

## Homogenization of a parabolic equation in perforated domain with Neumann boundary condition

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**Abstract.** In this article, we study the homogenization of the family of parabolic equations over periodically perforated domains

$$\begin{aligned} \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - \operatorname{div} a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) &= f(x, t) \text{ in } \Omega_\varepsilon \times (0, T), \\ a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) \cdot \nu_\varepsilon &= 0 \text{ on } \partial S_\varepsilon \times (0, T), \\ u_\varepsilon &= 0 \text{ on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, 0) &= u_0(x) \text{ in } \Omega_\varepsilon. \end{aligned}$$

Here,  $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$  is a periodically perforated domain. We obtain the homogenized equation and corrector results. The homogenization of the equations on a fixed domain was studied by the authors [15]. The homogenization for a fixed domain and  $b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \equiv b(u_\varepsilon)$  has been done by Jian [11].

**Keywords.** Homogenization; perforated domain; two-scale convergence; correctors.

### 1. Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Let  $T > 0$  be a constant and let  $\varepsilon > 0$  be a small parameter which eventually tends to zero. Let  $Y = (0, 1)^n$  and  $S(\text{closed}) \subset Y$ . We define a periodically perforated domain  $\Omega_\varepsilon$  as follows: First define

$$I_\varepsilon = \{k \in \mathbb{Z}^n : \varepsilon(k + S) \subset \Omega\} \text{ and,} \quad (1.1)$$

$$S_\varepsilon = \bigcup_{k \in I_\varepsilon} \varepsilon(k + S). \quad (1.2)$$

Set

$$\Omega_\varepsilon = \Omega \setminus S_\varepsilon. \quad (1.3)$$

We assume that, (i)  $\partial\Omega_\varepsilon$  is smooth and, (ii)  $\Omega_\varepsilon$  is connected. Note that in the definition of the holes  $S_\varepsilon$  we have taken them to have no intersection with the external boundary  $\partial\Omega$ . This is only done to avoid technicalities and the results of our paper will remain valid without this assumption (cf. [2] for a refinement of our arguments).

We consider the following family of equations with Neumann conditions on the boundary of the holes

$$\begin{aligned} \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - \operatorname{div} a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) &= f(x, t) \text{ in } \Omega_\varepsilon \times (0, T), \\ a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) \cdot \nu_\varepsilon &= 0 \text{ on } \partial S_\varepsilon \times (0, T), \\ u_\varepsilon &= 0 \text{ on } \partial\Omega \times (0, T), \\ u_\varepsilon(x, 0) &= u_0(x) \text{ in } \Omega_\varepsilon, \end{aligned} \tag{1.4}$$

where  $u_0$  is a given function on  $\Omega$  and  $f$  is a given function on  $\Omega \times (0, T)$ . For given  $\varepsilon$ , the Cauchy problem (1.4) will also be denoted by  $(P_\varepsilon)$ . It is known that under suitable assumptions on  $a$  and  $b$  (cf. assumptions (A1)–(A4) below), that the problem  $(P_\varepsilon)$  has a solution  $u_\varepsilon$ . Our aim in this paper is to study the homogenization of the equations  $(P_\varepsilon)$  as  $\varepsilon \rightarrow 0$  i.e. to study the limiting behaviour of  $u_\varepsilon$  as  $\varepsilon \rightarrow 0$  and obtain the limiting equation satisfied by the limit. The theory of homogenization is a well-developed area of research [5,12] and we are not going into the details of this theory. The case when  $b$  is linear (i.e.  $b(y, s) = s$ ) has been studied quite widely [4, 6–8, 10,16]. When  $b$  is not linear, the homogenization of the equation in a fixed domain was studied by Jian [11] for  $b(y, s) \equiv b(s)$  and in the general case, by the authors [15].

Assuming that  $\tilde{u}_\varepsilon$  are *a priori* bounded in  $L^\infty(\Omega \times (0, T))$  (where  $\tilde{\cdot}$  denotes the extension of  $u_\varepsilon$  by zero in the holes), we show that  $\tilde{u}_\varepsilon$  converges, for a subsequence, to a solution of the homogenized problem

$$\begin{aligned} \partial_t b^*(u) - \operatorname{div} A(u, \nabla u) &= m^* f(x, t) \text{ in } \Omega \times (0, T), \\ u &= 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= 0 \text{ in } \Omega, \end{aligned} \tag{1.5}$$

where  $b^*$  and  $A$  have been identified (cf. eqs (2.13)–(2.15));  $m^*$  denotes the volume fraction of  $Y \setminus S$  in  $Y$ , i.e.  $m^* = |Y \setminus S|$ , the Lebesgue measure of  $Y \setminus S$ . We also improve the weak convergence of  $\tilde{u}_\varepsilon$  by obtaining correctors. Such results are very useful in obtaining numerical approximations and they form an important aspect of homogenization.

The layout of the paper is as follows: In §2, we give the weak formulation for the problem  $(P_\varepsilon)$ . Then we state our main results viz. Theorems 2.4 and 2.6. In §3, we sketch some preliminary results and recall facts about the two-scale convergence method. Finally, in §4, we prove our main results in the framework of two-scale convergence.

