trace ideal functional is bounded independently of $|\Lambda|$. Hence, from (3.5)–(3.7), we obtain

$$\mathbb{E}\{\mathrm{Tr}\ (E_\Lambda(I))\} \ \leq \ C_2|I|^{1/q}\|h_0\|_\infty|\Lambda|, \tag{3.10}$$

proving the theorem. □

The trace estimate used above is the following: We let H_0 be the Schrödinger operator

$$H_0 = (-i\nabla - A)^2 + W, \tag{3.11}$$

where A is a vector potential with $A \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and $W = W_+ - W_-$ is a background potential with $W_- \in K_d(\mathbb{R}^d)$ and $W_+ \in K_d^{\mathrm{loc}}(\mathbb{R}^d)$. We denote by $H = H_0 + V$, for suitable real-valued functions V. We are interested in a bounded potential V with compact support. The proof of the following proposition is given in [9].

PROPOSITION 3.2

Let H_0 be as above, and let V_1 be a Kato-class potential such that $\|V_1\|_{K_d} \leq M_1$. Let $H_1 \equiv H_0 + V_1$, and let $M > 0$ be a sufficiently large constant given in the proof. Let V be a Kato-class function supported in $B(R)$, the ball of radius $R > 0$ with center at the origin. Then, for any $p > 0$, we have

$$V_{\mathrm{eff}} \equiv (H_1 + V + M)^{-k} - (H_1 + M)^{-k} \in \mathcal{I}_{1/p}, \tag{3.12}$$

provided $k > \mathrm{d}p/2 + 2$. Under these conditions, there exists a constant C_0, depending on $p, k, H_0, M_1, \|V\|_{K_d}$, and R, so that

$$\|V_{\mathrm{eff}}\|_{1/p} \ \leq \ C_0. \tag{3.13}$$

We remark that for the case of a locally perturbed Schrödinger operator H_0, with $H = H_0 + V_\Lambda$, Kostrykin and Schrader [25] showed that the constant C_0 in (3.13) is bounded above by $C_1|\Lambda|^p$, for a constant C_1 independent of $|\Lambda|$.

3.2 *Application to Anderson-type models*

We indicate how to verify the assumptions (A1)–(A3) for the Anderson-type additive models described in section one. This provides a simpler proof than the one presented in [9] provided we add hypothesis (H3b). *To simplify the notation, we will drop the notation H_0^A and H_Λ^A, and write H_0 and H_Λ, respectively, for the additive case.*

As with the proof in [9], we note that this proof of the Wegner estimate does not require spectral averaging [7]. It does, however, rely upon some monotonicity of the eigenvalues with respect to the random variables (for comparison, see the work [5] in one dimension). Furthermore, the comparison theorem of Kirsch, Stollmann, and Stolz [21], used in [9] is not needed for this version of the proof. In the next section, we present a technique that removes the positivity assumption.

PROPOSITION 3.3

Let us suppose that H_Λ satisfies hypotheses (H1) *or* (H1a), (H2), (H3b), *and* (H4). *Then, the additively-perturbed Anderson model satisfies assumptions* (A1)–(A3).

Proof. Assumption (A1) is obviously satisfied by the Anderson-type potentials with $N = |\tilde{\Lambda}|$, the number of lattice points in Λ. We turn to the proof of (A2). Hypothesis (H3b) implies that the single-site potentials satisfy $u_i u_j = \delta_{ij} u_j$. Let $E_0 \in G$, where $G \subset \rho(H_0)$ is a subset of an unperturbed spectral gap for H_0, and choose $\eta > 0$ so that the interval I of (A2) is $I = I_\eta \equiv [E_0 - \eta, E_0 + \eta] \subset G$. Since the perturbation V_Λ is relatively H_0 compact, we know that $\sigma(H_0) \cap G$ is discrete. Let $\phi \in E_\Lambda(I_\eta) L^2(\mathbb{R}^d)$ be a normalized eigenfunction of H_Λ with eigenvalue $E \in I_\eta$. Using the eigenvalue equation, we easily verify that

$$\|(H_0 - E)\phi\| = \|V_\Lambda \phi\|. \tag{3.14}$$

Furthermore, we can expand the right side as

$$\begin{aligned} \|V_\Lambda \phi\|^2 &= \langle \phi, V_\Lambda^2 \phi \rangle \\ &= \sum_{j \in \tilde{\Lambda}} \lambda_j^2 \langle \phi, u_j^2 \phi \rangle \\ &= \|(H_0 - E)\phi\|^2 \\ &\geq [\text{dist}\,(\sigma(H_0), I_\eta)]^2, \end{aligned} \tag{3.15}$$

using hypothesis (H3b). Now, we know that

$$\frac{\partial H_\Lambda}{\partial \lambda_j} = \frac{\partial V_\Lambda}{\partial \lambda_j} = u_j(\cdot - j), \tag{3.16}$$

so that, as $u_j(\cdot - j) \geq C_0 u_j(\cdot - j)^2$, we have

$$\sum_{j\in\tilde{\Lambda}} \left\langle \phi, \left(\frac{\partial H_\Lambda}{\partial \lambda_j}\right)\phi \right\rangle \geq \sum_{j\in\tilde{\Lambda}} C_0\langle \phi, u_j(\cdot - j)^2\phi\rangle$$
$$\geq C_0(\lambda^+)^{-2}\langle \phi, V_\Lambda^2\phi\rangle$$
$$\geq C_0(\lambda^+)^{-2}[\text{dist } (\sigma(H_0), I_\eta)]^2\|\phi\|^2, \tag{3.17}$$

