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On Ricci curvature of *C*-totally real submanifolds in Sasakian space forms

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Abstract. Let M^n be a Riemannian n-manifold. Denote by $S(p)$ and $\overline{\mathrm{Ric}}(p)$ the Ricci tensor and the maximum Ricci curvature on M^n, respectively. In this paper we prove that every C-totally real submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ satisfies $S \leq (\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2)g$, where H^2 and g are the square mean curvature function and metric tensor on M^n, respectively. The equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a C-totally real submanifold M^n of $\bar{M}^{2n+1}(c)$ satisfies $\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2$ identically, then it is minimal.

Keywords. Ricci curvature; C-totally real submanifold; Sasakian space form.

1. Introduction

Let M^n be a Riemannian n-manifold isometrically immersed in a Riemannian m-manifold $\bar{M}^m(c)$ of constant sectional curvature c. Denote by g, R and h the metric tensor, Riemann curvature tensor and the second fundamental form of M^n, respectively. Then the mean curvature vector H of M^n is given by $H = \frac{1}{n}\mathrm{trace}\, h$. The Ricci tensor S and the scalar curvature ρ at a point $p \in M^n$ are given by $S(X, Y) = \sum_{i=1}^n \langle R(e_i, X)Y, e_i\rangle$ and $\rho = \sum_{i=1}^n S(e_i, e_i)$, respectively, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM^n. A submanifold M^n is called totally umbilical if h, H and g satisfy $h(X, Y) = g(X, Y)H$ for X, Y tangent to M^n.

The equation of Gauss for the submanifold M^n is given by

$$\begin{aligned} g(R(X, Y)Z, W) &= c(g(X, W)g(Y, Z) - g(X, Z)g(Y, W)) \\ &\quad + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{1}$$

where $X, Y, Z, W \in TM^n$. From (1) we have

$$\rho = n(n-1)c + n^2H^2 - |h|^2, \tag{2}$$

where $|h|^2$ is the squared norm of the second fundamental form. From (2) we have

$$\rho \leq n(n-1)c + n^2H^2,$$

with equality holding identically if and only if M^n is totally geodesic.

Let $\overline{\mathrm{Ric}}(p)$ denote the maximum Ricci curvature function on M^n defined by

$$\overline{\mathrm{Ric}}(p) = \max\{S(u,u) | u \in T_p^1 M^n, \quad p \in M^n\},$$

where $T_p^1 M^n = \{v \in T_p M^n | \langle v, v\rangle = 1\}$.

In [3], Chen proves that there exists a basic inequality on Ricci tensor S for any submanifold M^n in $\bar{M}^m(c)$, i.e.

$$S \leq \left((n-1)c + \frac{n^2}{4} H^2\right) g, \tag{3}$$

with the equality holding if and only if either M^n is a totally geodesic submanifold or $n = 2$ and M^n is a totally umbilical submanifold. And in [4], Chen proves that every isotropic submanifold M^n in a complex space form $\bar{M}^m(4c)$ satisfies $\overline{\mathrm{Ric}} \leq (n-1)c + \frac{n^2}{4} H^2$, and every Lagrangian submanifold of a complex space form satisfying the equality case identically is a minimal submanifold. In the present paper, we would like to extend the above results to the C-totally real submanifolds of a Sasakian space form, namely, we prove that every C-totally real submanifold of a Sasakian space form $\bar{M}^{2m+1}(c)$ satisfies $S \leq (\frac{(n-1)(c+3)}{4} + \frac{n^2}{4} H^2) g$, and the equality holds identically if and only if either M^n is totally geodesic submanifold or $n = 2$ and M^n is totally umbilical submanifold. Also we show that if a C-totally real submanifold M^n of a Sasakian space form $\bar{M}^{2n+1}(c)$ satisfies $\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4} H^2$ identically, then it is minimal.