## 2. Assumptions and main results

For  $p > 1$ ,  $p^*$  will denote the conjugate exponent  $p/(p - 1)$ . Define the space  $V_\varepsilon \doteq \{v \in W^{1,p}(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$ ;  $V_\varepsilon^*$  is its dual and  $E_\varepsilon \doteq L^p(0, T; V_\varepsilon)$ . We define  $u_\varepsilon \in E_\varepsilon$  to be a weak solution of  $(P_\varepsilon)$  if it satisfies:

$$b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^\infty(0, T; L^1(\Omega_\varepsilon)), \quad \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \in L^{p^*}(0, T; V_\varepsilon^*), \tag{2.1}$$

that is

$$\int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \xi(x, t) \right\rangle_\varepsilon dt + \int_0^T \int_{\Omega_\varepsilon} \left\langle b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u_0\right) \right\rangle \partial_t \xi dx dt = 0 \tag{2.2}$$

for all  $\xi \in E_\varepsilon \cap W^{1,1}(0, T; L^\infty(\Omega_\varepsilon))$  with  $\xi(T) = 0$ ; and,

$$\begin{aligned} \int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \xi(x, t) \right\rangle_\varepsilon dt + \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, u_\varepsilon, \nabla u_\varepsilon\right) \cdot \nabla \xi(x, t) dx dt \\ = \int_0^T \int_{\Omega_\varepsilon} f(x, t) \xi(x, t) dx dt \end{aligned} \quad (2.3)$$

for all  $\xi \in E_\varepsilon$ . Here  $\langle \cdot, \cdot \rangle_\varepsilon$  denotes the duality bracket with respect to  $E_\varepsilon^*$ ,  $E_\varepsilon$ .

*Note:* For any  $\xi \in L^p(0, T; W_0^{1,p}(\Omega))$ , its restriction to  $\Omega_\varepsilon$  can be used as a test function in (2.3) for the problem  $(P_\varepsilon)$ . ■

We make the following assumptions on  $a$  and  $b$ :

- (A1) The function  $b(y, s)$  is continuous in  $y$  and  $s$ ,  $Y$ -periodic in  $y$  and non-decreasing in  $s$  and  $b(y, 0) = 0$ .
- (A2) There exists a constant  $\theta > 0$  such that for every  $\delta$  and  $R$  with  $0 < \delta < R$ , there exists  $C(\delta, R) > 0$  such that

$$|b(y, s_1) - b(y, s_2)| > C(\delta, R)|s_1 - s_2|^\theta \quad (2.4)$$

for all  $y \in Y$  and  $s_1, s_2 \in [-R, R]$  with  $\delta < |s_1|$ .

*Remark 2.1.* The prototype for  $b$  is a function of the form  $c(y)|s|^k \text{sgn}(s)$  for some positive real number  $k$  and continuous and  $Y$ -periodic function,  $c(\cdot)$ , which is positive on  $Y$ . ■

- (A3) The mapping  $(y, \mu, \lambda) \mapsto a(y, \mu, \lambda)$  defined from  $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$  to  $\mathbb{R}^n$  is measurable and  $Y$ -periodic in  $y$  and continuous in  $(\mu, \lambda)$ . Further, it is assumed that there exists positive constants  $\alpha, r$  such that

$$a(y, \mu, \lambda) \cdot \lambda \geq \alpha |\lambda|^p, \quad (2.5)$$

$$(a(y, \mu, \lambda_1) - a(y, \mu, \lambda_2))(\lambda_1 - \lambda_2) > 0, \quad \forall \lambda_1 \neq \lambda_2 \quad (2.6)$$

$$|a(y, \mu, \lambda)| \leq \alpha^{-1}(1 + |\mu|^{p-1} + |\lambda|^{p-1}), \quad (2.7)$$

$$\begin{aligned} |a(y, \mu_1, \lambda) - a(y, \mu_2, \lambda)| \leq \alpha^{-1} |\mu_1 - \mu_2|^r (1 + |\mu_1|^{p-1-r} \\ + |\mu_2|^{p-1-r} + |\lambda|^{p-1-r}). \end{aligned} \quad (2.8)$$

- (A4) We assume that the data,  $f \in L^\infty(\Omega \times T)$ . Under the above assumptions, it is known that  $(P_\varepsilon)$  admits a solution  $u_\varepsilon$  (cf. [3]).

*Remark 2.2.* From hereon, we make the following simplifying assumption to reduce the technicalities in the proof of the homogenization, viz. that  $a(y, \mu, \lambda)$  is of the form  $a(y, \lambda)$ . The same results can be carried over for general  $a$  (cf. see [11,15] for suitable modifications).

- (A5) The strong monotonicity condition;

$$(a(y, \lambda_1) - a(y, \lambda_2)) \cdot (\lambda_1 - \lambda_2) \geq \alpha |\lambda_1 - \lambda_2|^p. \quad (2.9)$$

*Remark 2.3.* We use the condition (A5) only for obtaining the corrector result. It will not be required for identifying the homogenized problem.

We now state our main theorems.

