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Multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$

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Abstract. The orthonormal basis generated by a wavelet of $L^2(\mathbb{R})$ has poor frequency localization. To overcome this disadvantage Coifman, Meyer, and Wickerhauser constructed wavelet packets. We extend this concept to the higher dimensions where we consider arbitrary dilation matrices. The resulting basis of $L^2(\mathbb{R}^d)$ is called the multiwavelet packet basis. The concept of wavelet frame packet is also generalized to this setting. Further, we show how to construct various orthonormal bases of $L^2(\mathbb{R}^d)$ from the multiwavelet packets.

Keywords. Wavelet; wavelet packets; frame packets; dilation matrix.

1. Introduction

Consider an orthonormal wavelet of $L^2(\mathbb{R})$. At the jth resolution level, the orthonormal basis $\{\psi_{jk} : j, k \in \mathbb{Z}\}$ generated by the wavelet has a frequency localization proportional to 2^j. For example, if the wavelet ψ is band-limited (i.e., $\hat{\psi}$ is compactly supported), then the measure of the support of $(\psi_{jk})^\wedge$ is 2^j times the measure of the support of $\hat{\psi}$, since

$$(\psi_{jk})^\wedge(\xi) = 2^{-j/2}\hat{\psi}(2^{-j}\xi)\mathrm{e}^{-i2^{-j}k\xi}, \quad j, k \in \mathbb{Z},$$

where

$$\psi_{jk} = 2^{j/2}\psi(2^j \cdot -k), \quad j, k \in \mathbb{Z}.$$

So when j is large, the wavelet bases have poor frequency localization. Better frequency localization can be achieved by a suitable construction starting from an MRA wavelet basis.

Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R})$ with corresponding scaling function φ and wavelet ψ. Let W_j be the corresponding wavelet subspaces: $W_j = \overline{sp}\{\psi_{jk} : k \in \mathbb{Z}\}$. In the construction of a wavelet from an MRA, essentially the space V_1 was split into two orthogonal components V_0 and W_0. Note that V_1 is the closure of the linear span of the functions $\{2^{\frac{1}{2}}\varphi(2 \cdot -k) : k \in \mathbb{Z}\}$, whereas V_0 and W_0 are respectively the closure of the span of $\{\varphi(\cdot - k) : k\}$ and $\{\psi(\cdot - k) : k\}$. Since $\varphi(2 \cdot -k) = \varphi\left(2(\cdot - \frac{k}{2})\right)$, we see that the above procedure splits the half-integer translates of a function into integer translates of two functions.

In fact, the splitting is not confined to V_1 alone: we can choose to split W_j, which is the span of $\{\psi(2^j \cdot -k) : k\} = \{\psi\left(2^j(\cdot - \frac{k}{2^j})\right) : k\}$, to get two functions whose $2^{-(j-1)}k$ translates will span the same space W_j. Repeating the splitting procedure j times, we get

2^j functions whose integer translates alone span the space W_j. If we apply this to each W_j, then the resulting basis of $L^2(\mathbb{R})$, which will consist of integer translates of a countable number of functions (instead of all dilations and translations of the wavelet ψ), will give us a better frequency localization. This basis is called ‘wavelet packet basis’. The concept of wavelet packet was introduced by Coifman, Meyer and Wickerhauser [6, 7]. For a nice exposition of wavelet packets of $L^2(\mathbb{R})$ with dilation 2, see [11].

The concept of wavelet packet was subsequently generalized to $\mathbb{R}^d$ by taking tensor products [5]. The non-tensor product version is due to Shen [16]. Other notable generalizations are the biorthogonal wavelet packets [4], non-orthogonal version of wavelet packets [3], the wavelet frame packets [2] on $\mathbb{R}$ for dilation 2, and the orthogonal, biorthogonal and frame packets on $\mathbb{R}^d$ by Long and Chen [13] for the dyadic dilation.

In this article we generalize these concepts to $\mathbb{R}^d$ for arbitrary dilation matrices and we will not restrict ourselves to one scaling function: we consider the case of those MRAs for which the central space is generated by several scaling functions.

DEFINITION 1.1

A $d \times d$ matrix A is said to be a dilation matrix for $\mathbb{R}^d$ if

(i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$ and

(ii) all eigenvalues λ of A satisfy $|\lambda| > 1$.

Property (i) implies that A has integer entries and hence $|\det A|$ is an integer, and (ii) says that $|\det A|$ is greater than 1. Let $B = A^t$, the transpose of A and $a = |\det A| = |\det B|$.

Considering $\mathbb{Z}^d$ as an additive group, we see that $A\mathbb{Z}^d$ is a normal subgroup of $\mathbb{Z}^d$. So we can form the cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$. It is a well-known fact that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is equal to $a = |\det A|$ ([10, 17]). A subset of $\mathbb{Z}^d$ which consists of exactly one element from each of the a cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ will be called a *set of digits* for the dilation matrix A. Therefore, if K_A is a set of digits for A, then we can write

$$\mathbb{Z}^d = \bigcup_{\mu \in K_A} (A\mathbb{Z}^d + \mu),$$

where $\{A\mathbb{Z}^d + \mu : \mu \in K_A\}$ are pairwise disjoint. A set of digits for A need not be a set of digits for its transpose. For example, for the dilation matrix $M = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ of $\mathbb{R}^2$, the set $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a set of digits for M but not for M^t. It is easy to see that if K is a set of digits for A, then so is $K - \mu$, where $\mu \in K$. Therefore, we can assume, without loss of generality, that $0 \in K$.

The notion of a multiresolution analysis can be extended to $L^2(\mathbb{R}^d)$ by replacing the dyadic dilation by a dilation matrix and allowing the resolution spaces to be spanned by more than one scaling function.

DEFINITION 1.2

A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ will be called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

(M1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$

(M2) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

(M3) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$

(M4) there exist L functions $\{\varphi_1, \varphi_2, \dots, \varphi_L\}$ in V_0, called the *scaling functions*, such that the system of functions $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ forms an orthonormal basis for V_0.

The concept of multiplicity was introduced by Hervé [12] in his Ph.D. thesis.

Since $\{\varphi_l(\cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_0, it follows from property (M3) that $\{a^{j/2}\varphi_l(A^j \cdot - k) : 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V_j. Observe that if $f \in L^2(\mathbb{R}^d)$, then

$$\big(a^{j/2} f(A^j \cdot - k)\big)^\wedge(\xi) = a^{-j/2} \mathrm{e}^{-i\langle B^{-j}\xi, k\rangle} \hat{f}(B^{-j}\xi), \quad \xi \in \mathbb{R}^d,\ k \in \mathbb{Z}^d.$$

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-i\langle \xi, x\rangle} \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

To define the Fourier transform for functions of $L^2(\mathbb{R}^d)$, the operator $\mathcal{F}$ is extended from $L^1 \cap L^2(\mathbb{R}^d)$, which is dense in $L^2(\mathbb{R}^d)$ in the L^2-norm, to the whole of $L^2(\mathbb{R}^d)$. For this definition of the Fourier transform, Plancherel theorem takes the form

$$\langle f, g\rangle = \frac{1}{(2\pi)^d}\langle \hat{f}, \hat{g}\rangle; \quad f, g \in L^2(\mathbb{R}^d).$$

First of all we will prove a lemma, the splitting lemma (see [8]), which is essential for the construction of wavelet packets. We need the following facts for the proof of the splitting lemma.

(a) Let $\mathbb{T}^d = [-\pi, \pi]^d$ and $f \in L^1(\mathbb{R}^d)$. Since $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d}(\mathbb{T}^d + 2k\pi)$, we can write

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \int_{\mathbb{T}^d} \Big\{\sum_{k\in\mathbb{Z}^d} f(x + 2k\pi)\Big\}\mathrm{d}x. \tag{1}$$

(b) Let $\{s_k : k \in \mathbb{Z}^d\} \in l^1(\mathbb{Z}^d)$ and K_B be a set of digits for the dilation matrix B. As $\mathbb{Z}^d$ can be decomposed as $\mathbb{Z}^d = \cup_{\mu\in K_B}(B\mathbb{Z}^d + \mu)$, we can write

$$\sum_{k\in\mathbb{Z}^d} s_k = \sum_{\mu\in K_B}\sum_{k\in\mathbb{Z}^d} s_{\mu+Bk}. \tag{2}$$

(c) Let K_B be a set of digits for B. Define

$$Q_0 = \bigcup_{\mu\in K_B} B^{-1}(\mathbb{T}^d + 2\mu\pi).$$

Since K_B is a set of digits for B, the set Q_0 satisfies $\cup_{k\in\mathbb{Z}^d}(Q_0 + 2k\pi) = \mathbb{R}^d$. This fact, together with $|Q_0| = (2\pi)^d$, implies that $\{Q_0 + 2k\pi : k \in \mathbb{Z}^d\}$ is a pairwise disjoint collection (see Lemma 1 of [10]). Therefore,

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}x = \int_{Q_0} \Big\{\sum_{k\in\mathbb{Z}^d} f(x + 2k\pi)\Big\}\mathrm{d}x, \quad \text{for } f \in L^1(\mathbb{R}^d). \tag{3}$$

A function f is said to be $2\pi\mathbb{Z}^d$-periodic if $f(x + 2k\pi) = f(x)$ for all $k \in \mathbb{Z}^d$ and for a.e. $x \in \mathbb{R}^d$.