where we assume, without loss of generality, that $|\lambda^+| \geq |\lambda^-|$. This inequality immediately implies (A2) since

$$\operatorname{Tr}\left\{\sum_{j\in\tilde{\Lambda}}\left(\frac{\partial H_\Lambda}{\partial\lambda_j}\right)E_\Lambda(I)\right\}$$
$$= \sum_k \sum_{j\in\tilde{\Lambda}} \left\langle \phi_k, \left(\frac{\partial H_\Lambda}{\partial\lambda_j}\right)\phi_k\right\rangle$$
$$\geq C_0(\lambda^+)^{-2}[\text{dist } (\sigma(H_0), I_\eta)]^2 \sum_k \|\phi_k\|^2$$
$$\geq C_0(\lambda^+)^{-2}[\text{dist } (\sigma(H_0), I_\eta)]^2 \operatorname{Tr} E_\Lambda(I_\eta). \tag{3.18}$$

Finally, we verify (A3). We note that $(H^\Lambda_{\omega_j^+} - H^\Lambda_{\omega_j^-}) = (\lambda^+ - \lambda^-)u_j(\cdot - j)$. It is proved in [9] that given any $p > 1$, for any $k > (pd/2) + 2$, if we set $g(E) = (E + M_0)^{-k}$, then

$$\|g(H^\Lambda_{\omega_j^+}) - g(H^\Lambda_{\omega_j^-})\|_{1/p} \leq C_k < \infty, \tag{3.19}$$

where the constant C_k is independent of the index j, and it is independent of $|\Lambda|$ and depends only on $|\text{supp } u_j|$. □

The verification of assumption (A2) is more difficult in the general case. In the absence of hypothesis (H3b), there are two possibilities: (1) the u_j satisfies hypothesis (H3a), i.e. $u_j \geq C_0\chi_{\Lambda_1(0)}$, or (2) the u_j satisfies hypothesis (H3), i.e. u_j is nonnegative and the support of u_j is compact. In the first case, assumption (H3a), the single-site potentials satisfy

$$\sum_{j\in\tilde{\Lambda}} u_j(x - j) \geq C_0\chi_\Lambda, \tag{3.20}$$

which is a strong monotonicity condition. Under this condition, we have the following global result.

PROPOSITION 3.3

We define the local Hamiltonian H_Λ by $H_\Lambda = (H_0 + V_\omega)|\Lambda$, with Dirichlet boundary conditions on $\partial\Lambda$. Suppose that the local Hamiltonian H_Λ satisfies (H1), (H2), (H3a), *and* (H4). *Then, for any $E_0 \in \mathbb{R}$, and any interval $I_\eta = [E_0 - \eta, E_0 + \eta] \subset \mathbb{R}$, there exists a finite constant $C_W > 0$, depending on (d, η, E_0), so that we have*

$$\mathbb{E}\{\operatorname{Tr}(E_\Lambda(I_\eta))\} \leq C_W\eta|\Lambda|. \tag{3.21}$$

Consequently, the IDS is Lipschitz continuous at all energies.

Sketch of the Proof. The proof of this proposition follows the lines of the proof given in [6]. We suppose that Λ is a cube, and work on the Hilbert space $L^2(\Lambda)$. Since V_ω is bounded, there is a finite, positive constant V_0 so that $-V_0 \leq V_\Lambda$, $H_\Lambda \geq H_0^\Lambda - V_0$, where $H_0^\Lambda \equiv H_0|\Lambda$, with Dirichlet boundary conditions. This lower bound and Jensen's inequality lead to the bound

$$\begin{aligned} \mathrm{Tr}\, E_\Lambda(I_\eta) &\leq \mathrm{e}^{(E_0+\eta)}\, \mathrm{Tr}\{\mathrm{e}^{-H_\Lambda} E_\Lambda(I_\eta)\} \\ &\leq \mathrm{e}^{(E_0+\eta+V_0)}\, \mathrm{Tr}\{\mathrm{e}^{-H_0^\Lambda} E_\Lambda(I_\eta)\}. \end{aligned} \tag{3.22}$$

We decompose the cube Λ into unit cubes Λ_j, so that $\Lambda = \mathrm{Int}\, \overline{\cup_j \Lambda_j}$. Dirichlet–Neumann bracketing and the diamagnetic inequality imply that

$$\mathrm{e}^{-H_0^\Lambda} \leq \mathrm{e}^{\oplus_j \Delta_{\Lambda_j}^N} = \sum_{j \in \tilde{\Lambda}} \chi_j \mathrm{e}^{\Delta_{\Lambda_j}^N} \chi_j, \tag{3.23}$$

where χ_j is the characteristic function on Λ_j and $-\Delta_j^N$ is the nonnegative Neumann Laplacian on Λ_j. Substituting this into (3.22), we obtain

$$\mathrm{Tr} E_\Lambda(I_\eta) \leq \mathrm{e}^{(E_0+\eta+V_0)} \sum_{j \in \tilde{\Lambda}} \mathrm{Tr}\{\mathrm{e}^{\Delta_{\Lambda_j}^N} \chi_j E_\Lambda(I_\eta) \chi_j\}. \tag{3.24}$$

We now expand the trace in the eigenfunctions of $\Delta_{\Lambda_j}^N$ and use spectral averaging. The result follows by noting that $\mathrm{Tr}\{\mathrm{e}^{\Delta_{\Lambda_j}^N}\}$ is bounded. □

The general case of hypothesis (H3), was treated in [9] using a result of Kirsch, Stollmann, and Stolz [21] on the localization of the eigenfunctions of the local Hamiltonian H_Λ. This theorem provides precise information about the eigenfunctions in the region Λ. The proof of this theorem is simple and we refer the reader to [21,9].