**Theorem 2.4.** *Let  $u_\varepsilon$  be a family of solutions of  $(P_\varepsilon)$ . Assume that there is a constant  $C > 0$ , such that*

$$\sup_\varepsilon \|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon \times (0, T))} \leq C. \tag{2.10}$$

*Then there exists a subsequence of  $\varepsilon$ , still denoted by  $\varepsilon$ , such that for all  $q$  with  $0 < q < \infty$ , we have*

$$\tilde{u}_\varepsilon \rightarrow u \text{ strongly in } L^q(\Omega_T) \tag{2.11}$$

*and  $u$  solves,*

$$\begin{aligned} \partial_t b^*(u) - \operatorname{div} A(\nabla u) &= m^* f \text{ in } \Omega_T, \\ u &= 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0 \text{ in } \Omega, \end{aligned} \tag{2.12}$$

*where  $b^*$  and  $A$  are defined below by (2.13)–(2.15),  $m^* = |Y \setminus S|$  and  $\Omega_T$  denotes the set  $\Omega \times (0, T)$ . ■*

The functions  $b^*$  and  $A$  are defined through,

$$b^*(s) = \int_{Y \setminus S} b(y, s) \, dy, \tag{2.13}$$

$$A(\lambda) = \int_{Y \setminus S} a(y, \lambda + \nabla \Phi_\lambda(y)) \, dy, \tag{2.14}$$

where  $\lambda \in \mathbb{R}^n$  and  $\Phi_\lambda \in W_{\text{per}}^{1,p}(Y)$  solves the periodic boundary value problem

$$\int_{Y \setminus S} a(y, \lambda + \nabla \Phi_\lambda(y)) \cdot \nabla \phi(y) \, dy = 0 \tag{2.15}$$

for all  $\phi \in W_{\text{per}}^{1,p}(Y)$ . See [9] for further details about eq. (2.15) and for the properties of  $A$ .

*Remark 2.5.* The assumption (2.10) is true in special cases (see [13]) and it is reasonable on physical grounds (see [11]). ■

We see that by extending  $u_\varepsilon$  by zero in the holes we do not get functions in  $L^p(0, T; W_0^{1,p}(\Omega))$ . However,  $\tilde{u}_\varepsilon$  can be shown to be a bounded sequence in  $L^p(\Omega_T)$  and has a weak limit. We will improve the weak convergence through a corrector result. For this, we consider the solution  $U_1(x, t, y) \in L^p(\Omega \times (0, T); W_{\text{per}}^{1,p}(Y))$  of

$$\int_0^T \int_\Omega \int_{Y \setminus S} a(y, \nabla_x u(x, t) + \nabla_y U_1(x, t, y)) \cdot \nabla_y \Phi(x, t, y) \, dy \, dx \, dt = 0$$

for all  $\Phi \in L^p(\Omega \times (0, T); W_{\text{per}}^{1,p}(Y))$  (we will see that such a  $U_1$  exists). Let  $\chi$  be the characteristic function of  $Y \setminus S$  defined on  $Y$  and extended  $Y$ -periodically to  $\mathbb{R}^n$ . Then,  $\chi(x/\varepsilon)$  is the characteristic function of  $\Omega_\varepsilon$ . The corrector result is as follows:

**Theorem 2.6.** *Let the functions  $u_\varepsilon, u, U_1$  be as above. Assume further that (A5) holds and  $u, U_1$  are sufficiently smooth, i.e. that they belong to  $C^1(\Omega \times (0, T))$  and  $C(\Omega \times (0, T); C^1_{\text{per}}(Y))$  respectively. Then*

$$\begin{aligned} \tilde{u}_\varepsilon - u - \varepsilon U_1\left(x, t, \frac{x}{\varepsilon}\right) &\rightarrow 0 \text{ strongly in } L^p(\Omega_T) \text{ and,} \\ \widetilde{\nabla} u_\varepsilon - \chi\left(\frac{x}{\varepsilon}\right)\left(\nabla_x u + \nabla_y U_1\left(x, t, \frac{x}{\varepsilon}\right)\right) &\rightarrow 0 \text{ strongly in } L^p(\Omega_T). \quad \blacksquare \end{aligned}$$

*Remark 2.7.* Note that, in particular, the  $L^p$  weak limit of  $\widetilde{\nabla} u_\varepsilon$  is  $m^* \nabla u(x, t) + \int_{\partial S} \frac{\partial U_1}{\partial \nu}(x, t, y) d\sigma(y)$ . ■

### 3. Some preliminary results

As we closely follow the treatment in [15], many results will only be sketched and we refer the reader to [15] for more details as and when necessary.

We first obtain *a priori* bounds under the assumption (2.10). From now on,  $C$  will denote a generic positive constant which is independent of  $\varepsilon$ .

*Lemma 3.1.* *Let  $u_\varepsilon$  be a family of solutions of  $(P_\varepsilon)$  and assume that (2.10) holds. Then,*

$$\sup_\varepsilon \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon \times (0, T))} \leq C, \tag{3.1}$$

$$\sup_\varepsilon \left\| a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \right\|_{L^{p^*}(\Omega_\varepsilon \times (0, T))} \leq C, \tag{3.2}$$

$$\sup_\varepsilon \left\| \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \right\|_{E^*_\varepsilon} \leq C. \tag{3.3}$$

*Proof.* Define the function  $B(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$B(y, s) = b(y, s)s - \int_0^s b(y, \tau) d\tau. \tag{3.4}$$