2. The splitting lemma

Let $\{\varphi_l : 1 \leq l \leq L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ is an orthonormal system. Let $V = \overline{sp}\{a^{1/2}\varphi_l(A \cdot -k) : l, k\}$. For $1 \leq l, j \leq L$ and $0 \leq r \leq a-1$, suppose that there exist sequences $\{h^r_{ljk} : k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$. Define

$$f^r_l(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h^r_{ljk} a^{1/2} \varphi_j(Ax - k). \tag{4}$$

Taking Fourier transform of both sides

$$\begin{aligned} \hat{f}^r_l(\xi) &= \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} h^r_{ljk} a^{-1/2} \mathrm{e}^{-i\langle B^{-1}\xi, k\rangle} \hat{\varphi}_j(B^{-1}\xi) \\ &= \sum_{j=1}^{L} h^r_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi), \end{aligned} \tag{5}$$

where

$$h^r_{lj}(\xi) = \sum_{k \in \mathbb{Z}^d} a^{-1/2} h^r_{ljk} \mathrm{e}^{-i\langle \xi, k\rangle}, \quad 1 \leq l, j \leq L, \ 0 \leq r \leq a-1, \tag{6}$$

and h^r_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$. Now, for $0 \leq r \leq a-1$, define the $L \times L$ matrices

$$H_r(\xi) = \left(h^r_{lj}(\xi)\right)_{1 \leq l, j \leq L}. \tag{7}$$

By denoting

$$\Phi(x) = (\varphi_1(x), \ldots, \varphi_L(x))^t \tag{8}$$

$$\hat{\Phi}(\xi) = \left(\hat{\varphi}_1(\xi), \ldots, \hat{\varphi}_L(\xi)\right)^t, \tag{9}$$

we can write (5) as

$$\hat{F}_r(\xi) = H_r(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi), \ \ 0 \leq r \leq a-1, \tag{10}$$

where $F_r(x) = (f^r_1(x), f^r_2(x), \ldots, f^r_L(x))^t$ and $\hat{F}_r(\xi) = (\hat{f}^r_1(\xi), \hat{f}^r_2(\xi), \ldots, \hat{f}^r_L(\xi))^t$.

The following well-known lemma characterizes the orthonormality of the system $\{\varphi_l(\cdot - k) : \ 1 \leq l \leq L, \ k \in \mathbb{Z}^d\}$. We give a proof for the sake of completeness.

Lemma 2.1. *The system* $\{\varphi_l(\cdot - k)\colon \ 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ *is orthonormal if and only if*

$$\sum_{k \in \mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)} = \delta_{jl}, \quad 1 \leq j, l \leq L.$$

Proof. Suppose that the system $\{\varphi_l(\cdot - k) : \ 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ is orthonormal. Note that $\langle \varphi_j(\cdot - p), \varphi_l(\cdot - q)\rangle = \langle \varphi_j, \varphi_l(\cdot - (q-p))\rangle$ for $1 \leq j, l \leq L$ and $p, q \in \mathbb{Z}^d$. Now

$$\begin{aligned} \delta_{jl}\delta_{0p} &= \langle \varphi_j, \varphi_l(\cdot - p)\rangle = \frac{1}{(2\pi)^d}\langle \hat{\varphi}_j, (\varphi_l(\cdot - p))^\wedge\rangle \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{\varphi}_j(\xi)\overline{\hat{\varphi}_l(\xi)}\mathrm{e}^{i\langle p, \xi\rangle}\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left\{\sum_{k \in \mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)}\right\}\mathrm{e}^{i\langle p, \xi\rangle}\mathrm{d}\xi, \quad \text{by (1)}. \end{aligned}$$

Therefore, the $2\pi\mathbb{Z}^d$-periodic function $G_{jl}(\xi) = \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi + 2k\pi)\overline{\hat{\varphi}_l(\xi + 2k\pi)}$ has Fourier coefficients $\hat{G}_{jl}(-p) = \delta_{jl}\delta_{0p}$, $p \in \mathbb{Z}^d$ which implies that $G_{jl} = \delta_{jl}$ a.e. By reversing the above steps we can prove the converse. $\square$

Let $M^*(\xi)$ be the conjugate transpose of the matrix $M(\xi)$ and I_L denote the identity matrix of order L.

Lemma 2.2. (*The splitting lemma*) *Let $\{\varphi_l : 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that the system $\{a^{1/2}\varphi_j(A\cdot -k) : 1 \le j \le L,\ \ k \in \mathbb{Z}^d\}$ is orthonormal. Let V be its closed linear span. Let K be a set of digits for B. Also let f_l^r, H_r be as above. Then*

$$\{f_l^r(\cdot - k) :\ 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$$

is an orthonormal system if and only if

$$\sum_{\mu\in K} H_r(\xi + 2B^{-1}\mu\pi)H_s^*(\xi + 2B^{-1}\mu\pi) = \delta_{rs}I_L, \quad 0 \le r, s \le a-1. \tag{11}$$

Moreover, $\{f_l^r(\cdot - k) :\ 0 \le r \le a-1,\ 1 \le l \le L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of V whenever it is orthonormal.

Proof. For $1 \le l, j \le L$, $0 \le r, s \le a-1$ and $p \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\left\langle f_j^r, f_l^s(\cdot - p)\right\rangle \\
&\quad = \frac{1}{(2\pi)^d}\left\langle (f_j^r)^\wedge, (f_l^s(\cdot - p))^\wedge\right\rangle \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (f_j^r)^\wedge(\xi)\overline{(f_l^s)^\wedge(\xi)\mathrm{e}^{-i\langle p,\xi\rangle}}\mathrm{d}\xi \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi)\overline{h_{ln}^s(B^{-1}\xi)}\hat{\varphi}_m(B^{-1}\xi)\overline{\hat{\varphi}_n(B^{-1}\xi)}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (5))} \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d}\sum_{m=1}^{L}\sum_{n=1}^{L} \Big\{h_{jm}^r(B^{-1}(\xi + 2k\pi))\overline{h_{ln}^s(B^{-1}(\xi + 2k\pi))} \\
&\qquad\qquad \cdot\hat{\varphi}_m(B^{-1}(\xi + 2k\pi))\overline{\hat{\varphi}_n(B^{-1}(\xi + 2k\pi))}\Big\}\mathrm{e}^{i\langle p,\, \xi+2k\pi\rangle}\mathrm{d}\xi \quad \text{(by (1))} \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} \\
&\qquad\qquad \cdot\left\{\sum_{k\in\mathbb{Z}^d} \hat{\varphi}_m(B^{-1}(\xi + 2\mu\pi) + 2k\pi)\overline{\hat{\varphi}_n(B^{-1}(\xi + 2\mu\pi) + 2k\pi)}\right\}\mathrm{e}^{i\langle p,\, \xi\rangle}\mathrm{d}\xi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2))} \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \sum_{\mu\in K}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{ln}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\delta_{mn}\mathrm{e}^{i\langle p,\, \xi\rangle}\mathrm{d}\xi \quad \text{(by Lemma 2.1)} \\
&\quad = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \left\{\sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi + 2B^{-1}\mu\pi)}\right\}\mathrm{e}^{i\langle p,\, \xi\rangle}\mathrm{d}\xi.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\left\langle f_j^r, f_l^s(\cdot - p)\right\rangle &= \delta_{rs}\delta_{jl}\delta_{0p}\\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(B^{-1}\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(B^{-1}\xi + 2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \text{ for a.e. } \xi\in\mathbb{R}^d\\
\Leftrightarrow \sum_{\mu\in K}\sum_{m=1}^{L} h_{jm}^r(\xi + 2B^{-1}\mu\pi)\overline{h_{lm}^s(\xi + 2B^{-1}\mu\pi)} &= \delta_{rs}\delta_{jl} \text{ for a.e. } \xi\in\mathbb{R}^d\\
\Leftrightarrow \sum_{\mu\in K} H_r(\xi + 2B^{-1}\mu\pi)H_s^*(\xi + 2B^{-1}\mu\pi) &= \delta_{rs}I_L \text{ for a.e. } \xi\in\mathbb{R}^d.
\end{aligned}
$$

We have proved the first part of the lemma.

Now assume that $\{f_l^r(\cdot - k) :\ 0 \le r \le a-1,\ 1\le l\le L,\ k\in\mathbb{Z}^d\}$ is an orthonormal system. We want to show that this is an orthonormal basis of V. Let $f\in V$. So there exists $\{c_{jp} : p\in\mathbb{Z}^d\}\in l^2(Z^d)$, $1\le j\le L$ such that

$$f(x) = \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(Ax - p).$$

Assume that $f\perp f_l^r(\cdot - k)$ for all r, l, k.

Claim. $f = 0$.

For all r, l, k such that $0\le r\le a-1, 1\le l\le L, k\in\mathbb{Z}^d$, we have

$$
\begin{aligned}
0 &= \left\langle f_l^r(\cdot - k), f\right\rangle = \left\langle f_l^r(\cdot - k), \sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\cdot - p)\right\rangle\\
&= \frac{1}{(2\pi)^d}\left\langle \left(f_l^r(\cdot - k)\right)^\wedge, \left(\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} c_{jp}a^{1/2}\varphi_j(A\cdot - p)\right)^\wedge\right\rangle\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} (f_l^r)^\wedge(\xi)\mathrm{e}^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}a^{-1/2}\mathrm{e}^{i\langle B^{-1}\xi, p\rangle}\overline{\hat\varphi_j(B^{-1}\xi)}\mathrm{d}\xi\\
&= \frac{a^{-1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(B^{-1}\xi)\hat\varphi_m(B^{-1}\xi)\mathrm{e}^{-i\langle k,\xi\rangle}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\mathrm{e}^{i\langle B^{-1}\xi, p\rangle}\overline{\hat\varphi_j(B^{-1}\xi)}\mathrm{d}\xi\\
&\qquad\text{(by (5))}\\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{\mathbb{R}^d}\sum_{m=1}^{L} h_{lm}^r(\xi)\hat\varphi_m(\xi)\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat\varphi_j(\xi)}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \quad (\xi\to B\xi)\\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{q\in\mathbb{Z}^d}\sum_{m=1}^{L} h_{lm}^r(\xi + 2q\pi)\hat\varphi_m(\xi + 2q\pi)\\
&\qquad\cdot\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{jp}}\overline{\hat\varphi_j(\xi + 2q\pi)}\mathrm{e}^{-i\langle k,B(\xi+2q\pi)\rangle}\mathrm{e}^{i\langle p,\xi+2q\pi\rangle}\mathrm{d}\xi \quad \text{(by (3))}\\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{j=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{jp}}\left\{\sum_{q\in\mathbb{Z}^d}\hat\varphi_m(\xi + 2q\pi)\overline{\hat\varphi_j(\xi + 2q\pi)}\right\}\\
&\qquad\cdot\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a^{1/2}}{(2\pi)^d}\int_{Q_0}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{mp}}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi \quad \text{(by Lemma 2.1)}\\
&= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}(\mathbb{T}^d+2\mu\pi)}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi)\overline{c_{mp}}\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{e}^{i\langle p,\xi\rangle}\mathrm{d}\xi\\
&= \frac{a^{1/2}}{(2\pi)^d}\sum_{\mu\in K}\int_{B^{-1}\mathbb{T}^d}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}\mathrm{e}^{-i\left\langle k,B(\xi+2B^{-1}\mu\pi)\right\rangle}\\
&\qquad\qquad \cdot\mathrm{e}^{i\left\langle p,\xi+2B^{-1}\mu\pi\right\rangle}\mathrm{d}\xi\\
&= \frac{a^{1/2}}{(2\pi)^d}\int_{B^{-1}\mathbb{T}^d}\left\{\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d} h_{lm}^r(\xi+2B^{-1}\mu\pi)\overline{c_{mp}}\mathrm{e}^{i\left\langle p,\xi+2B^{-1}\mu\pi\right\rangle}\right\}\\
&\qquad\qquad \cdot\mathrm{e}^{-i\langle k,B\xi\rangle}\mathrm{d}\xi.
\end{aligned}
$$