2. Preliminary

Let $\bar{M}^{2m+1}$ be an odd dimensional Riemannian manifold with metric g. Let ϕ be a (1,1)-tensor field, ξ a vector field, and η a 1-form on $\bar{M}^{2m+1}$, such that

$$\phi^2 X = -X + \eta(X)\xi\,, \quad \phi\xi = 0\,, \quad \eta(\phi X) = 0\,, \quad \eta(\xi) = 1\,,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \quad \eta(X) = g(X, \xi).$$

If, in addition, $d\eta(X, Y) = g(\phi X, Y)$, for all vector fields X, Y on $\bar{M}^{2m+1}$, then $\bar{M}^{2m+1}$ is said to have a contact metric structure (ϕ, ξ, η, g), and $\bar{M}^{2m+1}$ is called a contact metric manifold. If moreover the structure is normal, that is if $[\phi X, \phi Y] + \phi^2[X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a Sasakian structure (normal contact metric structure) and $\bar{M}^{2m+1}$ is called a Sasakian manifold. For more details and background, see the standard references [1] and [8].

A plane section σ in $T_p\bar{M}^{2m+1}$ of a Sasakian manifold $\bar{M}^{2m+1}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\bar{K}(\sigma)$ with respect to a ϕ-section σ is called a ϕ-sectional curvature. If a Sasakian manifold $\bar{M}^{2m+1}$ has constant ϕ-sectional curvature c, $\bar{M}^{2m+1}$ is called a Sasakian space form and is denoted by $\bar{M}^{2m+1}(c)$.

The curvature tensor $\bar{R}$ of a Sasakian space form $\tilde{M}^{2m+1}(c)$ is given by [8]

$$\begin{aligned}
\tilde{R}(X, Y)Z \quad = \quad & \frac{c+3}{4}(g(Y, Z)X - g(X, Z)Y) \\
& + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\
& + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z),
\end{aligned}$$

for any tangent vector fields X, Y, Z to $\bar{M}^{2m+1}(c)$.

An n-dimensional submanifold M^n of a Sasakian space form $\bar{M}^{2m+1}(c)$ is called a C-totally real submanifold of $\bar{M}^{2m+1}(c)$ if ξ is a normal vector field on M^n. A direct consequence of this definition is that $\phi(TM^n) \subset T^{\perp}M^n$, which means that M^n is an anti-invariant submanifold of $\bar{M}^{2m+1}(c)$. So we have $n \leq m$.

The Gauss equation implies that

$$\begin{aligned} R(X, Y, Z, W) &= \frac{1}{4}(c+3)(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ &\quad + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \end{aligned} \tag{4}$$

for all vector fields X, Y, Z, W tangent to M^n, where h denotes the second fundamental form and R the curvature tensor of M^n.

Let A denote the shape operator on M^n in $\bar{M}^{2m+1}(c)$. Then A is related to the second fundamental form h by

$$g(h(X, Y), \alpha) = g(A_\alpha X, Y), \tag{5}$$

where α is a normal vector field on M^n.

For C-totally real submanifold in $\bar{M}^{2m+1}(c)$, we also have (for example, see [7])

$$A_{\phi Y} X = -\phi h(X, Y) = A_{\phi X} Y, \quad A_\xi = 0. \tag{6}$$

$$g(h(X, Y), \phi Z) = g(h(X, Z), \phi Y). \tag{7}$$

3. Ricci tensor of C-totally real submanifolds

We will need the following algebraic lemma due to Chen [2].

Lemma 3.1. *Let $a_1, \ldots, a_n, c$ be $n+1$ ($n \geq 2$) real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right). \tag{8}$$

Then $2a_1a_2 \geq c$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

For a C-totally real submanifold M^n of $\bar{M}^{2m+1}(c)$, we have

Theorem 3.1. *If M^n is a C-totally real submanifold of $\bar{M}^{2m+1}(c)$, then the Ricci tensor of M^n satisfies*

$$S \leq \left(\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2\right)g, \tag{9}$$

and the equality holds identically if and only if either M^n is totally geodesic or $n = 2$ and M^n is totally umbilical.