As in [15] we deduce that

$$\begin{aligned} &\int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}, u_\varepsilon(x, T)\right) dx + \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \cdot \nabla u_\varepsilon dx dt \\ &= \int_{\Omega_\varepsilon} B\left(\frac{x}{\varepsilon}, u_0\right) dx + \int_0^T \int_{\Omega_\varepsilon} f u_\varepsilon dx dt \end{aligned}$$

and from this we obtain

$$\int_{\Omega} B\left(\frac{x}{\varepsilon}, u_\varepsilon(x, T)\right) dx + \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \cdot \nabla u_\varepsilon dx dt \leq C \tag{3.5}$$

by (2.10) and the assumptions on  $b$  and  $a$ . Then, (3.1) follows from (3.5) and (2.5), as  $B$  is non-negative, while (3.2) follows from (3.1) and (2.7). The estimate (3.3) may be obtained from (3.1), (3.2) and (2.3). Thus the lemma. ■

We now show that  $\tilde{u}_\varepsilon \rightarrow u$  a.e.  $\Omega \times (0, T)$ , for a subsequence and for some  $u$ . This is the crucial part of the present analysis. This result is to be seen in the context of the estimate

(3.1) where there is no time gradient estimate. Hence, we cannot use any ready-made compact imbedding theorem to get the strong convergence of  $u_\varepsilon$  to  $u$  in some  $L^r(\Omega_T)$ . However, we are able to achieve this by adapting a technique found in [3]. This is given in the following lemma.

*Lemma 3.2.* *Let  $u_\varepsilon$  be as above. Then, the sequence  $\{\tilde{u}_\varepsilon\}_{\varepsilon>0}$  is relatively compact in  $L^\theta(\Omega_T)$ , where  $\theta$  is as in (A2). As a result, there is a subsequence of  $u_\varepsilon$  such that,*

$$\tilde{u}_\varepsilon \rightarrow u \text{ a. e. in } \Omega_T. \quad (3.6)$$

*Proof.* Analogous to Step 1, Lemma 3.3 in [15], we can derive the following estimate

$$h^{-1} \int_0^{T-h} \int_{\Omega_\varepsilon} \left( b\left(\frac{x}{\varepsilon}, u_\varepsilon(t+h)\right) - b\left(\frac{x}{\varepsilon}, u_\varepsilon(t)\right) \right) (u_\varepsilon(t+h) - u_\varepsilon(t)) \, dx \, dt \leq C$$

for some constant  $C$  which is independent of  $\varepsilon$  and  $h$ . Thus, as we have assumed in (A1) that  $b(y, 0) = 0$ , we get

$$h^{-1} \int_0^{T-h} \int_{\Omega} \left( b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon(t+h)\right) - b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon(t)\right) \right) (\tilde{u}_\varepsilon(t+h) - \tilde{u}_\varepsilon(t)) \, dx \, dt \leq C.$$

Proceeding as in [15] we obtain the desired result. ■

From Lemma 3.2 above, the continuity of  $b$  and the assumption (2.10) we derive the following corollary.

#### COROLLARY 3.3

We have,  $b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right) \rightarrow 0$  strongly in  $L^q(\Omega_T) \forall q, 0 < q < \infty$ .

*Proof.* By the *a priori* bound (2.10), it is enough to consider the function  $b$  on  $Y \times [-M, M]$  for a large  $M > 0$ . As  $b$  is continuous, it is uniformly continuous on  $Y \times [-M, M]$ . Therefore, given  $h_0 > 0$ , there exists a  $\delta > 0$  such that,

$$|b(y, s) - b(y', s')| < h_0,$$

whenever  $|y - y'| + |s - s'| < \delta$ .

Now, since  $\tilde{u}_\varepsilon \rightarrow u$  a.e. in  $\Omega_T$ , by Egoroff's theorem, given  $h_1 > 0$ , there exists  $E \subset \Omega_T$  such that its Lebesgue measure  $m(E) < h_1$  and  $\tilde{u}_\varepsilon$  converges uniformly to  $u$  on  $(\Omega_T \setminus E)$ , which we denote by  $E'$ . Therefore, we can find  $\varepsilon_1 > 0$  such that

$$\|\tilde{u}_\varepsilon - u\|_{\infty, E'} < \delta \quad \forall \varepsilon < \varepsilon_1. \quad (3.7)$$

Therefore, for  $\varepsilon < \varepsilon_1$  we have

$$\begin{aligned} & \int_{\Omega_T} \left| b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right) \right|^q \, dx \, dt \\ &= \int_{E'} \left| b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right) \right|^q \, dx \, dt \\ & \quad + \int_E \left| b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right) \right|^q \, dx \, dt \\ &\leq h_0^q m(\Omega_T) + 2^q \sup(|b|^q) m(E) \\ &\leq h_0^q m(\Omega_T) + 2^q \sup(|b|^q) h_1. \end{aligned}$$

This completes the proof as  $h_0$  and  $h_1$  can be chosen arbitrarily small. ■

As a consequence, we have the following corollary.

**COROLLARY 3.4**

We have the following convergences:

$$\begin{aligned} b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) &\rightharpoonup \bar{b}(u) \text{ weakly in } L^q(\Omega_T), \\ \chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) &\rightharpoonup b^*(u) \text{ weakly in } L^q(\Omega_T) \end{aligned}$$

for  $q > 1$ . Further,  $\bar{b}(u) = b^*(u)$ .