Since $\left\{\frac{a^{1/2}}{(2\pi)^d}\mathrm{e}^{-i\langle k,B\cdot\rangle} : k\in\mathbb{Z}^d\right\}$ is an orthonormal basis for $L^2(B^{-1}\mathbb{T}^d)$, the above equations give

$$
\sum_{\mu\in K}\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d}\overline{c_{mp}}\mathrm{e}^{i\left\langle \xi+2B^{-1}\mu\pi,p\right\rangle}h_{lm}^r(\xi+2B^{-1}\mu\pi)=0 \text{ a.e.} \quad \text{for all } r,\ l.
$$

For $m=1,2,\ldots,L$, define

$$
C_m(\xi)=\sum_{p\in\mathbb{Z}^d}c_{mp}\mathrm{e}^{-i\langle\xi,p\rangle}. \tag{12}
$$

So we have

$$
\sum_{\mu\in K}\sum_{m=1}^{L}\overline{C_m(\xi+2B^{-1}\mu\pi)}h_{lm}^r(\xi+2B^{-1}\mu\pi)=0,\quad 0\le r\le a-1,\quad 1\le l\le L. \tag{13}
$$

Equations (11) are equivalent to saying that for $0\le r\le a-1,\ 1\le l\le L$ and for a.e. $\xi\in\mathbb{R}^d$, the vectors

$$
\left(h_{lm}^r(\xi+2B^{-1}\mu\pi) : 1\le m\le L,\ \mu\in K\right)
$$

are mutually orthogonal and each has norm 1, considered as a vector in the aL-dimensional space $\mathbb{C}^{aL}$, so that they form an orthonormal basis for $\mathbb{C}^{aL}$. Equation (13) says that the vector

$$
\left(C_m(\xi+2B^{-1}\mu\pi) : 1\le m\le L,\ \mu\in K\right) \tag{14}
$$

is orthogonal to each member of the above orthonormal basis of $\mathbb{C}^{aL}$. Hence, the vector in the expression (14) is zero. In particular, $C_m(\xi)=0$, for all $m,\ 1\le m\le L$. That is, $c_{mp}=0,\ 1\le m\le L,\ p\in\mathbb{Z}^d$. Therefore, $f=0$. This ends the proof. □

The splitting lemma can be used to decompose an arbitrary Hilbert space into mutually orthogonal subspaces, as in [7]. We will use the following corollary later.

COROLLARY 2.3

Let $\left\{E_{lk}:\ 1\leq l\leq L,\ k\in\mathbb{Z}^d\right\}$ *be an orthonormal basis of a separable Hilbert space* $\mathcal{H}$. *Let* $H_r,\ 0\leq r\leq a-1$ *be as above and satisfy* (11). *Define*

$$F^r_{lk}=\sum_{m=1}^{L}\sum_{p\in\mathbb{Z}^d}h^r_{l,m,p-Ak}E_{mp};\qquad 0\leq r\leq a-1,\ 1\leq l\leq L,\ k\in\mathbb{Z}^d.$$

Then $\left\{F^r_{lk}:\ 1\leq l\leq L,\ k\in\mathbb{Z}^d\right\}$ *is an orthonormal basis for its closed linear span* $\mathcal{H}^r$ *and* $\mathcal{H}=\oplus_{r=0}^{a-1}\mathcal{H}^r$.

Proof. Let $\varphi_1,\varphi_2,\ldots,\varphi_L$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot-k):1\leq l\leq L,k\in\mathbb{Z}^d\}$ is an orthonormal system. Let $V=\overline{sp}\{a^{1/2}\varphi_l(A\cdot-k):l,\ k\}$. Define a linear operator T from the Hilbert space V to $\mathcal{H}$ by $T(a^{1/2}\varphi_l(A\cdot-k))=E_{l,k}$. Let f^r_l are as in (4). Then, $T(f^r_l(\cdot-k))=F^r_{l,k}$. Now the corollary follows from the splitting lemma. □

3. Construction of multiwavelet packets

Let $\{V_j:j\in\mathbb{Z}\}$ be an MRA of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A. Let $\{\varphi_l:1\leq l\leq L\}$ be the scaling functions. Since $\varphi_l, 1\leq l\leq L$ are in $V_0\subset V_1$ and $\{a^{1/2}\varphi_j(A\cdot-k):1\leq j\leq L,k\in\mathbb{Z}^d\}$ forms an orthonormal basis of V_1, there exist $\{h_{ljk}:k\in\mathbb{Z}^d\}\in l^2(\mathbb{Z}^d)$ for $1\leq l,j\leq L$ such that

$$\varphi_l(x)=\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d}h_{ljk}a^{1/2}\varphi_j(Ax-k).$$

Taking Fourier transform, we get

$$\begin{aligned}\hat{\varphi}_l(\xi)&=\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d}h_{ljk}a^{-1/2}\mathrm{e}^{-i\langle B^{-1}\xi,k\rangle}\hat{\varphi}_j(B^{-1}\xi)\\&=\sum_{j=1}^{L}h_{lj}(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi),\end{aligned}\tag{15}$$

where $h_{lj}(\xi)=\sum_{k\in\mathbb{Z}^d}a^{-1/2}h_{ljk}\mathrm{e}^{-i\langle\xi,k\rangle}$, and h_{lj} is $2\pi\mathbb{Z}^d$-periodic and is in $L^2(\mathbb{T}^d)$. Let $H_0(\xi)$ be the $L\times L$ matrix defined by

$$H_0(\xi)=\Big((h_{lj}(\xi)\Big)_{1\leq l,j\leq L}.$$

We will call H_0 the *low-pass filter matrix*. Rewriting (15) in the vector notations (8) and (9), we have

$$\hat{\Phi}(\xi)=H_0(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi).\tag{16}$$

Let W_j be the wavelet subspaces, the orthogonal complement of V_j in V_{j+1}:

$$W_j=V_{j+1}\ominus V_j.$$

Properties (M1) and (M3) of Definition 1.2 now imply that

$$W_j\perp W_{j'},\quad j\neq j'$$

and

$$f \in W_j \Leftrightarrow f(A^{-j}\cdot) \in W_0. \tag{17}$$

Moreover, by (M2), $L^2(\mathbb{R}^d)$ can be decomposed into orthogonal direct sums as

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j \tag{18}$$

$$= V_0 \oplus \left(\bigoplus_{j\geq 0} W_j\right). \tag{19}$$

By Lemma 2.1 and eq. (15), we have (for $1 \leq l, j \leq L$)

$$\begin{aligned}\delta_{jl} &= \sum_{k\in\mathbb{Z}^d} \hat{\varphi}_j(\xi+2k\pi)\overline{\hat{\varphi}_l(\xi+2k\pi)} \\ &= \sum_{k\in\mathbb{Z}^d}\left\{\sum_{m=1}^{L} h_{jm}(B^{-1}(\xi+2k\pi))\hat{\varphi}_m(B^{-1}(\xi+2k\pi))\right\} \\ &\quad \cdot\left\{\sum_{n=1}^{L} \overline{h_{ln}(B^{-1}(\xi+2k\pi))\hat{\varphi}_n(B^{-1}(\xi+2k\pi))}\right\}.\end{aligned}$$

Now, using (2), we have

$$\begin{aligned}\delta_{jl} &= \sum_{\mu\in K_B}\sum_{m=1}^{L}\sum_{n=1}^{L} h_{jm}(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{ln}(B^{-1}\xi+2B^{-1}\mu\pi)} \\ &\quad \cdot\sum_{k\in\mathbb{Z}^d}\left\{\hat{\varphi}_m(B^{-1}(\xi+2\mu\pi)+2k\pi)\overline{\hat{\varphi}_n(B^{-1}(\xi+2\mu\pi)+2k\pi)}\right\},\end{aligned}$$

where K_B is a set of digits for B. Using Lemma 2.1 again, we get

$$\delta_{jl} = \sum_{\mu\in K_B}\sum_{m=1}^{L} h_{jm}(B^{-1}\xi+2B^{-1}\mu\pi)\overline{h_{lm}(B^{-1}\xi+2B^{-1}\mu\pi)}. \tag{20}$$

This is equivalent to saying that

$$\sum_{\mu\in K_B} H_0(\xi+2B^{-1}\mu\pi)H_0^*(\xi+2B^{-1}\mu\pi) = I_L \quad \text{for a.e. } \xi.$$

Equation (20) is also equivalent to the orthonormality of the vectors

$$\Big(h_{lj}(\xi+2B^{-1}\mu\pi):\ 1\leq j\leq L,\ \mu\in K_B\Big), \qquad 1\leq l\leq L,\ \xi\in\mathbb{T}^d.$$