PROPOSITION 3.4

Let H_0 and V_Λ be as above and $H_\Lambda \phi = E\phi$ with $E \in G$ and $\phi \in L^2(\mathbb{R}^d)$. Suppose that the following two conditions are satisfied:

1. *There exists a potential V_0 such that, with $H_\Lambda^0 \equiv H_0 + V_0$, we have $E \in \rho(H_\Lambda^0)$;*
2. *There exists a subset $F \subset \Lambda$ and a constant $\theta > 0$ so that* dist $(F \cup \Lambda^c, \{x \mid V_\Lambda(x) \neq V_0(x)\}) > \theta > 0$.

We then have

$$\|\phi\| \leq (1 + \|(H_\Lambda^0 - E)^{-1} W_1\|)\|(1 - \chi_F)\phi\|, \tag{3.25}$$

where $W_1 \equiv [H_0, \chi_1]$, with χ_1 is defined in the proof, and χ_F is the characteristic function of F.

4. The nonsign-definite case

Although the proof presented in §3 is elementary, it does require that the single-site potentials u_j have a definite sign. The case of nonsign definite single-site potentials is more delicate since the eigenvalues are no longer monotonic functions of the random variables. We have two main results in the nonsign-definite case. The first, and more general result applies to energies below the bottom of the spectrum of the background operator H_0. The second result concerns the Wegner estimate at energies in an internal gap of the spectrum of H_0. This requires the disorder to be small. The basic idea of the proofs is to combine the vector field method of Klopp [18] with the techniques of §3.

The single-site potentials u_j must satisfy hypothesis (H3c), that is weaker than the other hypotheses (H3), (H3a), or (H3b). Basically, we need that u_j is continuous and nonvanishing on some bounded, open set. As we mention in the proof below, we need a slightly stronger hypothesis on the common distribution h_0 of the random variables. This is given in hypothesis (H4a).

4.1 *Below the infimum of the spectrum of H_0^A*

For energies $E < \inf \sigma(H_0^A) \equiv \Sigma_0^A$, the operator $(H_0 - E)$ is strictly positive. This allows us to reformulate the Wegner estimate as a statement concerning a Birman–Schwinger-type operator. The main result, under hypotheses (H1a), (H2), (H3c), and (H4a) on the unperturbed operator H_0^A and the local perturbation V_Λ, is the following theorem. We recall that for multiplicative perturbations, we have $\Sigma_0^M = \inf \Sigma^M = 0$, where $\Sigma^X \equiv \sigma(H_\omega^X)$ almost surely, so these results apply only to additive perturbations.

Theorem 4.1. *Assume* (H1a), (H2), (H3c), *and* (H4a). *For any $q > 1$, and for any $E_0 \in (-\infty, \Sigma_0^A)$, there exists a finite, positive constant C_{E_0}, depending only on* $[\text{dist}\,(\sigma(H_\Lambda^A), E_0)]^{-1}$, *the dimension d, and $q > 1$, so that for any $\eta <$ dist $(\sigma(H_0^A), E_0)$, we have*

$$\mathbb{P}\left\{\text{dist}\,(E_0, \sigma(H_\Lambda^A)) \leq \eta\right\} \leq C_{E_0} \eta^{1/q}\, |\Lambda|\, . \tag{4.1}$$

As an immediate corollary of Theorem 4.1, and of the definition of the density of states, we obtain

COROLLARY 4.2

Assume (H1a), (H2), (H3c), *and* (H4a), *and that the model H_ω^A is ergodic. The integrated density of states is locally Hölder continuous of order $1/q$, for any $q > 1$, on the interval $(-\infty, \Sigma_0^A)$.*

Following [18], we formulate the Wegner estimate in terms of the resolvent of H_Λ^A using the fact that if $E_0 < \inf \sigma(H_0^A)$, we have that $(H_0^A - E_0) > 0$. So, for an energy E_0 in the resolvent set of H_Λ^A, we have

$$R_\Lambda(E_0) = (H_\Lambda^A - E_0)^{-1} = (H_0^A - E_0)^{-1/2}(1 + \Gamma_\Lambda(E_0;\omega))^{-1}(H_0^A - E_0)^{-1/2}. \tag{4.2}$$

The Birman–Schwinger-type operator $\Gamma_\Lambda(E_0;\omega)$ is defined by

$$\begin{aligned}\Gamma_\Lambda(E_0;\omega) &= (H_0^A - E_0)^{-1/2} V_\Lambda (H_0^A - E_0)^{-1/2}\\ &= \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega)(H_0^A - E_0)^{-1/2} u_j (H_0^A - E_0)^{-1/2}.\end{aligned} \tag{4.3}$$