*Proof.* We note that

$$\begin{aligned} b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) &= \left(b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right)\right) + b\left(\frac{x}{\varepsilon}, u\right) \\ &\rightarrow 0 + \bar{b}(u) \end{aligned}$$

by Corollary 3.3 and the averaging principle for periodic functions mentioned below.

Similarly,

$$\begin{aligned} \chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) &= \chi\left(\frac{x}{\varepsilon}\right) \left(b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) - b\left(\frac{x}{\varepsilon}, u\right)\right) + \chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, u\right) \\ &\rightarrow 0 + b^*(u). \end{aligned}$$

From  $\chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right) = b\left(\frac{x}{\varepsilon}, \tilde{u}_\varepsilon\right)$ , we readily obtain the last of the conclusions in the corollary. ■

The homogenization will be done in the framework of two-scale convergence. So to conclude this section, we recall an averaging principle which is at the basis of this method, the definition of two-scale convergence and a few related results which we will use in the sequel. For an exposition of this method and its main features, see [14,1,15].

For  $f(x, y) \in L^q_{\text{loc}}(\mathbb{R}^n; C_{\text{per}}(Y))$ , the oscillatory function  $f\left(x, \frac{x}{\varepsilon}\right)$  converges weakly in  $L^q_{\text{loc}}(\mathbb{R}^n)$  to  $\int_Y f(x, y) dy$ , for all  $q > 1$ . A similar result holds for  $f(x, y) \in L^\infty_{\text{loc}}(\mathbb{R}^n; C_{\text{per}}(Y))$ , with weak convergence replaced by weak-\* convergence.

**DEFINITION 3.5**

Let  $1 < q < \infty$ . A sequence of functions  $v_\varepsilon \in L^q(\Omega_T)$  is said to two-scale converge to a function  $v \in L^q(\Omega_T \times Y)$  if

$$\int_{\Omega_T} v_\varepsilon \psi\left(x, t, \frac{x}{\varepsilon}\right) dx dt \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega_T} \int_Y v(x, t, y) \psi(x, t, y) dy dx dt$$

for all  $\psi \in L^{q^*}(\Omega_T; C_{\text{per}}(Y))$ . We write  $v_\varepsilon \xrightarrow{2-s} v$ .

**Theorem 3.6.** Let  $1 < q < \infty$  and let  $v_\varepsilon$  be a bounded sequence in  $L^q(\Omega_T)$ . Then there exists a function  $v \in L^q(\Omega_T \times Y)$  such that, up to a subsequence,  $v_\varepsilon \xrightarrow{2-s} v(x, t, y)$ .

The following theorem [1] is useful in obtaining the limit of the product of two two-scale convergent sequences. Let  $1 < q < \infty$ .

**Theorem 3.7.** *Let  $q > 1$ . Let  $v_\varepsilon$  be a sequence in  $L^q(\Omega_T)$  and  $w_\varepsilon$  be a sequence in  $L^{q^*}(\Omega_T)$  such that  $v_\varepsilon \xrightarrow{2-s} v$  and  $w_\varepsilon \xrightarrow{2-s} w$ . Further, assume that the sequence  $w_\varepsilon$  satisfies*

$$\int_{\Omega_T} |w_\varepsilon|^{q^*}(x, t) \, dx \, dt \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega_T} \int_Y |w(x, t, y)|^{q^*} \, dy \, dx \, dt. \quad (3.8)$$

Then,

$$\int_{\Omega_T} v_\varepsilon w_\varepsilon \, dx \, dt \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega_T} \int_Y v(x, t, y) w(x, t, y) \, dy \, dx \, dt. \quad (3.9)$$

**Remark 3.8.** A sequence  $w_\varepsilon$  which two-scale converges and satisfies (3.8) is said to be strongly two-scale convergent. Examples of strongly two-scale convergent sequences are  $\{\psi(x, t, \frac{x}{\varepsilon})\}_\varepsilon$  for any  $\psi \in L^q_{\text{per}}(Y; C(\Omega_T))$ . ■

We now state a result for perforated domains the proof of which may be found in [1].

**Theorem 3.9.** *Let  $q > 1$ . Let  $v_\varepsilon \in L^q(\Omega_\varepsilon \times (0, T))$  and let  $\nabla v_\varepsilon \in L^q(\Omega_\varepsilon \times (0, T))$  be such that*

$$\begin{aligned} \sup_\varepsilon \|v_\varepsilon\|_{L^q(\Omega_\varepsilon \times (0, T))} &\leq C, \\ \sup_\varepsilon \|\nabla v_\varepsilon\|_{L^q(\Omega_\varepsilon \times (0, T))} &\leq C. \end{aligned} \quad (3.10)$$

Then there exist  $v \in L^q(0, T; W^{1,q}(\Omega))$  and  $V_1 \in L^q(\Omega_T; W^{1,q}_{\text{per}}(Y))$  such that, up to a subsequence

$$\begin{aligned} \widetilde{v}_\varepsilon &\xrightarrow{2-s} v(x, t), \\ \widetilde{\nabla v}_\varepsilon &\xrightarrow{2-s} \chi(y)(\nabla_x v(x, t) + \nabla_y V_1(x, t, y)). \end{aligned} \quad (3.11)$$