These L orthonormal vectors in the aL-dimensional space $\mathbb{C}^{aL}$ can be completed, by Gram–Schmidt orthonormalization process, to produce an orthonormal basis for $\mathbb{C}^{aL}$. Let us denote the new vectors by

$$\Big(h_{lj}^r(\xi+2B^{-1}\mu\pi):\ 1\leq j\leq L,\ \mu\in K_B\Big), \qquad 1\leq l\leq L,\ 1\leq r\leq a-1,\ \xi\in\mathbb{T}^d,$$

and extend the functions h_{lj}^r $(1\leq r\leq a-1,\ 1\leq l, j\leq L)$ $2\pi\mathbb{Z}^d$-periodically (see [9] for the one-dimensional dyadic dilation). Denoting by $H_r(\xi)$, $1\leq r\leq a-1$ the $L\times L$ matrix

$$\Big(h_{lj}^r(\xi)\Big)_{1\leq l, j\leq L},$$

we have

$$\sum_{\mu\in K_B} H_r(\xi+2B^{-1}\mu\pi)H_s^*(\xi+2B^{-1}\mu\pi)=\delta_{rs}I_L \quad \text{for a.e. } \xi.$$

Now, for $1\le r\le a-1,\ 1\le l\le L$, define

$$\hat{f}_l^r(\xi)=\sum_{j=1}^{L} h_{lj}^r(B^{-1}\xi)\hat{\varphi}_j(B^{-1}\xi). \tag{21}$$

Since h_{lj}^r are $2\pi\mathbb{Z}^d$-periodic, there exist $\{h_{ljk}^r:\ k\in\mathbb{Z}^d\}\in l^2(\mathbb{Z}^d)$ such that

$$h_{lj}^r(\xi)=\sum_{k\in\mathbb{Z}^d} a^{-1/2}h_{ljk}^r\mathrm{e}^{-i\langle\xi,k\rangle}.$$

Now, applying the splitting lemma to V_1, we see that $\{f_l^r(\cdot-k):0\le r\le a-1,\ 1\le l\le L, k\in\mathbb{Z}^d\}$ is an orthonormal basis for V_1. We use the convention $\varphi_l=f_l^0$, $1\le l\le L$ with $h_{lj}=h_{lj}^0$ and $h_{ljk}=h_{ljk}^0$. The decomposition $V_1=V_0\oplus W_0$, and the fact that $\{f_l^0(\cdot-k):1\le l\le L,\ k\in\mathbb{Z}^d\}$ is an orthonormal basis of V_0, imply that

$$\{f_l^r(\cdot-k):1\le r\le a-1,\ 1\le l\le L,\ k\in\mathbb{Z}^d\}$$

is an orthonormal basis for W_0. By (17) and (18), we see that

$$\{a^{j/2}f_l^r(A^j\cdot -k):1\le r\le a-1,\ 1\le l\le L,\ j\in\mathbb{Z},\ k\in\mathbb{Z}^d\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$. This basis is called the *multiwavelet basis* and the functions $\{f_l^r:1\le r\le a-1,\ 1\le l\le L\}$ are the *multiwavelets* associated with the MRA $\{V_j:j\in\mathbb{Z}\}$ of multiplicity L. For $0\le r\le a-1$, by denoting $F_r(x)=\big(f_1^r(x),f_2^r(x),\ldots,f_L^r(x)\big)^t$ and $\hat{F}_r(\xi)=\big(\hat{f}_1^r(\xi),\hat{f}_2^r(\xi),\ldots,\hat{f}_L^r(\xi)\big)^t$, we can write (16) and (21) as

$$\hat{F}_r(\xi)=H_r(B^{-1}\xi)\hat{\Phi}(B^{-1}\xi),\quad 0\le r\le a-1. \tag{22}$$

This equation is known as the *scaling relation* satisfied by the scaling functions ($r=0$) and the multiwavelets ($1\le r\le a-1$).

As we observed, applying splitting lemma to the space $V_1=\overline{sp}\{a^{1/2}\varphi_l(A\cdot -k):1\le l\le L,\ k\in\mathbb{Z}^d\}$, we get the functions $f_l^r,\ \ 0\le r\le a-1,\ 1\le l\le L$. Now, for any $n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, we define $f_l^n,\ 1\le l\le L$ recursively as follows. Suppose that f_l^r, $r\in\mathbb{N}_0,\ 1\le l\le L$ are defined already. Then define

$$f_l^{s+ar}(x)=\sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{ljk}^s a^{1/2}f_j^r(Ax-k);\ \ 0\le s\le a-1,\ 1\le l\le L. \tag{23}$$

Taking Fourier transform

$$(f_l^{s+ar})^\wedge(\xi)=\sum_{j=1}^{L} h_{lj}^s(B^{-1}\xi)(f_j^r)^\wedge(B^{-1}\xi). \tag{24}$$

In vector notation, (24) can be written as

$$(F_{s+ar})^\wedge(\xi)=H_s(B^{-1}\xi)\hat{F}_r(B^{-1}\xi). \tag{25}$$

Note that (23) defines f_l^n for every non-negative integer n and every l such that $1 \leq l \leq L$. Observe that $f_l^0 = \varphi_l, 1 \leq l \leq L$ are the scaling functions and $f_l^r, 1 \leq r \leq a-1, 1 \leq l \leq L$ are the multiwavelets. So this definition is consistent with the scaling relation (22) satisfied by the scaling functions and the multiwavelets.

DEFINITION 3.1

The functions $\left\{f_l^n : n \geq 0,\ 1 \leq l \leq L\right\}$ as defined above will be called the *basic multiwavelet packets* corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation A.

The Fourier transforms of the multiwavelet packets

Our aim is to find an expression for the Fourier transform of the basic multiwavelet packets in terms of the Fourier transform of the scaling functions. For an integer $n \geq 1$, we consider the unique *'a-adic expansion'* (i.e., expansion in the base a):

$$n = \mu_1 + \mu_2 a + \mu_3 a^2 + \cdots + \mu_j a^{j-1}, \tag{26}$$

where $0 \leq \mu_i \leq a-1$ for all $i = 1, 2, \ldots, j$ and $\mu_j \neq 0$.

If n can be expressed as in (26) then we will say n has a-adic length j. We claim that if n has length j and has expansion (26), then

$$\hat{F}_n(\xi) = H_{\mu_1}(B^{-1}\xi) H_{\mu_2}(B^{-2}\xi) \cdots H_{\mu_j}(B^{-j}\xi) \hat{\Phi}(B^{-j}\xi), \tag{27}$$

so that $(f_l^n)^\wedge(\xi)$ is the lth component of the column vector in the right hand side of (27). We will prove the claim by induction.

From (22) we see that the claim is true for all n of length 1. Assume it for length j. Then an integer m of a-adic length $j+1$ is of the form $m = \mu + an$, where $0 \leq \mu \leq a-1$ and n has length j. Suppose n has the expansion (26). Then from (25) and(27), we have

$$\begin{aligned} (F_m)^\wedge(\xi) &= (F_{\mu+an})^\wedge(\xi) \\ &= H_\mu(B^{-1}\xi)\hat{F}_n(B^{-1}\xi) \\ &= H_\mu(B^{-1}\xi) H_{\mu_1}(B^{-2}\xi) \cdots H_{\mu_j}(B^{-(j+1)}\xi)\hat{\Phi}(B^{-(j+1)}\xi). \end{aligned}$$

Since $m = \mu + an = \mu + \mu_1 a + \mu_2 a^2 + \cdots + \mu_j a^j$, $\hat{F}_m(\xi)$ has the desired form. Hence, the induction is complete.

The first theorem regarding the multiwavelet packets is the following.

Theorem 3.2. *Let $\{f_l^n : n \geq 0,\ 1 \leq l \leq L\}$ be the basic multiwavelet packets constructed above. Then*

(i) *$\{f_l^n(\cdot - k) : a^j \leq n \leq a^{j+1} - 1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $W_j,\ j \geq 0$.*

(ii) *$\{f_l^n(\cdot - k) : 0 \leq n \leq a^j - 1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $V_j,\ j \geq 0$.*

(iii) *$\{f_l^n(\cdot - k) : n \geq 0,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.*

Proof. Since $\{f_l^n : 1 \leq n \leq a-1,\ \ 1 \leq l \leq L\}$ are the multiwavelets, their $\mathbb{Z}^d$-translates form an orthonormal basis for W_0. So (i) is verified for $j = 0$. Assume for j. We will prove for $j+1$. By assumption, the functions $\{f_l^n(\cdot - k) : a^j \leq n \leq a^{j+1} - 1,$ $1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of W_j. Since $f \in W_j \Leftrightarrow f(A\cdot) \in W_{j+1}$, the system of functions

$$\{a^{1/2} f_l^n(A \cdot -k) : a^j \leq n \leq a^{j+1} - 1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$$

is an orthonormal basis of W_{j+1}. Let

$$E_n = \overline{sp}\{a^{1/2} f_l^n(A \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}.$$

Hence,

$$W_{j+1} = \bigoplus_{n=a^j}^{a^{j+1}-1} E_n. \tag{28}$$

Applying the splitting lemma to E_n, we get the functions

$$g_l^{n,r}(x) = \sum_{m=1}^{L} \sum_{k \in \mathbb{Z}^d} h_{lmk}^r a^{1/2} f_m^n(Ax - k) \quad (0 \leq r \leq a-1,\ \ 1 \leq l \leq L) \tag{29}$$

so that $\{g_l^{n,r}(\cdot - k) :\ 0 \leq r \leq a-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}$ is an orthonormal basis of E_n. But by (23), we have

$$g_l^{n,r} = f_l^{r+an}.$$

This fact, together with (28), shows that

$$\begin{aligned} &\{f_l^{r+an}(\cdot - k) :\ 0 \leq r \leq a-1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d,\ a^j \leq n \leq a^{j+1} - 1\} \\ &\quad = \ \ \{f_l^n(\cdot - k) : a^{j+1} \leq n \leq a^{j+2} - 1,\ \ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\} \end{aligned}$$

is an orthonormal basis of W_{j+1}. So (i) is proved. Item (ii) follows from the observation that $V_j = V_0 \oplus W_0 \oplus \cdots \oplus W_{j-1}$ and (iii) follows from the fact that $\overline{\cup V_j} = L^2(\mathbb{R}^d)$. $\square$

4. Construction of orthonormal bases from the multiwavelet packets

We now take *all* dilations by the matrix A and *all* $\mathbb{Z}^d$-translations of the basic multiwavelet packet functions.