Since supp u_j is compact and the sum over $j \in \tilde{\Lambda}$ is finite, the operator $\Gamma(E_0; \omega_\Lambda)$ is compact, self-adjoint, and uniformly bounded. Let us write δ for dist$(E_0, \inf \sigma(H_0^A))$. It follows from (4.2) that

$$\begin{aligned} \|R_\Lambda(E_0)\| \leq & \quad \{\text{dist } (\sigma(H_0^A), E_0)\}^{-1} \|(1 + \Gamma_\Lambda(E_0; \omega))^{-1}\| \\ \leq & \quad \delta^{-1} \|(1 + \Gamma_\Lambda(E_0; \omega))^{-1}\|. \end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$\mathbb{P}\{\|R_\Lambda(E_0)\| \leq 1/\eta\} \geq \mathbb{P}\{\|(1 + \Gamma_\Lambda(E_0; \omega))^{-1}\| \leq \delta/\eta\}. \tag{4.5}$$

Consequently, Wegner's estimate can be reformulated as

$$\begin{aligned} \mathbb{P}\{\text{dist } (\sigma(H_\Lambda^A), E_0) < \eta\} = & \quad \mathbb{P}\{ \|R_\Lambda(E_0)\| > 1/\eta\} \\ \leq & \quad \mathbb{P}\{ \|(1 + \Gamma_\Lambda(E_0; \omega))^{-1}\| > \delta/\eta\} \\ = & \quad \mathbb{P}\{ \text{dist } (\sigma(\Gamma_\Lambda(E_0; \omega)), -1) < \eta/\delta\}. \end{aligned} \tag{4.6}$$

Hence, it suffices to compute

$$\mathbb{P}\{ \text{dist } (\sigma(\Gamma_\Lambda(E_0; \omega)), -1) < \eta/\delta\}. \tag{4.7}$$

The key observation of [18] that takes the place of monotonicity and the eigenfunction localization theorem of Kirsch, Stollmann, and Stolz [21], Proposition 3.4, is the following. We define a vector field A_Λ on $L^2([m, M]^{\tilde{\Lambda}}, \Pi_{j \in \tilde{\Lambda}} h_0(\lambda_j) \, d\lambda_j)$ by

$$A_\Lambda \equiv \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega) \frac{\partial}{\partial \lambda_j(\omega)}. \tag{4.8}$$

Then, the operator $\Gamma_\Lambda(E_0; \omega)$ is an eigenvector of A_Λ in that

$$A_\Lambda \Gamma_\Lambda(E_0; \omega) = \Gamma_\Lambda(E_0; \omega). \tag{4.9}$$

It is this relationship that replaces the positivity used in [9] since, if $\Gamma_\Lambda(E_0; \omega)$ is restricted to the spectral subspace where the operator is smaller than $(-1 + 3\kappa/2)$, we have that $-\Gamma_\Lambda(E_0; \omega)$ is strictly positive, and hence invertible. We will use this below.

Sketch of the Proof of Theorem 4.1.

1. It follows from the reduction given above that we need to estimate the probability in (4.7). Let $G = (-\infty, \inf \sigma(H_0^A))$ be the unperturbed spectral gap. Since the local potential V_Λ is a relatively compact perturbation of H_0^A, the operator $\Gamma_\Lambda(E_0; \omega)$ has only discrete spectrum with zero the only possible accumulation point. Let us write $\kappa \equiv \eta/\delta$. We choose $\eta > 0$ small enough so that $[E_0 - \eta, E_0 + \eta] \subset G$, and that $[-1 - 2\kappa, -1 + 2\kappa] \subset \mathbb{R}^-$. We denote by I_κ the interval $[-1 - \kappa, -1 + \kappa]$. The probability in (4.7) is expressible in terms of the finite-rank spectral projector for the interval I_κ and $\Gamma_\Lambda(E_0; \omega)$, which we write as $E_\Lambda(I_\kappa)$. Like $\Gamma_\Lambda(E_0; \omega)$, this projection is a random variable, but we will suppress any reference to ω in the notation. We now apply Chebyshev's inequality to the random variable Tr $(E_\Lambda(I_\kappa))$ and obtain

$$\begin{aligned} \mathbb{P}\{ \text{dist } (\sigma(\Gamma_\Lambda(E_0)), -1) < \kappa\} = & \quad \mathbb{P}\{\text{Tr } (E_\Lambda(I_\kappa)) \geq 1\} \\ \leq & \quad \mathbb{E}\{\text{Tr } (E_\Lambda(I_\kappa))\}. \end{aligned} \tag{4.10}$$

2. We now proceed to estimate the expectation of the trace in (4.10), following the original argument of Wegner [35] as modified by Kirsch [20]. Let ρ be a nonnegative, smooth function such that $\rho(x) = 1$, for $-M_1 < x < -\kappa/2$, and $\rho(x) = 0$, for $x \geq \kappa/2$ and for $x \leq -M_1$, for some $M_1 > 0$. We can assume that $M_1 < \infty$, so that ρ has compact support, since $\Gamma_\Lambda(E_0)$ is lower semibounded and independent of Λ. We further assume that ρ is monotone decreasing for $x > -M_1$. As in [9], we have

$$\begin{aligned} \mathbb{E}_\Lambda\{\mathrm{Tr}\,(E_\Lambda(I_\kappa))\} &\leq \mathbb{E}_\Lambda\{\mathrm{Tr}\,[\rho(\Gamma_\Lambda(E_0)+1-3\kappa/2) - \rho(\Gamma_\Lambda(E_0)+1+3\kappa/2)]\} \\ &\leq \mathbb{E}_\Lambda\left\{\mathrm{Tr}\left[\int_{-3\kappa/2}^{3\kappa/2} \frac{\mathrm{d}}{\mathrm{d}t}\rho(\Gamma_\Lambda(E_0)+1-t)\,\mathrm{d}t\right]\right\}. \end{aligned} \tag{4.11}$$