Proof. From Gauss' equation (4), we have

$$\rho = \frac{n(n-1)(c+3)}{4} + n^2H^2 - |h|^2. \tag{10}$$

Put $\delta = \rho - \frac{n(n-1)(c+3)}{4} - \frac{n^2}{2}H^2$. Then from (10) we obtain

$$n^2H^2 = 2(\delta + |h|^2). \tag{11}$$

Let L be a linear $(n-1)$-subspace of T_pM^n, $p \in M^n$, and $\{e_1, \ldots, e_{2m}, e_{2m+1} = \xi\}$ an orthonormal basis such that (1) $e_1, \ldots, e_n$ are tangent to M^n, (2) $e_1, \ldots, e_{n-1} \in L$ and (3) if $H(p) \neq 0$, e_{n+1} is in the direction of the mean curvature vector at p.

Put $a_i = h_{ii}^{n+1}$, $i = 1, \ldots, n$. Then from (11) we get

$$\left(\sum_{i=1}^{n} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{n} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\right\}. \tag{12}$$

Equation (12) is equivalent to

$$\left(\sum_{i=1}^{3} \bar{a}_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} \bar{a}_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \sum_{2\le i\neq j\le n-1} a_i a_j\right\}, \tag{13}$$

where $\bar{a}_1 = a_1$, $\bar{a}_2 = a_2 + \cdots + a_{n-1}$, $\bar{a}_3 = a_n$.

By Lemma 3.1 we know that if $(\sum_{i=1}^{3}\bar{a}_i)^2 = 2(c + \sum_{i=1}^{3}\bar{a}_i^2)$, then $2\bar{a}_1\bar{a}_2 \ge c$ with equality holding if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$. Hence from (13) we can get

$$\sum_{1\le i\neq j\le n-1} a_i a_j \ge \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2, \tag{14}$$

which gives

$$\frac{n(n-1)(c+3)}{4} + \frac{n^2}{2}H^2 \ge \rho - \sum_{1\le i\neq j\le n-1} a_i a_j + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2. \tag{15}$$

Using Gauss' equation we have

$$\begin{aligned}
&\rho - \sum_{1\le i\neq j\le n-1} a_i a_j + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\\
&= 2S(e_n, e_n) + \frac{(n-1)(n-2)(c+3)}{4} + 2\sum_{i<n}(h_{in}^{n+1})^2\\
&+ \sum_{r=n+2}^{2m+1}\left[(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right].
\end{aligned} \tag{16}$$

From (15) and (16) we have

$$\begin{aligned}
\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2 \quad \ge \quad & S(e_n, e_n) + 2\sum_{i<n}(h_{in}^{n+1})^2\\
& + \sum_{r=n+2}^{2m+1}\left[\sum_{i=1}^{n}(h_{in}^r)^2 + \left(\sum_{j=1}^{n-1}h_{jj}^r\right)^2\right].
\end{aligned} \tag{17}$$

So we have

$$\frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2 \geq S(e_n, e_n) \tag{18}$$

with equality holding if and only if

$$h^s_{jn} = 0, \quad h^r_{in} = 0, \quad \sum_{j=1}^{n-1} h^s_{jj} = h^s_{nn} \tag{19}$$

for $1 \leq j \leq n-1$, $1 \leq i \leq n$ and $n+2 \leq r \leq 2m+1$ and, since Lemma 3.1 states that $2\bar{a}_1\bar{a}_2 = c$ if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3$, we also have $h^{n+1}_{nn} = \sum_{j=1}^{n-1} h^{n+1}_{jj}$. Since e_n can be any unit tangent vector of M^n, then (18) implies inequality (9).