DEFINITION 4.1

Let $\{f_l^n : n \geq 0,\ \ 1 \leq l \leq L\}$ be the basic multiwavelet packets. The collection of functions

$$\mathcal{P} = \{a^{j/2} f_l^n(A^j \cdot -k) : n \geq 0,\ \ 1 \leq l \leq L,\ \ j \in \mathbb{Z},\ \ k \in \mathbb{Z}^d\}$$

will be called the '*general multiwavelet packets*' associated with the MRA $\{V_j\}$ of $L^2(\mathbb{R}^d)$ of multiplicity L.

Remark 4.2. Obviously the collection $\mathcal{P}$ is overcomplete in $L^2(\mathbb{R}^d)$. For example

(i) The subcollection with $j = 0,\ n \geq 0,\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d$ gives us the basic multiwavelet packet basis constructed in the previous section.

(ii) The subcollection with $n = 1, 2, \ldots, a-1;\ 1 \leq l \leq L,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d$ is the usual multiwavelet basis.

So it will be interesting to find out other subcollections of $\mathcal{P}$ which form orthonormal bases for $L^2(\mathbb{R}^d)$.

For $n \geq 0$ and $j \in \mathbb{Z}$, define the subspaces

$$U_j^n = \overline{sp}\{a^{j/2} f_l^n(A^j \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\}. \tag{30}$$

Observe that

$$U_j^0 = V_j \quad \text{and} \quad \bigoplus_{r=1}^{a-1} U_j^r = W_j, \quad j \in \mathbb{Z}.$$

Hence, the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = \bigoplus_{r=0}^{a-1} U_j^r.$$

We can generalize this fact to other values of n.

PROPOSITION 4.3

For $n \geq 0$ and $j \in \mathbb{Z}$, we have

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_j^{an+r}. \tag{31}$$

Proof. By definition

$$U_{j+1}^n = \overline{sp}\left\{a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d\right\}.$$

Let

$$E_{l,k}(x) = a^{\frac{j+1}{2}} f_l^n(A^{j+1} \cdot -k), \quad \text{for } 1 \leq l \leq L,\ k \in \mathbb{Z}^d.$$

Then $\{E_{l,k} :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$ is an orthonormal basis of the Hilbert space U_{j+1}^n. For $0 \leq r \leq a-1$, let

$$F_{l,k}^r(x) = \sum_{m=1}^{L} \sum_{\beta \in \mathbb{Z}^d} h_{l,m,\beta - Ak}^r E_{m,\beta}(x), \quad 1 \leq l \leq L,\ k \in \mathbb{Z}^d,$$

and

$$\mathcal{H}^r = \overline{sp}\{F_{l,k}^r :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}.$$

Then, by Corollary 2.3 we have

$$U_{j+1}^n = \bigoplus_{r=0}^{a-1} \mathcal{H}^r.$$

Now

$$\begin{aligned} F^r_{l,k}(x) &= \sum_{m=1}^{L}\sum_{\beta\in\mathbb{Z}^d} h^r_{l,m,\beta-Ak} E_{m,\beta}(x) \\ &= \sum_{m=1}^{L}\sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} E_{m,Ak+\alpha}(x) \\ &= \sum_{m=1}^{L}\sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} a^{\frac{j+1}{2}} f^n_m(A^{j+1}x - Ak - \alpha) \\ &= a^{\frac{j}{2}} \sum_{m=1}^{L}\sum_{\alpha\in\mathbb{Z}^d} h^r_{l,m,\alpha} a^{\frac{1}{2}} f^n_m\left(A(A^j x - k) - \alpha\right) \\ &= a^{\frac{j}{2}} f^{an+r}_l (A^j x - k), \quad \text{by (23)}. \end{aligned}$$

Therefore,

$$\mathcal{H}^r = U^{an+r}_j$$

and

$$U^n_{j+1} = \bigoplus_{r=0}^{a-1} U^{an+r}_j.$$

□

Using Proposition 4.3 we can get various decompositions of the wavelet subspaces $W_j,\ j \geq 0$, which in turn will give rise to various orthonormal bases of $L^2(\mathbb{R}^d)$.

Theorem 4.4. *Let* $j \geq 0$. *Then, we have*

$$\begin{aligned} W_j &= \bigoplus_{r=1}^{a-1} U^r_j \\ W_j &= \bigoplus_{r=a}^{a^2-1} U^r_{j-1} \\ &\vdots \\ W_j &= \bigoplus_{r=a^l}^{a^{l+1}-1} U^r_{j-l}, \quad l \leq j \\ W_j &= \bigoplus_{r=a^j}^{a^{j+1}-1} U^r_0, \end{aligned} \tag{32}$$

where U^n_j *is defined in* (30).

Proof. Since $W_j = \bigoplus_{r=1}^{a-1} U^r_j$, we can apply Proposition 4.3 repeatedly to get (32). □

Theorem 4.4 can be used to construct many orthonormal bases of $L^2(\mathbb{R}^d)$. We have the following orthogonal decomposition (see (19)):

$$L^2(\mathbb{R}^d) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots.$$

For each $j \geq 0$, we can choose any of the decompositions of W_j described in (32). For example, if we do not want to decompose any W_j, then we have the usual multiwavelet decomposition. On the other hand, if we prefer the last decomposition in (32) for each W_j, then we get the multiwavelet packet decomposition. There are other decompositions as well. Observe that in (32), the lower index of U_j^n's are decreased by 1 in each successive step. If we keep some of these spaces fixed and choose to decompose others by using (31), then we get decompositions of W_j which do not appear in (32). So there is certain interplay between the indices $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

Let S be a subset of $\mathbb{N}_0 \times \mathbb{Z}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Our aim is to characterize those S for which the collection

$$\mathcal{P}_S = \left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d,\ (n, j) \in S\right\}$$

will be an orthonormal basis of $L^2(\mathbb{R}^d)$. In other words, we want to find out those subsets S of $\mathbb{N}_0 \times \mathbb{Z}$ for which

$$\bigoplus_{(n,j)\in S} U_j^n = L^2(\mathbb{R}^d). \tag{33}$$

By using (31) repeatedly, we have

$$\begin{aligned} U_j^n &= \bigoplus_{r=0}^{a-1} U_{j-1}^{an+r} \\ &= \bigoplus_{r=an}^{a(n+1)-1} U_{j-1}^r = \bigoplus_{r=an}^{a(n+1)-1} \left[\bigoplus_{s=0}^{a-1} U_{j-2}^{ar+s}\right] \\ &= \bigoplus_{r=a^2 n}^{a^2(n+1)-1} U_{j-2}^r = \cdots = \bigoplus_{r=a^j n}^{a^j(n+1)-1} U_0^r. \end{aligned} \tag{34}$$

Let $I_{n,j} = \{r \in \mathbb{N}_0 : a^j n \leq r \leq a^j(n+1) - 1\}$. Hence,

$$U_j^n = \bigoplus_{r\in I_{n,j}} U_0^r. \tag{35}$$

That is,

$$\bigoplus_{(n,j)\in S} U_j^n = \bigoplus_{(n,j)\in S}\ \bigoplus_{r\in I_{(n,j)}} U_0^r.$$

But we have already proved in Theorem 3.2 that

$$L^2(\mathbb{R}^d) = \bigoplus_{r\in\mathbb{N}_0} U_0^r.$$

Thus, for (33) to be true, it is necessary and sufficient that $\{I_{n,j} : (n, j) \in S\}$ is a partition of $\mathbb{N}_0$. We say $\{A_l : l \in I\}$ is a partition of $\mathbb{N}_0$ if $A_l \subset \mathbb{N}_0$, A_l's are pairwise disjoint, and $\cup_{l\in I} A_l = \mathbb{N}_0$. We summarize the above discussion in the following theorem.

Theorem 4.5. *Let $\{f_l^n : n \geq 0,\ \ 1 \leq l \leq L\}$ be the basic multiwavelet packets and $S \subset \mathbb{N}_0 \times \mathbb{Z}$. Then the collection of functions*

$$\left\{a^{\frac{j}{2}} f_l^n(A^j \cdot -k) :\ 1 \leq l \leq L,\ \ k \in \mathbb{Z}^d,\ (n, j) \in S\right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\{I_{n,j} : (n, j) \in S\}$ is a partition of $\mathbb{N}_0$.

5. Wavelet frame packets

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{x_k : k \in \mathbb{Z}\}$ of $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist constants C_1 and C_2, $0 < C_1 \leq C_2 < \infty$ such that for all $x \in \mathcal{H}$,

$$C_1 \|x\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle x, x_k \rangle|^2 \leq C_2 \|x\|^2. \tag{36}$$

The largest C_1 and the smallest C_2 for which (36) holds are called the frame bounds.

Suppose that $\Phi = \left\{\varphi^1, \varphi^2, \ldots, \varphi^N\right\} \subset L^2(\mathbb{R}^d)$ such that $\{\varphi^l(\cdot - k) : 1 \leq l \leq N, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span $S(\Phi)$. Let $\psi^1, \psi^2, \ldots, \psi^N$ be elements in $S(\Phi)$ so that each ψ^j is a linear combination of $\varphi^l(\cdot - k)$; $1 \leq l \leq L$, $k \in \mathbb{Z}^d$. A natural question to ask is the following: when can we say that $\{\psi^j(\cdot - k) : 1 \leq j \leq N, k \in \mathbb{Z}^d\}$ is also a frame for $S(\Phi)$?