In order to evaluate the ρ' term, we use the fact that $\Gamma_\Lambda(E_0)$ is an eigenfunction for the vector field A_Λ, as expressed in (4.9). We write ρ' as

$$\begin{aligned} A_\Lambda\rho(\Gamma_\Lambda(E_0)+1-t) &= \rho'(\Gamma_\Lambda(E_0)+1-t)\; A_\Lambda\Gamma_\Lambda(E_0) \\ &= \rho'(\Gamma_\Lambda(E_0)+1-t)\Gamma_\Lambda(E_0). \end{aligned} \tag{4.12}$$

We now note that $\rho' \leq 0$ (in the region of interest), and that on supp ρ', the operator $\Gamma_\Lambda(E_0) \leq (-1+2\kappa)$, so we obtain

$$-\rho'(\Gamma_\Lambda(E_0)+1-t) \leq -\frac{1}{(1-2\kappa)}\sum_{k\in\Lambda}\lambda_k\,\frac{\partial\rho}{\partial\lambda_k}\,(\Gamma_\Lambda(E_0)+1-t). \tag{4.13}$$

With this estimate, and the fact that $\mathrm{d}\rho(x+1-t)/\mathrm{d}t = -\rho'(x+1-t)$, the right side of (4.11) can be bounded above by

$$-\frac{1}{(1-2\kappa)}\sum_{k\in\tilde\Lambda}\int_{-3\kappa/2}^{3\kappa/2}\mathbb{E}\left\{\lambda_k\frac{\partial}{\partial\lambda_k}\,\mathrm{Tr}\,[\rho(\Gamma_\Lambda(E_0)+1-t)]\right\}\,\mathrm{d}t. \tag{4.14}$$

As in the proof of Theorem 3.1, we select one random variable, say λ_k, with $k\in\tilde\Lambda$, and first integrate with respect to this variable using hypothesis (H4a). The local absolute continuity property is necessary here because a single term in the sum of (4.14) is not necessarily positive. Let us suppose that there is a decomposition $[0, M] = \cup_{l=0}^{N-1}(M_l, M_{l+1})$ so that h_0 is absolutely continuous on each subinterval. We denote by $\tilde h_0$ the function $\tilde h_0(\lambda) \equiv \lambda h_0(\lambda)$. As $\tilde h_0$ is locally absolutely continuous, we can integrate by parts and obtain

$$\begin{aligned} &\left|\int_0^M \mathrm{d}\lambda_k\tilde h_0(\lambda_k)\frac{\partial}{\partial\lambda_k}\,\mathrm{Tr}\{\rho(\Gamma_\Lambda(E_0)+1-t) - \rho(\Gamma_\Lambda(E_0)^{0,k}+1-t\}\right| \\ &\quad= \left|\sum_{l=0}^{N-1}\int_{M_l}^{M_{l+1}}\mathrm{d}\lambda_k\tilde h_0(\lambda_k)\frac{\partial}{\partial\lambda_k}\mathrm{Tr}\{\rho(\lambda_k)-\rho(\lambda_k=0)\}\right| \\ &\quad\leq \tilde h_0(M)|\,\mathrm{Tr}\,\{\rho(\Gamma_\Lambda(E_0)^{M,k}+1-t)-\rho(\Gamma_\Lambda(E_0)^{0,k}+1-t)\}| \\ &\qquad+ \|\tilde h_0'\|_\infty \sup_{\lambda\in[0,M]}|\,\mathrm{Tr}\,\{\rho(\Gamma_\Lambda(E_0)^{\lambda,k}+1-t)-\rho(\Gamma_\Lambda(E_0)^{0,k}+1-t)\}|, \end{aligned} \tag{4.15}$$

where $\Gamma_\Lambda(E_0)^{\lambda,k}$ is the operator $\Gamma_\Lambda(E_0)$ with the coupling constant λ_k at the kth-site fixed at the value $\lambda_k = \lambda$. Similarly, the value 0 or M denotes the coupling constant λ_k fixed at those values. Consequently, we are left with the task of estimating

$$\frac{\max\,(\|\tilde{h}_0'\|_\infty, \tilde{h}_0(M))}{(1-2\kappa)} \sum_{k\in\tilde{\Lambda}} \int_{-3\kappa/2}^{3\kappa/2} \mathrm{d}t \int_0^M \Pi_{l\neq k}\, h_0(\lambda_l)\ \mathrm{d}\,\lambda_l\ |\ \mathrm{Tr}\,\{D(k, E_0, 0, \lambda_k^+)\}|, \tag{4.16}$$

where $D(k, E_0, 0, \lambda_k^+)$ denotes the operator

$$D(k, E_0, 0, \lambda_k^+) \equiv \rho(\Gamma_\Lambda(E_0)^{0,k} + 1 - t) - \rho(\Gamma_\Lambda(E_0)^{\lambda_k^+,k} + 1 - t), \tag{4.17}$$

and $\lambda_k^+ \in [0, M]$ denotes the value of the coupling constant λ_k where the maximum in (4.15) is obtained. We remark that each term in (4.16) is easily seen to be trace-class since the operator $\Gamma_\Lambda(E_0)$ has discrete spectrum with zero the only accumulation point, and the function $\rho(x+1-t)$ is supported in x in a compact interval away from 0 for $t \in [-3\kappa/2, 3\kappa/2]$.