If the equality sign case of (9) holds identically, then we have

$$\begin{aligned} &h^{n+1}_{ij} = 0 \qquad (1 \leq i \neq j \leq n),\\ &h^r_{ij} = 0 \qquad (1 \leq i, j \leq n; n+2 \leq r \leq 2m+1),\\ &h^{n+1}_{ii} = \sum_{k \neq i} h^{n+1}_{kk}, \quad \sum_{k \neq i} h^r_{kk} = 0, \qquad (n+2 \leq r \leq 2m+1). \end{aligned} \tag{20}$$

If $\lambda_i = h^{n+1}_{ii} (1 \leq i \leq n)$, we find $\sum_{k \neq i} \lambda_k = \lambda_i (1 \leq i \leq n)$ and, since the matrix $A^{(n)} = (a^{(n)}_{ij})$ with $a^{(n)}_{ij} = 1 - 2\delta_{ij}$ is regular for $n \neq 2$ and has kernel $R(1, 1)$ for $n = 2$, we conclude that M^n is either totally geodesic or $n = 2$ and M^n is totally umbilical.

The converse is easy to prove. This completes the proof of Theorem 3.1.

4. Minimality of *C*-totally real submanifolds

Theorem 4.1. *If M^n is an n-dimensional C-totally real submanifold in a Sasakian space form $\bar{M}^{2n+1}(c)$, then*

$$\overline{\text{Ric}} \leq \frac{(n-1)(c+3)}{4} + \frac{n^2}{4}H^2. \tag{21}$$

If M^n satisfies the equality case of (21) *identically, then M^n is minimal.*

Clearly Theorem 4.1 follows immediately from the following Lemma.

Lemma 4.1. *If M^n is an n-dimensional totally real submanifold in a Sasakian space form $\bar{M}^{2m+1}(c)$, then we have* (21)*. If a C-totally real submanifold M^n in $\bar{M}^{2m+1}(c)$ satisfies the equality case of* (21) *at a point p, then the mean curvature vector H at p is perpendicular to $\phi(T_pM^n)$.*

Proof. Inequality (21) is an immediate consequence of inequality (9).

Now let us assume that M^n is a C-totally real submanifold of $\bar{M}^{2m+1}(c)$ which satisfies the equality sign of (21) at a point $p \in M^n$. Without loss of the generality we may choose an orthonormal basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ of T_pM^n such that $\overline{\text{Ric}}(p) = S(\bar{e}_n, \bar{e}_n)$. From the proof of Theorem 3.1, we get

$$h^s_{in} = 0, \quad \sum_{i=1}^{n-1} h^s_{ii} = h^s_{nn}, \quad i = 1, \ldots, n-1; s = n+1, \ldots, 2m+1, \tag{22}$$

where h_{ij}^s denote the coefficients of the second fundamental form with respect to the orthonormal basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ and $\{\bar{e}_{n+1}, \ldots, \bar{e}_{2m+1} = \xi\}$.

If for all tangent vectors u, v and w at p, $g(h(u, v), \phi w) = 0$, there is nothing to prove. So we assume that this is not the case. We define a function f_p by

$$f_p: \ T_p^1 M^n \to R: \quad v \mapsto f_p(v) = g(h(v, v), \phi v). \tag{23}$$

Since $T_p^1 M^n$ is a compact set, there exists a vector $v \in T_p^1 M^n$ such that f_p attains an absolute maximum at v. Then $f_p(v) > 0$ and $g(h(v, v), \phi w) = 0$ for all w perpendicular to v. So from (5), we know that v is an eigenvector of $A_{\phi v}$. Choose a frame $\{e_1, e_2, \ldots, e_n\}$ of $T_p M^n$ such that $e_1 = v$ and e_i be an eigenvector of $A_{\phi e_1}$ with eigenvalue λ_i. The function f_i, $i \geq 2$, defined by $f_i(t) = f_p(\cos t\, e_1 + \sin t\, e_2)$ has relative maximum at $t = 0$, so $f_i''(0) \leq 0$. This will lead to the inequality $\lambda_1 \geq 2\lambda_i$. Since $\lambda_1 > 0$, we have

$$\lambda_i \neq \lambda_1, \quad \lambda_1 \geq 2\lambda_i, \quad i \geq 2. \tag{24}$$

Thus, the eigenspace of $A_{\phi e_1}$ with eigenvalue λ_1 is 1-dimensional.