■

#### 4. Proofs of the main Theorems

We first obtain the two-scale homogenized equation. From the estimates (3.1) and (3.2) and Theorems 3.8 and 3.5, there exists  $U_1 \in L^p(\Omega_T; W^{1,p}_{\text{per}}(Y))$  and  $a_0 \in L^{p^*}(\Omega_T \times Y)$  such that, up to a subsequence,

$$\widetilde{\nabla u}_\varepsilon \xrightarrow{2-s} \chi(y)(\nabla_x u(x, t) + \nabla_y U_1(x, t, y)), \quad (4.1)$$

$$\chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \widetilde{\nabla u}_\varepsilon\right) \xrightarrow{2-s} a_0(x, t, y). \quad (4.2)$$

**Theorem 4.1.** *The limits  $u, a_0$  satisfy the following two-scale homogenized problem*

$$\begin{aligned} &\int_0^T \langle \partial_t b^*(u), \phi \rangle \, dt + \int_{\Omega_T} \int_{Y \setminus S} a_0(x, t, y) \cdot (\nabla_x \phi + \nabla_y \Phi(x, t, y)) \, dy \, dx \, dt \\ &= \int_{\Omega_T} f \phi \, dx \, dt \end{aligned} \quad (4.3)$$

for all  $\phi \in C_0^\infty(\Omega_T)$  and  $\Phi \in C_0^\infty(\Omega_T; C^\infty_{\text{per}}(Y))$ .

*Proof.* We pass to the limit in (2.3) in the presence of appropriate test functions. Let  $\phi \in C_0^\infty(\Omega_T)$  and  $\Phi \in C_0^\infty(\Omega_T; C_{\text{per}}^\infty(Y))$ . Set,

$$\phi_\varepsilon = \phi(x, t) + \varepsilon \Phi\left(x, t, \frac{x}{\varepsilon}\right).$$

We may use these as test functions in (2.3) as the restrictions of  $\phi_\varepsilon$  to  $\Omega_\varepsilon$  are in  $E_\varepsilon$  and we note

$$\begin{aligned} & \int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \phi_\varepsilon \right\rangle_\varepsilon dt + \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \widetilde{\nabla} u_\varepsilon\right) \\ & \times \left(\nabla_x \phi + \nabla_y \Phi\left(x, t, \frac{x}{\varepsilon}\right)\right) dx dt + o(1) = \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) f(x, t) \phi(x, t) dx dt + o(1). \end{aligned} \tag{4.4}$$

Now we pass to the limit in the above identity. We may handle the first term as follows. Observing that  $\phi_\varepsilon(\cdot, T) = 0 = \phi_\varepsilon(\cdot, 0)$ , we have from (2.2),

$$\begin{aligned} \int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), \phi_\varepsilon \right\rangle_\varepsilon dt &= - \int_0^T \int_{\Omega_\varepsilon} b\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \partial_t \phi_\varepsilon dx dt \\ &= - \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) b\left(\frac{x}{\varepsilon}, \widetilde{u}_\varepsilon\right) \partial_t \phi dx dt + o(1) \\ &\rightarrow - \int_0^T \int_\Omega b^*(u) \partial_t \phi dx dt \\ &= \int_0^T \langle \partial_t b^*(u), \phi \rangle dx dt, \end{aligned}$$

where the convergence in the above follows from Corollary 3.4.

For passing to the limit in the other two terms in (4.4) we observe that

$$\begin{aligned} & \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \cdot \left(\nabla_x \phi + \nabla_y \Phi\left(x, t, \frac{x}{\varepsilon}\right)\right) dx dt \\ &= \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \widetilde{\nabla} u_\varepsilon\right) \cdot \left(\nabla_x \phi + \nabla_y \Phi\left(x, t, \frac{x}{\varepsilon}\right)\right) dx dt \\ &\rightarrow \int_0^T \int_\Omega \int_Y a_0(x, t, y) \cdot (\nabla_x \phi + \nabla_y \Phi(x, t, y)) dy dx dt, \end{aligned}$$

where the convergence in the last step follows from Theorem 3.6. For this, one observes from Remark 3.7 that  $\nabla_x \phi + \nabla_y \Phi\left(x, t, \frac{x}{\varepsilon}\right)$  is strongly two-scale convergent to  $\nabla_x \phi + \nabla_y \Phi(x, t, y)$ . Further, observe that since  $\chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \widetilde{\nabla} u_\varepsilon\right)$  is zero in the holes, we have  $a_0(x, t, y)$  to be zero in  $S$  and the integral over  $Y$  can be replaced by that over  $Y \setminus S$ .

Finally, note that

$$\begin{aligned} \int_0^T \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) f(x, t) \phi(x, t) dx dt &\rightarrow \int_0^T \int_\Omega \int_Y \chi(y) f(x, t) \phi(x, t) dy dx dt \\ &= \int_0^T \int_\Omega m^* f \phi dx dt. \end{aligned}$$

Thus, from the above we deduce that the limit of the equations (2.3) with  $\phi_\varepsilon$  as test functions is nothing but (4.3). This is referred to as the two-scale homogenized problem. ■