3. The trace in (4.16) can be rewritten in terms of a spectral shift function as follows: We let $H_1 \equiv \Gamma_\Lambda(E_0)^{0,k}$ be the unperturbed operator, and write

$$\begin{aligned}\Gamma_\Lambda(E_0)^{\lambda_k^+,k} &= H_1 + \lambda_k^+(H_0^A - E_0)^{-1/2} u_k (H_0^A - E_0)^{-1/2} \\ &= H_1 + V.\end{aligned} \tag{4.18}$$

Although the difference V is not trace class, the single-site potential u_k does have compact support. A result similar to Proposition 3.2 holds in this case, and the difference of sufficiently large powers of the bounded operators $H_1 = \Gamma_\Lambda(E_0)^{0,k}$ and $H_1+V = \Gamma_\Lambda(E_0)^{\lambda_k^+,k}$ is not only in the trace class, but is in the super-trace class $\mathcal{I}_{1/p}$, for all $p \geq 1$. Specifically, let us define the function $g(\lambda) = \lambda^k$. We prove that for $k > pd/2 + 1$, and $p > 1$,

$$g(H_1 + V) - g(H_1) \in \mathcal{I}_{1/p}. \tag{4.19}$$

The spectral shift function $\xi(\lambda\ ; H_1 + V, H_1)$ is defined for the pair $(H_1, H_1 + V)$ by the invariance principle (2.12). Recall that both ρ and ρ' have compact support. Because of this, and the fact that the difference $\{g(H_1 + V) - g(H_1)\}$ is super-trace class, we can apply the Birman–Krein identity [4] to the trace in (4.16). This gives

$$\begin{aligned}&\mathrm{Tr}\,\{\rho(\Gamma_\Lambda(E_0)^{\lambda_k^+,k} + 1 - t) - \rho(\Gamma_\Lambda(E_0)^{0,k} + 1 - t)\} \\ &\quad = -\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\lambda}\rho(\lambda + 1 - t)\ \xi(\lambda; H_1 + V, H_1)\ \mathrm{d}\lambda \\ &\quad = -\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\lambda}\rho(\lambda + 1 - t)\ \xi(g(\lambda);\ \ g(H_1 + V), g(H_1)).\end{aligned} \tag{4.20}$$

We estimate the integral using the Hölder inequality and the L^p-theory of § 2. Let $\tilde{\xi}(\lambda) = \xi(g(\lambda);\ \ g(H_1 + V), g(H_1))$, for notational convenience. Let $\chi(x)$ be the characteristic function for the support of $\rho'(x)$ for $x > 0$, and we write $\tilde{\chi}(x) \equiv \chi(\lambda + 1 - t)$, so that the support of $\tilde{\chi}$ is contained in $[-1-2\kappa, -1+2\kappa]$. For any $p > 1$, and q such that $(1/p) + (1/q) = 1$, the right side of (4.20) can be bounded above by

$$\left\{\int |\rho'|^q\right\}^{1/q} \left\{\int |\tilde{\xi}(\lambda)\ \tilde{\chi}(\lambda)|^p\right\}^{1/p} \leq C_0 \kappa^{(1-q)/q}\ \|\tilde{\xi}\,\tilde{\chi}\|_{L^p}. \tag{4.21}$$

Here, we integrated one power of ρ', using the fact that $-\rho' > 0$ in the region of interest, and used the fact that $|\rho'| = \mathcal{O}(\kappa^{-1})$, to obtain

$$\left\{ \int |\rho'|^{q-1} \, |\rho'| \right\}^{1/q} \leq \kappa^{(1-q)/q} \, \left\{ - \int \rho' \right\}^{1/q} \leq C_0 \kappa^{(1-q)/q}. \tag{4.22}$$

By a simple change of variables, we find

$$\begin{aligned} \|\tilde{\xi} \tilde{\chi}\|_p &= \left\{ \int |\xi(g(\lambda);\; g(H_1+V), g(H_1))|^p \; \tilde{\chi}(\lambda) \, \mathrm{d}\lambda \right\}^{1/p} \\ &\leq \quad C_1 \left\{ \int_{\mathbb{R}} |\xi(\lambda; g(H_1+V), g(H_1))|^p \, \mathrm{d}\lambda \right\}^{1/p} \\ &\leq \quad C_1 \, \|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}. \end{aligned} \tag{4.23}$$

We recall that

$$V = \lambda_k^+ (H_0^A - E_0)^{-1/2} u_k (H_0^A - E_0)^{-1/2}. \tag{4.24}$$

In particular, the volume of the support of V has order one, and is independent of $|\Lambda|$. As in § 3, one can prove that the constant $\|g(H_1+V) - g(H_1)\|_{1/p}^{1/p}$ depends only on the single-site potential u_k and dist$(E_0, \inf \sigma(H_0^A))$, and is independent of $|\Lambda|$. Consequently, the right side of (4.23) is bounded above by $C_0 \kappa^{(1-q)/q}$, independent of $|\Lambda|$. This estimate, eqs (4.16) and (4.20), lead us to the result

$$\mathbb{P}\{\text{dist}\,(-1, \sigma(\Gamma_\Lambda(E_0))) < \kappa\} \leq C_W \kappa^{1/q} \|g\|_\infty |\Lambda|, \tag{4.25}$$

for any $q > 1$. □

4.2 *The case of a general band edge and small disorder*

Suppose now that the background operator H_0 has an open, internal spectral gap, as in hypothesis (H1). In the case of nonsign-definite single-site potentials, the behavior of the eigenvalues created by V_Λ, as a coupling constant $\lambda_j(\omega)$ varies, may be very complicated. In order to compensate for this, we must work in the weak disorder regime. The main result is the following.