If $\psi^j \in S(\Phi)$, then there exists $\left\{p_{jlk} : k \in \mathbb{Z}^d\right\}$ in $l^2(\mathbb{Z}^d)$ such that

$$\psi^j(x) = \sum_{l=1}^{N} \sum_{k \in \mathbb{Z}^d} p_{jlk} \varphi^l(x - k).$$

In terms of Fourier transform

$$\begin{aligned} \hat{\psi}^j(\xi) &= \sum_{l=1}^{N} \sum_{k \in \mathbb{Z}^d} p_{jlk} \mathrm{e}^{-i\langle k, \xi\rangle} \hat{\varphi}^l(\xi) \\ &= \sum_{l=1}^{N} P_{jl}(\xi) \hat{\varphi}^l(\xi) \quad (1 \leq j \leq N), \end{aligned} \tag{37}$$

where $P_{jl}(\xi) = \sum_{k \in \mathbb{Z}^d} p_{jlk} \mathrm{e}^{-i\langle k, \xi\rangle}$. Let $P(\xi)$ be the $N \times N$ matrix:

$$P(\xi) = \Big(P_{jl}(\xi)\Big)_{1 \leq j, l \leq N}.$$

Let S and T be two positive definite matrices of order N. We say $S \leq T$ if $\langle x, Sx\rangle \leq \langle x, Tx\rangle$ for all $x \in \mathbb{R}^N$. The following lemma is the generalization of Lemma 3.1 in [2].

Lemma 5.1. *Let φ^l, ψ^l for $1 \leq l \leq N$, and $P(\xi)$ be as above. Suppose that there exist constants C_1 and C_2, $0 < C_1 \leq C_2 < \infty$ such that*

$$C_1 I \leq P^*(\xi) P(\xi) \leq C_2 I \quad \text{for a.e. } \xi \in \mathbb{T}^d. \tag{38}$$

Then, for all $f \in L^2(\mathbb{R}^d)$, we have

$$C_1 \sum_{l=1}^{N} \sum_{k \in \mathbb{Z}^d} \left|\left\langle f, \varphi^l(\cdot - k)\right\rangle\right|^2 \leq \sum_{l=1}^{N} \sum_{k \in \mathbb{Z}^d} \left|\left\langle f, \psi^l(\cdot - k)\right\rangle\right|^2 \leq C_2 \sum_{l=1}^{N} \sum_{k \in \mathbb{Z}^d} \left|\left\langle f, \varphi^l(\cdot - k)\right\rangle\right|^2. \tag{39}$$

Let A be a dilation matrix, $B = A^t$ and $a = |\det A| = |\det B|$. Let

$$K_A = \{\alpha_0, \alpha_1, \ldots, \alpha_{a-1}\} \tag{40}$$

and

$$K_B = \{\beta_0, \beta_1, \ldots, \beta_{a-1}\} \tag{41}$$

be fixed sets of digits for A and B respectively. For $0 \le r, s \le a-1$ and $1 \le l, j \le L$, define for a.e. ξ,

$$\mathcal{E}_{lj}^{rs}(\xi) = \delta_{lj} a^{-\frac{1}{2}} \mathrm{e}^{-i\langle \xi + 2B^{-1}\beta_s \pi, \alpha_r \rangle}. \tag{42}$$

Let

$$E^{rs}(\xi) = \left(\mathcal{E}_{lj}^{rs}(\xi)\right)_{1\le l, j\le L} \tag{43}$$

and

$$E(\xi) = \left(E^{rs}(\xi)\right)_{0\le r,s\le a-1}. \tag{44}$$

So $E(\xi)$ is block matrix with a blocks in each row and each column, and each block is a square matrix of order L, so that $E(\xi)$ is a square matrix of order aL. We have the following lemma which will be useful for the splitting trick for frames.

Lemma 5.2. (i) *If* $\nu \in K_A$, *then* $\sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle} = a\delta_{0\nu}$.
(ii) *The matrix* $E(\xi)$, *defined in* (44), *is unitary*.

Proof. Item (i) is the orthogonal relation for the characters of the finite group $\mathbb{Z}^d/B\mathbb{Z}^d$ (see [14]). Observe that the mapping

$$\mu + B\mathbb{Z}^d \mapsto \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle}, \quad \nu \in K_A$$

is a character of the (finite) coset group $\mathbb{Z}^d/B\mathbb{Z}^d$. If $\nu = 0$ (i.e., if $\nu \in A\mathbb{Z}^d$), then there is nothing to prove. Suppose that $\nu \ne 0$, then there exists a $\mu' \in K_B$ such that $\mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu\rangle} \ne 1$. Since K_B is a set of digits for B, so is $K_B - \mu'$. Hence,

$$\sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}(\mu-\mu'), \nu\rangle} = \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle}. \tag{45}$$

Now

$$\begin{aligned} \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle} &= \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu\rangle} \cdot \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}(\mu-\mu'), \nu\rangle} \\ &= \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu\rangle} \cdot \sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle}, \quad \text{by (45)}. \end{aligned}$$

Therefore,

$$\sum_{\mu \in K_B} \mathrm{e}^{-i2\pi\langle B^{-1}\mu, \nu\rangle} = 0, \quad \text{since } \mathrm{e}^{-i2\pi\langle B^{-1}\mu', \nu\rangle} \ne 1.$$

To prove (ii), observe that the (r, s)th block of the matrix $E(\xi)E^*(\xi)$ is

$$\sum_{t=0}^{a-1} E^{rt}(\xi)\left(E^{ts}(\xi)\right)^*.$$

The (l, j)th entry in this block is

$$\sum_{t=0}^{a-1}\sum_{m=1}^{L} \mathcal{E}_{lm}^{rt}(\xi)\left(\mathcal{E}_{mj}^{ts}(\xi)\right)^*$$

$$
\begin{aligned}
&= \sum_{t=0}^{a-1}\sum_{m=1}^{L} \delta_{lm} a^{-1/2} \mathrm{e}^{-i\langle \xi + 2B^{-1}\beta_t\pi, \alpha_r\rangle} \cdot \delta_{jm} a^{-1/2} \mathrm{e}^{i\langle \xi + 2B^{-1}\beta_t\pi, \alpha_s\rangle} \\
&= \sum_{m=1}^{L} \delta_{lm}\delta_{jm} \sum_{t=0}^{a-1} a^{-1} \mathrm{e}^{-i\langle \xi + 2B^{-1}\beta_t\pi, \alpha_r - \alpha_s\rangle} \\
&= \sum_{m=1}^{L} \delta_{lm}\delta_{jm}\delta_{rs}, \quad \text{(by (i) of the lemma)} \\
&= \delta_{lj}\delta_{rs}.
\end{aligned}
$$

This proves that $E(\xi)E^*(\xi) = I$. Similarly, $E^*(\xi)E(\xi) = I$. Therefore, $E(\xi)$ is a unitary matrix. □

6. Splitting lemma for frame packets

Let $\{\varphi_l : 1 \le l \le L\}$ be functions in $L^2(\mathbb{R}^d)$ such that $\{\varphi_l(\cdot - k) : 1 \le l \le L, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span V. For $0 \le r \le a-1$ and $1 \le l \le L$, suppose that there exist sequences $\{h^r_{ljk} :\ k \in \mathbb{Z}^d\} \in l^2(\mathbb{Z}^d)$. Define f^r_l as in (4) and (5). That is,

$$
f^r_l(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2}\varphi_j(Ax - k). \tag{46}
$$

Let $H_r(\xi)$ be the matrix defined in (7). Let K_A and K_B be respectively fixed sets of digits for A and B as in (40) and (41). Let $H(\xi)$ be the matrix

$$
H(\xi) = \Big(H_r(\xi + 2B^{-1}\beta_s\pi)\Big)_{0\le r,s\le a-1}. \tag{47}
$$

$H(\xi)$ is a block matrix with a blocks in each row and each column, and each block is of order L so that $H(\xi)$ is a square matrix of order aL. Assume that there exist constants C_1 and C_2, $0 < C_1 \le C_2 < \infty$ such that

$$
C_1 I \le H^*(\xi)H(\xi) \le C_2 I \quad \text{for a.e. } \xi \in \mathbb{T}^d. \tag{48}
$$

We can write f^r_l as

$$
\begin{aligned}
f^r_l(x) &= \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h^r_{ljk} a^{1/2}\varphi_j(Ax - k) \\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s + Ak} a^{1/2}\varphi_j(Ax - \alpha_s - Ak), \text{ by (2)} \\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s + Ak} \varphi^{(s)}_j(x - k),
\end{aligned}
$$

where

$$
\varphi^{(s)}_j(x) = a^{1/2}\varphi_j(Ax - \alpha_s), \qquad 0 \le s \le a-1. \tag{49}
$$

Taking Fourier transform, we obtain

$$
\begin{aligned}
(f^r_l)^\wedge(\xi) &= \sum_{j=1}^{L}\sum_{s=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h^r_{l,j,\alpha_s + Ak} \mathrm{e}^{-i\langle \xi, k\rangle} (\varphi^{(s)}_j)^\wedge(\xi) \\
&= \sum_{j=1}^{L}\sum_{s=0}^{a-1} p^{rs}_{lj}(\xi)(\varphi^{(s)}_j)^\wedge(\xi),
\end{aligned}
$$

where $p_{lj}^{rs}(\xi) = \sum_{k\in\mathbb{Z}^d} h_{l,j,\alpha_s+Ak}^r \mathrm{e}^{-i\langle\xi,k\rangle}$. Define

$$P^{rs}(\xi) = \left(p_{lj}^{rs}(\xi)\right)_{1\le l,j\le L} \tag{50}$$

and

$$P(\xi) = \left(P^{rs}(\xi)\right)_{0\le r,s\le a-1}. \tag{51}$$

Claim.

$$H(\xi) = P(B\xi)E(\xi), \tag{52}$$

where $E(\xi)$ is defined in (42)–(44).