*Remark 4.2.* We note from the two-scale homogenized problem (4.3), by setting  $\Phi \equiv 0$  that

$$\partial_t b^*(u) - \operatorname{div} \left( \int_{Y \setminus S} a_0(x, t, y) dy \right) = m^* f \text{ in } \Omega \times (0, T) \tag{4.5}$$

and by setting  $\phi \equiv 0$  we get

$$\int_0^T \int_{\Omega} \int_{Y \setminus S} a_0(x, t, y) \cdot \nabla_y \Phi(x, t, y) dy dx dt = 0 \tag{4.6}$$

for all  $\Phi \in C_0^\infty(\Omega_T; C_{\text{per}}^\infty(Y))$ . In view of (4.5) and (4.6), it is enough to show that  $a_0(x, t, y) = a(y, \nabla_x u(x, t) + \nabla_y U_1(x, t, y))$  to conclude Theorem 2.4. ■

*Proof of Theorem 2.4.* Note that (2.11) follows from the convergence (3.6) and the assumption (2.10). We now justify (2.12), for which it is enough to show that  $a_0(x, t, y)$  equals  $a(y, \nabla_x u(x, t) + \nabla_y U_1(x, t, y))$ , by the preceding remark.

Let  $\xi \in L^p(0, T; W_0^{1,p}(\Omega)) \cup W^{1,1}(0, T; L^\infty(\Omega))$  with  $\xi(\cdot, T) = 0 = \xi(\cdot, 0)$ . This may be used as a test function in (2.2) and we have

$$\int_0^T \left\langle \partial_t b \left( \frac{x}{\varepsilon}, u_\varepsilon \right), \xi(x, t) \right\rangle_\varepsilon dt + \int_0^T \int_{\Omega_\varepsilon} b \left( \frac{x}{\varepsilon}, u_\varepsilon \right) \partial_t \xi dx dt = 0.$$

This can be rewritten as

$$\int_0^T \left\langle R_\varepsilon^* \partial_t b \left( \frac{x}{\varepsilon}, u_\varepsilon \right), \xi \right\rangle dt + \int_0^T \int_{\Omega} \chi \left( \frac{x}{\varepsilon} \right) b \left( \frac{x}{\varepsilon}, \tilde{u}_\varepsilon \right) \partial_t \xi dx dt = 0. \tag{4.7}$$

Observe that  $\sup_\varepsilon \|R_\varepsilon^* \partial_t b \left( \frac{x}{\varepsilon}, u_\varepsilon \right)\|_{L^{p^*}(0, T; W^{-1, p^*}(\Omega))} \leq C$  by (3.3) where  $R_\varepsilon$  is the restriction operator from  $L^p(0, T; W_0^{1,p}(\Omega))$  to  $E_\varepsilon$ . Thus for some  $w \in L^{p^*}(0, T; W^{-1, p^*}(\Omega))$  and for a subsequence

$$R_\varepsilon^* \partial_t b \left( \frac{x}{\varepsilon}, u_\varepsilon \right) \rightharpoonup w \text{ weakly }^* \text{ in } L^{p^*}(0, T; W^{-1, p^*}(\Omega)).$$

Thus, passing to the limit in (4.7) we get

$$\int_0^T \langle w, \xi \rangle + \int_0^T \int_{\Omega} b^*(u) \partial_t \xi dx dt = 0$$

for all  $\xi$  as above. So,  $\partial_t b^*(u) = w$  in the sense of distributions.

We now show that  $a_0(x, t, y) = a(y, \nabla_x u(x, t) + \nabla_y U_1(x, t, y))$ . Let  $\phi, \Phi$  be as in the proof of Theorem 4.1. Let  $\lambda > 0$  and  $\psi \in C_0^\infty(\Omega_T; C_{\text{per}}^\infty(Y))^n$ . Set

$$\eta_\varepsilon(x, t) \doteq \nabla_x \phi(x, t) + \nabla_y \Phi \left( x, t, \frac{x}{\varepsilon} \right) + \lambda \psi \left( x, t, \frac{x}{\varepsilon} \right).$$

By (2.6) we note that

$$\int_0^T \int_{\Omega_\varepsilon} \left( a \left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) - a \left( \frac{x}{\varepsilon}, \eta_\varepsilon \right) \right) \cdot (\nabla u_\varepsilon - \eta_\varepsilon) dx dt \geq 0. \tag{4.8}$$

We have

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \cdot \nabla u_\varepsilon &= - \int_0^T \left\langle \partial_t b\left(\frac{x}{\varepsilon}, u_\varepsilon\right), u_\varepsilon \right\rangle dt + \int_0^T \int_{\Omega_\varepsilon} f u_\varepsilon dx dt \\ &\rightarrow - \int_0^T \langle \partial_t b^*(u), u \rangle + \int_0^T \int_{\Omega} m^* f u dx dt \\ &= \int_0^T \int_{\Omega} \int_{Y \setminus S} a_0(x, t, y) \cdot \nabla_x u dy dx dt, \end{aligned}$$

where the last step follows from (4.5).