Theorem 4.3. *We assume that H_0^X and V_ω satisfy* (H1), (H2), (H3c), *and* (H4a), *and let $H_\Lambda^A(\lambda) \equiv H_0^A + \lambda V_\Lambda$, and $H_\Lambda^M(\lambda) = (1+\lambda V_\Lambda)^{-1/2} H_0^M (1+\lambda V_\Lambda)^{-1/2}$. Let $E_0 \in (B_-, B_+)$ be any energy in the unperturbed spectral gap of H_0, and define $\delta_\pm(E_0) \equiv$* dist $(E_0, B_\pm)$. *We define a constant*

$$\lambda(E_0) \equiv \min \left(\frac{(B_+ - B_-)}{4\|V_\Lambda\|}, \frac{1}{4\|V_\Lambda\|} \left(\frac{\delta_+(E_0)\delta_-(E_0)}{2} \right)^{1/2} \right).$$

Then, for any $q > 1$, there exists a finite constant C_{E_0}, depending on λ_0, the dimension d, the index $q > 1$, and $[\text{dist}(\sigma(H_0), I)]^{-1}$, so that for all $|\lambda| < \lambda(E_0)$, and for all $\eta <$ min $(\delta_-(E_0), \delta_+(E_0))/32$, we have

$$\mathbb{P}\{\text{ dist}\,(\sigma(H_\Lambda^X(\lambda)), E_0) \leq \eta\} \leq C_{E_0} \eta^{1/q} |\Lambda|. \tag{4.26}$$

Consequently, for ergodic models, the IDS is Hölder continuous in a neighborhood of E_0.

We give some ideas concerning the proof. Formula (4.2) is no longer valid so we replace it with the Feshbach projection formula. Let $P_\pm$ denote the spectral projectors for H_0 corresponding to the components of the spectrum $[B_+, \infty)$ and $(-\infty, B_-]$, respectively, so that $P_+ + P_- = 1$, and $P_+P_- = 0$. The Feshbach method permits us to decompose the problem relative to these two orthogonal projectors. Let $H_0^\pm \equiv P_\pm H_0$, and denote by $H_\pm(\lambda) \equiv H_0^\pm + \lambda P_\pm V P_\pm$. We need the various projections of the potential between the subspaces $P_\pm L^2(\mathbb{R}^d)$, and we denote them by $V_\pm \equiv P_\pm V P_\pm$, and $V_{+-} \equiv P_+ V P_-$, with $V_{-+} = V_{+-}^* = P_- V P_+$. Let $z \in \mathbb{C}$, with $\operatorname{Im} z \neq 0$. We can write the resolvent $R_\Lambda(z) = (H_\Lambda(\lambda) - z)^{-1}$ in terms of the resolvents of the projected operators $H_\pm(\lambda)$. In order to write a formula valid for either P_+ or P_-, we let $P = P_\pm$, $Q = 1 - P_\pm$, and write $R_P(z) = (PH_0 + \lambda P V_\Lambda P - zP)^{-1}$. We then have

$$R_\Lambda(z) = PR_P(z)P + \{Q - \lambda PR_P(z)PV_\Lambda Q\}\mathcal{G}(z)\{Q - \lambda QV_\Lambda PR_P(z)^*P\}, \tag{4.27}$$

where the operator $\mathcal{G}(z)$ is given by

$$\mathcal{G}(z) = \{QH_0 + \lambda QV_\Lambda Q - zQ - \lambda^2 QV_\Lambda PR_P(z)PV_\Lambda Q\}^{-1}. \tag{4.28}$$

We notice that if $E_0 \in G$, then $(H_+ - E_0) > 0$, so that we can use the same ideas as in the previous subsection to treat this operator. For example, let us suppose that E_0 is close to the upper gap edge B_+. We then apply formula (4.27) with $Q = P_+$ and $P = P_-$. With this choice, we see that $R_P(E_0) = (P_-H_0 + \lambda P_- V_\Lambda P_- - zP_-)^{-1}$ is bounded provided $|\lambda|$ is small enough. This implies that the singularity of the resolvent comes from the operator $\mathcal{G}(z)$, for z near E_0. Following the general proof of Theorem 4.1, we reduce the statement of the Wegner estimate to a statement concerning the norm of the operator $\mathcal{G}(E_0)$:

$$\begin{aligned}\mathbb{P}\{\text{dist}\,(\sigma(H_\Lambda), E_0) < \eta\} &= \mathbb{P}\{\ \|R_\Lambda(E_0)\| > 1/\eta\} \\ &\leq \mathbb{P}\{\ \|\mathcal{G}(E_0)\|\ > 1/(8\eta)\}.\end{aligned} \tag{4.29}$$