Proof of the claim. The (r, s)th block of the matrix $P(B\xi)E(\xi)$ is the matrix

$$\sum_{t=0}^{a-1} P^{rt}(B\xi)E^{ts}(\xi).$$

The (l, j)th entry in this block is equal to

$$\begin{aligned}
&\sum_{t=0}^{a-1}\sum_{m=1}^{L} p_{lm}^{rt}(B\xi)\mathcal{E}_{mj}^{ts}(\xi)\\
&= \sum_{t=0}^{a-1}\sum_{m=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{l,m,\alpha_t+Ak}^r \mathrm{e}^{-i\langle B\xi,k\rangle}\delta_{mj}a^{-1/2}\mathrm{e}^{-i\left\langle\xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle}\\
&= \sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h_{l,j,\alpha_t+Ak}^r \mathrm{e}^{-i\langle B\xi,k\rangle}a^{-1/2}\mathrm{e}^{-i\left\langle\xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle}.
\end{aligned}$$

Now, the (l, j)th entry in the (r, s)th block of $H(\xi)$ is

$$\begin{aligned}
h_{lj}^r(\xi+2B^{-1}\beta_s\pi) &= a^{-1/2}\sum_{k\in\mathbb{Z}^d} h_{ljk}^r \mathrm{e}^{-i\left\langle\xi+2B^{-1}\beta_s\pi,k\right\rangle}\\
&= a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h_{l,j,\alpha_t+Ak}^r \mathrm{e}^{-i\left\langle\xi+2B^{-1}\beta_s\pi,\alpha_t+Ak\right\rangle}, \qquad \text{by (2)}\\
&= a^{-1/2}\sum_{t=0}^{a-1}\sum_{k\in\mathbb{Z}^d} h_{l,j,\alpha_t+Ak}^r \mathrm{e}^{-i\left\langle\xi+2B^{-1}\beta_s\pi,\alpha_t\right\rangle}\cdot \mathrm{e}^{-i\langle B\xi,k\rangle}.
\end{aligned}$$

So the claim is proved. In particular, we have

$$H^*(\xi)H(\xi) = E^*(\xi)P^*(B\xi)P(B\xi)E(\xi). \tag{53}$$

Since $E(\xi)$ is unitary by Lemma 5.2, $H^*(\xi)H(\xi)$ and $P^*(B\xi)P(B\xi)$ are similar matrices. Let $\lambda(\xi)$ and $\Lambda(\xi)$ respectively be the minimal and maximal eigenvalues of the positive definite matrix $H^*(\xi)H(\xi)$, and let $\lambda = \inf_\xi \lambda(\xi)$ and $\Lambda = \sup_\xi \Lambda(\xi)$. (It is clear from (52) that $\lambda(\xi)$ and $\Lambda(\xi)$ are $2\pi\mathbb{Z}^d$-periodic functions.) Suppose $0 < \lambda \le \Lambda < \infty$. Then we have, by (48) (in the sense of positive definite matrices),

$$\lambda I \le H^*(\xi)H(\xi) \le \Lambda I \quad \text{for a.e. } \xi \in \mathbb{T}^d$$

which is equivalent to

$$\lambda I \leq P^*(\xi)P(\xi) \leq \Lambda I \quad \text{for a.e. } \xi \in \mathbb{T}^d.$$

Then by Lemma 5.1, for all $g \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} \lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2 &\leq \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^s(\cdot - k)\right\rangle\right|^2 \\ &\leq \Lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2, \quad (54) \end{aligned}$$

where $\varphi_l^{(s)}$ is defined in (49). Since

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2}\varphi_l(A\cdot -k)\right\rangle\right|^2 = \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, \varphi_l^{(s)}(\cdot - k)\right\rangle\right|^2, \quad (55)$$

which follows from (49), inequality (54) can be written as

$$\begin{aligned} \lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2}\varphi_l(A\cdot -k)\right\rangle\right|^2 &\leq \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^s(\cdot - k)\right\rangle\right|^2 \\ &\leq \Lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2}\varphi_l(A\cdot -k)\right\rangle\right|^2. \quad (56) \end{aligned}$$

This is the splitting trick for frames: the $A^{-1}\mathbb{Z}^d$-translates of the L dilated functions $\varphi_l(A\cdot)$, $1 \leq l \leq L$, are 'decomposed' into $\mathbb{Z}^d$-translates of the aL functions f_l^s, $0 \leq s \leq a-1$, $1 \leq l \leq L$.

We now apply the splitting trick to the functions $\{f_l^s : 1 \leq l \leq L\}$ for each s, $0 \leq s \leq a-1$ to obtain

$$\begin{aligned} \lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2} f_l^s(A\cdot -k)\right\rangle\right|^2 &\leq \sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{s,r}(\cdot - k)\right\rangle\right|^2 \\ &\leq \Lambda \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2} f_l^s(A\cdot -k)\right\rangle\right|^2, \quad (57) \end{aligned}$$

where $f_l^{s,r}$, $0 \leq r \leq a-1$ are defined as in (46) (f_l^s now replaces φ_l):

$$f_l^{s,r}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{ljk}^s a^{1/2} f_j^r(Ax-k); \quad 0 \leq s \leq a-1, \ 1 \leq l \leq L. \quad (58)$$

Summing (57) over $0 \leq s \leq a-1$, we have

$$\begin{aligned} \lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2} f_l^s(A\cdot -k)\right\rangle\right|^2 &\leq \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{s,r}(\cdot - k)\right\rangle\right|^2 \\ &\leq \Lambda \sum_{s=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{1/2} f_l^s(A\cdot -k)\right\rangle\right|^2. \end{aligned}$$

Using (56), we obtain

$$\begin{aligned} \lambda^2 \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{2/2}\varphi_l(A^2\cdot -k)\right\rangle\right|^2 &\leq \sum_{s=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{s,r}(\cdot - k)\right\rangle\right|^2 \\ &\leq \Lambda^2 \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, a^{2/2}\varphi_l(A^2\cdot -k)\right\rangle\right|^2. \quad (59) \end{aligned}$$

Now as in the case of orthonormal wavelet packets, we can define f_l^n, for each $n \geq 0$ and $1 \leq l \leq L$ (see (23) and (27)). In order to ensure that f_l^n are in $L^2(\mathbb{R}^d)$, it is sufficient to assume that all the entries in the matrix $H(\xi)$, defined in (47), are bounded functions. Comparing (58) and (23), we see that

$$\{f_l^{s,r} : 0 \leq r, s \leq a-1\} = \{f_l^{s+ar} : 0 \leq r, s \leq a-1\} = \{f_l^n : 0 \leq n \leq a^2-1\}.$$

So (59) can be written as

$$\begin{aligned}\lambda^2 \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{2/2}\varphi_l(A^2\cdot -k)\rangle\right|^2 &\leq \sum_{n=0}^{a^2-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \\ &\leq \Lambda^2 \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{2/2}\varphi_l(A^2\cdot -k)\rangle\right|^2.\end{aligned}$$

By induction, we get for each $j \geq 1$,

$$\begin{aligned}\lambda^j \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{j/2}\varphi_l(A^j\cdot -k)\rangle\right|^2 &\leq \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \\ &\leq \Lambda^j \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{j/2}\varphi_l(A^j\cdot -k)\rangle\right|^2. \quad (60)\end{aligned}$$

We summarize the above discussion in the following theorem.

Note. $\{f_l^n : n \geq 0,\ 1 \leq l \leq L\}$ will be called the wavelet frame packets.

Theorem 6.1. *Let $\{\varphi_l : 1 \leq l \leq L\} \subset L^2(\mathbb{R}^d)$ be such that $\{\varphi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ is a frame for its closed linear span V_0, with frame bounds C_1 and C_2. Let $H(\xi)$, $H_r(\xi)$, λ and Λ be as above. A ssume that all entries of $H_r(\xi + 2B^{-1}\beta_s\pi)$ are bounded measurable functions such that $0 < \lambda \leq \Lambda < \infty$. Let $\{f_l^n : n \geq 0, 1 \leq l \leq L\}$ be the wavelet frame packets and let $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$. Then for all $j \geq 0$, the system of functions*

$$\{f_l^n(\cdot - k) : 0 \leq n \leq a^j - 1,\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j with frame bounds $\lambda^j C_1$ and $\Lambda^j C_2$.

Proof. Since $\{\varphi_l(\cdot - k) :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$ is a frame of V_0 with frame bounds C_1 and C_2 , it is clear that for all j

$$\{a^{j/2}\varphi_l(A^j\cdot -k) :\ 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$$

is a frame of V_j with the same bounds. So from (60), we have

$$\lambda^j C_1\|g\|^2 \leq \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^n(\cdot - k)\rangle\right|^2 \leq \Lambda^j C_2\|g\|^2 \quad \text{for all } g \in V_j. \quad (61)$$

$\square$

In Theorem 3.2 we proved that the basic multiwavelet packets form an orthonormal basis for $L^2(\mathbb{R}^d) = \overline{\cup V_j}$. An analogous result holds for the wavelet frame packets if the matrix $H(\xi)$, defined in (47), is unitary.

Before proving this result let us observe how the space $\overline{\cup_{j\geq 0} V_j}$ looks like. Let $V_0 = \overline{sp}\{\varphi_l(\cdot - k) : 1 \leq l \leq L,\ k \in \mathbb{Z}^d\}$, $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$ and $V_j \subset V_{j+1}$. Let $W = \cup V_j$. Then it is easy to check that $f \in W \Rightarrow f(\cdot - A^{-j}k) \in W$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. We claim that elements of the form $A^{-j}k$ are dense in $\mathbb{R}^d$. For $K = \{k_1, k_2, \ldots, k_a\}$ a set of digits for A, define the set

$$Q = Q(A, K) = \Big\{x \in \mathbb{R}^d : x = \sum_{j\geq 1} A^{-j}\epsilon_j; \epsilon_j \in K\Big\}.$$

In the above representation of x, ϵ_j's need not be distinct. We have

$$\|A^{-j}x\| \leq C\alpha^j\|x\|, \quad x \in \mathbb{R}^d,$$

where C is a constant and $0 < \alpha < 1$ (see [17], Chapter 5). Therefore, the series that defines x is convergent. For $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, $\|x\| = (|x_1|^2 + |x_2|^2 + \cdots + |x_d|^2)^{\frac{1}{2}}$. The set Q satisfies the following properties (see [10]):

(i) $Q = \cup_{i=1}^a A^{-1}(Q + k_i)$

(ii) $\cup_{k\in\mathbb{Z}}(Q + k) = \mathbb{R}^d$

(iii) Q is compact.