Also, since  $\eta_\varepsilon$  and  $a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right)$  are strongly two-scale convergent, Theorem 3.6 gives

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right) \cdot \nabla u_\varepsilon dx dt &= \int_0^T \int_{\Omega} a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right) \cdot \widetilde{\nabla} u_\varepsilon dx dt \\ &\rightarrow \int_0^T \int_{\Omega} \int_{Y \setminus S} a(y, \nabla_x \phi + \nabla_y \Phi + \lambda \psi) \cdot (\nabla_x u + \nabla_y U_1) dy dx dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right) \cdot \eta_\varepsilon dx dt &= \int_0^T \int_{\Omega} \chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right) \cdot \eta_\varepsilon dx dt \\ &\rightarrow \int_0^T \int_{\Omega} \int_{Y \setminus S} a(y, \nabla_x \phi + \nabla_y \Phi + \lambda \psi) \cdot (\nabla_x \phi + \nabla_y \Phi + \lambda \psi) dy dx dt \end{aligned}$$

and,

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) \cdot \eta_\varepsilon dx dt &= \int_0^T \int_{\Omega} \chi\left(\frac{x}{\varepsilon}\right) a\left(\frac{x}{\varepsilon}, \widetilde{\nabla} u_\varepsilon\right) \cdot \eta_\varepsilon dx dt \\ &\rightarrow \int_0^T \int_{\Omega} \int_{Y \setminus S} a_0(x, t, y) \cdot (\nabla_x \phi + \nabla_y \Phi + \lambda \psi) dy dx dt \\ &= \int_0^T \int_{\Omega} \int_{Y \setminus S} a_0(x, t, y) \cdot (\nabla_x \phi + \lambda \psi) dy dx dt. \end{aligned}$$

The equality in the last step follows from (4.6). Thus

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon} \left( a\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon\right) - a\left(\frac{x}{\varepsilon}, \eta_\varepsilon\right) \right) \cdot (\nabla u_\varepsilon - \eta_\varepsilon) = \int_0^T \int_{\Omega} \int_{Y \setminus S} a_0(x, t, y) \\ &\quad \times (\nabla_x(u - \phi) - \lambda \psi) - \int_0^T \int_{\Omega} \int_{Y \setminus S} a(y, \nabla_x \phi + \nabla_y \Phi + \lambda \psi) \\ &\quad \times (\nabla_x(u - \phi) + \nabla_y(U_1 - \Phi) - \lambda \psi). \end{aligned}$$

Letting  $\phi \rightarrow u$  and  $\Phi \rightarrow U_1$  in the above and using the continuity of  $a$  we get

$$\int_0^T \int_{\Omega} \int_{Y \setminus S} (a(y, \nabla_x u + \nabla_y U_1 + \lambda \psi) - a_0(x, t, y)) \cdot \lambda \psi dy dx dt \geq 0$$

for all  $\lambda$  and  $\psi$ . Dividing the above inequality by  $\lambda$  and letting  $\lambda \rightarrow 0$  we get,

$$\int_0^T \int_{\Omega} \int_{Y \setminus S} (a(y, \nabla_x u + \nabla_y U_1) - a_0(x, t, y)) \cdot \psi \, dy \, dx \, dt \geq 0$$

for all  $\psi$ . From this we conclude that  $a_0(x, t, y) = a(y, \nabla_x u + \nabla_y U_1)$ . ■

*Proof of Theorem 2.6.* Set

$$\eta_\varepsilon(x, t) \doteq \nabla_x u(x, t) + \nabla_y U_1 \left( x, t, \frac{x}{\varepsilon} \right).$$

From (2.9) it follows that

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} \left( a \left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) - a \left( \frac{x}{\varepsilon}, \eta_\varepsilon \right) \right) \cdot (\nabla u_\varepsilon - \eta_\varepsilon) \, dx \, dt \\ \geq \alpha \int_0^T \int_{\Omega_\varepsilon} |\nabla u_\varepsilon - \eta_\varepsilon|^p \, dx \, dt, \end{aligned}$$

that is,

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} \left( a \left( \frac{x}{\varepsilon}, \widetilde{\nabla u_\varepsilon} \right) - a \left( \frac{x}{\varepsilon}, \eta_\varepsilon \right) \right) \cdot \left( \widetilde{\nabla u_\varepsilon} - \chi \left( \frac{x}{\varepsilon} \right) \eta_\varepsilon \right) \, dx \, dt \\ \geq \alpha \int_0^T \int_{\Omega} \left| \widetilde{\nabla u_\varepsilon} - \chi \left( \frac{x}{\varepsilon} \right) \left[ \nabla_x u(x, t) + \nabla_y U_1 \left( x, t, \frac{x}{\varepsilon} \right) \right] \right|^p \, dx \, dt. \end{aligned}$$

As we have assumed that  $u, U_1$  are smooth, we may proceed as in the proof of Theorem 2.4 to obtain

$$\begin{aligned} \alpha \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega} \left| \widetilde{\nabla u_\varepsilon} - \chi \left( \frac{x}{\varepsilon} \right) \left( \nabla_x u(x, t) + \nabla_y U_1 \left( x, t, \frac{x}{\varepsilon} \right) \right) \right|^p \, dx \, dt \\ \leq \lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\Omega_\varepsilon} \left( a \left( \frac{x}{\varepsilon}, \widetilde{\nabla u_\varepsilon} \right) - a \left( \frac{x}{\varepsilon}, \eta_\varepsilon \right) \right) \cdot \left( \widetilde{\nabla u_\varepsilon} - \chi \left( \frac{x}{\varepsilon} \right) \eta_\varepsilon \right) \, dx \, dt \\ = 0. \end{aligned}$$

This completes the proof. ■

*Remark 4.3.* A weaker form of the corrector result with no smoothness assumptions can also be obtained. See [15] for the corresponding formulation. ■

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