From (22) we know that $\bar{e}_n$ is a common eigenvector for all shape operators at p. On the other hand, we have $e_1 \neq \pm\bar{e}_n$ since otherwise, from (22) and $A_{\phi e_i}\bar{e}_n = \pm A_{\phi e_i} e_1 = \pm A_{\phi e_1} e_i = \pm\lambda_i e_i \perp \bar{e}_n$ $(i = 2, \ldots, n)$, we obtain $\lambda_i = 0$, $i = 2, \ldots, n$; and hence $\lambda_1 = 0$ by (22), which is a contradiction. Consequently, without loss of generality we may assume $e_1 = \bar{e}_1, \ldots, e_n = \bar{e}_n$.

By (6), $A_{\phi e_n} e_1 = A_{\phi e_1} e_n = \lambda_n e_n$. Comparing this with (22) we obtain $\lambda_n = 0$. Thus, by applying (22) once more, we get $\lambda_1 + \cdots + \lambda_{n-1} = \lambda_n = 0$. Therefore, trace $A_{\phi e_1} = 0$.

For each $i = 2, \ldots, n$, we have

$$h_{nn}^{n+i} = g(A_{\phi e_i} e_n, e_n) = g(A_{\phi e_n} e_i, e_n) = h_{in}^{2n}.$$

Hence, by applying (22) again, we get $h_{nn}^{n+i} = 0$. Combining this with (22) yields trace $A_{\phi e_i} = 0$. So we have trace $A_{\phi X} = 0$ for any $X \in T_p M^n$. Therefore, we conclude that the mean curvature vector at p is perpendicular to $\phi(T_p M^n)$.

Remark 4.1. From the proof of Lemma 4.1 we know that if M^n is a C-totally real submanifold of $\bar{M}^{2n+1}(c)$ satisfying

$$\overline{\mathrm{Ric}} = \frac{(n-1)(c+3)}{4} + \frac{n^2}{4} H^2, \tag{25}$$

then M^n is minimal and $A_{\phi v} = 0$ for any unit tangent vector satisfying $S(v, v) = \overline{\mathrm{Ric}}$. Thus, by (6) we have $A_{\phi X} v = 0$. Hence, we obtain $h(v, X) = 0$ for any X tangent to M^n and any v satisfying $S(v, v) = \overline{\mathrm{Ric}}$. Conversely, if M^n is a minimal C-totally real submanifold of $\bar{M}^{2n+1}(c)$ such that for each $p \in M^n$ there exists a unit vector $v \in T_p M^n$ such that $h(v, X) = 0$ for all $X \in T_p M^n$, then it satisfies (25) indentically.

For each $p \in M^n$, the kernel of the second fundamental form is defined by

$$\mathcal{D}(p) = \{Y \in T_p M^n | h(X, Y) = 0, \forall X \in T_p M^n\}. \tag{26}$$

From the above discussion, we conclude that M^n is a minimal C-totally real submanifold of $\bar{M}^{2m+1}(c)$ satisfying (25) at p if and only if $\dim \mathcal{D}(p)$ is at least 1-dimensional.

Following the same argument as in [4], we can prove

Theorem 4.2. *Let M^n be a minimal C-totally real submanifold of $\bar{M}^{2n+1}(c)$. Then*

(1) *M^n satisfies* (25) *at a point p if and only if* $\dim\mathcal{D}(p) \geq 1$.

(2) *If the dimension of $\mathcal{D}(p)$ is positive constant d, then $\mathcal{D}$ is a completely integral distribution and M^n is d-ruled, i.e., for each point $p \in M^n$, M^n contains a d-dimensional totally geodesic submanifold N of $\bar{M}^{2n+1}(c)$ passing through p.*

(3) *A ruled minimal C-totally real submanifold M^n of $\bar{M}^{2n+1}(c)$ satisfies* (24) *identically if and only if, for each ruling N in M^n, the normal bundle $T^{\perp}M^n$ restricted to N is a parallel normal subbundle of the normal bundle $T^{\perp}N$ along N.*

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