Let $\epsilon > 0$ and $y \in Q$. We first show that there exist $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$ such that $\|y - A^{-j}k\| < \epsilon$. From (i) we have

$$\begin{aligned} Q &= \bigcup_{i=1}^a A^{-1}(Q + k_i) \\ &= \bigcup_{i=1}^a A^{-1}\left[\bigcup_{m=1}^a A^{-1}(Q + k_m) + k_i\right] \\ &= \bigcup_{i=1}^a \bigcup_{m=1}^a (A^{-2}Q + A^{-2}k_m + A^{-1}k_l). \end{aligned}$$

Hence, for any $j \geq 1$ and any $y \in Q$, there exist $y_j \in Q$ and $l_1, l_2, \ldots, l_j \in K$ such that

$$y = A^{-j}y_j + A^{-j}l_j + A^{-(j-1)}l_{j-1} + \cdots + A^{-1}l_1.$$

Therefore,

$$\begin{aligned} \|y - A^{-j}\{l_j + Al_{j-1} + \cdots + A^{j-1}l_1\}\| &= \|A^{-j}y_j\| \\ &\leq C\alpha^j\|y_j\| \\ &\leq C'\alpha^j \quad \text{(as } Q \text{ is compact)} \\ &< \epsilon, \quad \text{choosing } j \text{ suitably.} \end{aligned}$$

Now if $y \in \mathbb{R}^d$, then by (ii) $y = y_0 + p$ for some $y_0 \in Q$ and $p \in \mathbb{Z}^d$. For $y_0 \in Q$, there exist $j \geq 0$ and $k \in \mathbb{Z}^d$ such that $\|y_0 - A^{-j}k\| < \epsilon$. That is,

$$\|y_0 + p - A^{-j}(k + A^j p)\| < \epsilon$$

$$\Rightarrow \|y - A^{-j}(k + A^j p)\| < \epsilon.$$

So the claim is proved.

We have proved that W is invariant under translations by $A^{-j}k$ and these elements are dense in $\mathbb{R}^d$. Therefore, $\overline{W}$ is a closed translation invariant subspace of $L^2(\mathbb{R}^d)$. Hence, $\overline{W} = L^2_E(\mathbb{R}^d)$ for some $E \subset \mathbb{R}^d$ (see [15]), where

$$L^2_E(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : \operatorname{supp} \hat{f} \subset E\}.$$

Now let

$$E_0 = \bigcup_{l=1}^{L} \bigcup_{j \geq 0} B^j(\operatorname{supp} \hat{\varphi}_l).$$

Claim. $E = E_0$ a.e.

To prove the claim we will follow [1], Theorem 4.3. Since $\varphi_l(A^j \cdot) \in V_j \subset \overline{W}$, the function $\left(\varphi_l(A^j \cdot)\right)^\wedge = \frac{1}{a^j}\hat{\varphi}_l(B^{-j}\cdot) \in \widehat{\overline{W}} = \{\hat{f} : f \in \overline{W}\}$. Therefore, $B^j(\operatorname{supp} \hat{\varphi}_l) = \operatorname{supp} \left(\frac{1}{a^j}\hat{\varphi}_l(B^{-j}\cdot)\right) \subset E$ for all $j \geq 0$ and $1 \leq l \leq L$, which implies that $E_0 \subset E$. Let $E_1 = E \setminus E_0$. We have

$$f \in V_j \Leftrightarrow \hat{f} = \sum_{l=1}^{L} m_l(B^{-j}\xi)\hat{\varphi}_l(B^{-j}\xi), \tag{62}$$

for some $2\pi\mathbb{Z}^d$-periodic functions $m_l \in L^2(\mathbb{T}^d)$. Hence, (62) implies that $\hat{f} = 0$ on E_1 for all $f \in V_j$ and hence, for all $f \in \cup V_j = W$. Taking closure, we obtain that $\hat{f} = 0$ on E_1 for all $f \in \overline{W}$. But $\overline{W}$ is the set of all functions whose Fourier transform is supported in E. Since $E_1 \subset E$, we get that $E_1 = \emptyset$ a.e. Therefore, $E = E_0$ a.e. □

Theorem 6.2. *Let $\{\varphi_l(\cdot - k) : 1 \leq l \leq L, k \in \mathbb{Z}^d\} \subset L^2(\mathbb{R}^d)$ be a frame for its closed linear span V_0, with frame bounds C_1 and C_2 and let $V_0 \subset V_1$, where $V_j = \{f : f(A^{-j}\cdot) \in V_0\}$. Assume that $H(\xi)$ is unitary for a.e. ξ. Then $\{f_l^n(\cdot - k) : n \geq 0, 1 \leq l \leq L, k \in \mathbb{Z}^d\}$ is a frame for the space $\overline{\cup_{j \geq 0} V_j}$ with the same frame bounds.*

More generally, let $S = \{(n, j) \in \mathbb{N}_0 \times \mathbb{Z}\}$ be such that $\bigcup_{(n,j)\in S} I_{n,j}$ is a partition of $\mathbb{N}_0$. Then the collection of functions $\{a^{j/2} f_l^n(A^j \cdot - k) : 1 \leq l \leq L, (n, j) \in S, k \in \mathbb{Z}^d\}$ is a frame for $\overline{\cup_{j \geq 0} V_j}$ with the same bounds C_1 and C_2.

Proof. Since $H(\xi)$ is unitary, $\lambda = \Lambda = 1$ so that the inequalities in (60) are equalities, and from (61) we have

$$C_1\|g\|^2 \leq \sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\left\langle g, f_l^n(\cdot - k)\right\rangle\right|^2 \leq C_2\|g\|^2 \quad \text{for all } g \in V_j. \tag{63}$$

Now let $h \in \overline{\cup_{j \geq 0} V_j}$. Then there exists $h_j \in V_j$ such that $h_j \to h$ as $j \to \infty$. Fix j, then for $j < j'$, we have from (63)

$$\sum_{n=0}^{a^j-1} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\left\langle h_{j'}, f_l^n(\cdot - k)\right\rangle\right|^2 \leq C_2\|h_{j'}\|^2.$$

Letting $j' \to \infty$ first and then $j \to \infty$, we have for all $h \in \overline{\cup_{j \geq 0} V_j}$

$$\sum_{n \geq 0} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^d} \left|\left\langle h, f_l^n(\cdot - k)\right\rangle\right|^2 \leq C_2\|h\|^2. \tag{64}$$

To get the reverse inequality we again use (63):

$$\begin{aligned} C_1\|h_j\|^2 &\le \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h_j, f_l^n(\cdot-k)\rangle\right|^2 \\ &= \sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h_j-h, f_l^n(\cdot-k)\rangle + \langle h, f_l^n(\cdot-k)\rangle\right|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} C_1^{1/2}\|h_j\| &\le \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h_j-h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac{1}{2}} \\ &\quad + \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac{1}{2}} \\ &\le C_2^{1/2}\|h_j-h\| + \Big(\sum_{n=0}^{a^j-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h, f_l^n(\cdot-k)\rangle\right|^2\Big)^{\frac{1}{2}}, \qquad \text{by (64)}. \end{aligned}$$

Taking $j\to\infty$, we get

$$C_1\|h\|^2 \le \sum_{n\ge 0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle h, f_l^n(\cdot-k)\rangle\right|^2$$

for all $h\in\overline{\cup V_j}$. So the first part is proved.

Now let $U_j^n = \overline{sp}\{a^{j/2} f_l^n(A^j\cdot -k) : 1\le l\le L,\ \ k\in\mathbb{Z}^d\}$. Then we can prove as in the orthogonal case (see (35)) that

$$U_j^n = \bigoplus_{r\in I_{n,j}} U_0^r,$$

where $\bigoplus$ is just a direct sum not necessarily orthogonal, and $I_{n,j} = \{r\in\mathbb{N}_0 : a^j n\le r\le a^j(n+1)-1\}$. Now, since $H(\xi)$ is unitary, we have $\lambda=\Lambda=1$ and hence (57) is an equality. Therefore,

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{1/2} f_l^n(A\cdot -k)\rangle\right|^2 = \sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^{an+r}(\cdot-k)\rangle\right|^2.$$

From this we get

$$\begin{aligned} \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{2/2} f_l^n(A^2\cdot -k)\rangle\right|^2 &= \sum_{t=0}^{a-1}\sum_{r=0}^{a-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\left\langle g, f_l^{a(an+r)+t}(\cdot-k)\right\rangle\right|^2 \\ &= \sum_{r=a^2n}^{a^2(n+1)-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^r(\cdot-k)\rangle\right|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, a^{j/2} f_l^n(A^j\cdot -k)\rangle\right|^2 &= \sum_{r=a^jn}^{a^j(n+1)-1}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^r(\cdot-k)\rangle\right|^2 \\ &= \sum_{r\in I_{n,j}}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \left|\langle g, f_l^r(\cdot-k)\rangle\right|^2. \qquad (65) \end{aligned}$$

From the first part of the theorem, we have for all $f \in \overline{\cup V_j}$

$$C_1\|f\|^2 \leq \sum_{n\geq 0}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle f, f_l^n(\cdot - k)\right\rangle\right|^2 \leq C_2\|f\|^2.$$

But, the set S is such that $\bigcup_{(n,j)\in S} I_{n,j} = \mathbb{N}_0$. Therefore,

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S}\sum_{r\in I_{n,j}}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle f, f_l^r(\cdot - k)\right\rangle\right|^2 \leq C_2\|f\|^2.$$

Using (65), we get

$$C_1\|f\|^2 \leq \sum_{(n,j)\in S}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d}\left|\left\langle f, a^{j/2} f_l^n(A^j\cdot - k)\right\rangle\right|^2 \leq C_2\|f\|^2$$

for all $f \in \overline{\cup V_j}$. This completes the proof of the theorem. □

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