Looking closely at the operator $\mathcal{G}(E_0)$ in (4.28), we see that it can be written as

$$\mathcal{G}(E_0) = R_0^+(E_0)^{1/2}(1 + \tilde{\Gamma}_+(E_0))^{-1}R_0^+(E_0)^{1/2}, \tag{4.30}$$

where we define $\tilde{\Gamma}_+(E_0)$ by

$$\begin{aligned}\tilde{\Gamma}_+(E_0) &\equiv \lambda R_0^+(E_0)^{1/2}V_+R_0^+(E_0)^{1/2} \\ &\quad + \lambda^2 R_0^+(E_0)^{1/2}V_{+-}(E_0P_- - H_-(\lambda))^{-1}V_{-+}R_0^+(E_0)^{1/2}.\end{aligned} \tag{4.31}$$

This operator $\tilde{\Gamma}_+(E_0)$ is the analog of the operator $\Gamma_\Lambda(E_0;\omega)$ appearing in (4.3). Equations (4.29) and (4.30) show that we can, as in subsection 4.1, reduce the Wegner estimate as follows:

$$\begin{aligned}\mathbb{P}\{\text{ dist}\,(\sigma(H_\Lambda, E_0) < \eta\} &= \mathbb{P}\{\|R_\Lambda(E_0)\|\ > 1/\eta\} \\ &\leq \mathbb{P}\{\ \|(1 + \tilde{\Gamma}_+(E_0))^{-1}\|\ > \delta_+(E_0)/(8\eta)\} \\ &= \mathbb{P}\{\text{ dist}\,(\sigma(\tilde{\Gamma}_+(E_0)), -1) < 8\eta/\delta_+(E_0)\}.\end{aligned} \tag{4.32}$$

We can now proceed as in subsection 4.1. The final difficulty is that the operator $\tilde{\Gamma}_+(E_0)$ is no longer an eigenvector of the operator A_Λ defined in (4.8). Instead, a calculation yields the relation

$$A_\Lambda\tilde{\Gamma}_+(E_0) = \tilde{\Gamma}_+(E_0) + \lambda^2 W(E_0). \tag{4.33}$$

The second constraint of $|\lambda|$ originates with this expression. We want $|\lambda|$ small enough so that the leading term in (4.33) dominates. With this, the proof continues as in the proof of Theorem 4.1.

We conclude by mentioning the model studied by Veselić [33]. Let $\Gamma \subset \mathbb{Z}^d$, be a finite subset containing the origin $k = 0$, and consider a finite set of real numbers $\alpha \equiv \{\alpha_k \mid k \in \Gamma\}$. We assume that $\alpha_0 = 1$, and that the remaining terms α_k, $k \neq 0$, satisfy $\sum_{k \neq 0} |\alpha_k| < 1$. In the simplest case, let χ_0 be the characteristic function on the unit cell centered at the origin in $\mathbb{Z}^d$. We define a compactly-supported, single-site potential u by

$$u(x) \equiv \sum_{k \in \Gamma} \alpha_k \chi_0(x - k). \tag{4.34}$$

This potential has no fixed sign if some of the terms α_k, $k \neq 0$, are negative. Veselić considers the Anderson-type potentials (1.5) constructed with this single-site potential

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u(x - i), \tag{4.35}$$

with the coupling constants $\lambda_i(\omega)$ being independent and identically distributed with common density h_0, as considered in this paper. Veselić observes that the potential V_ω can be written as

$$\begin{aligned} V_\omega(x) &= \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) \left[\sum_{k \in \Gamma} \alpha_k \chi_0(x - k - i) \right] \\ &= \sum_{m \in \mathbb{Z}^d} \nu_m(\omega) \chi_0(x - m), \end{aligned} \tag{4.36}$$

where the new family of random variables $\{\nu_m(\omega) \mid m \in \mathbb{Z}^d\}$ is defined by

$$\nu_m(\omega) = \sum_{k \in \Gamma} \lambda_{m-k}(\omega) \alpha_k. \tag{4.37}$$

The Anderson-type potential V_ω in (4.36) is constructed from a sign-definite, single site potential χ_0, but the coupling constants $\nu_m(\omega)$ are not necessarily independent and have a different distribution that no longer has a product form. Note that if $\|k - m\|$ is sufficiently large, depending upon Γ, then the random variables $\nu_m(\omega)$ and $\nu_k(\omega)$ are independent. That is, the correlation is of finite range. The distribution of the family $\{\nu_m(\omega) \mid m \in \mathbb{Z}^d\}$ can be easily calculated. Let A be the infinite Toeplitz matrix with entries $A_{ij} = \alpha_{i-j}$. It follows from (4.37) that $\nu = A\lambda$. Formally, the probability distribution for the family ν is given by

$$\mathbb{P}\{\nu \in B\} = \int_B |\det(A^{-1})| \, \Pi_{k \in \mathbb{Z}^d} f((A^{-1}\nu)) \, \mathrm{d}\nu_k, \tag{4.38}$$

for any measurable subset $B \subset A[\operatorname{supp} h_0]^{\mathbb{Z}^d}$. These comments can be restricted to a finite cube. It follows that the conditional probability distribution of one random variable ν_k, conditioned on the others in a cube, is absolutely continuous. Consequently, the results of [8] apply, and, because hypothesis (H3a) is satisfied, one can prove a Wegner estimate at any energy (cf. Proposition 3.3).

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