Proc. Indian Acad. Sci. (Math. Sci.), Vol. 111, No. 3, August 2001, pp. 351–363.
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Monotone iterative technique for impulsive delay differential equations

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MS received 29 May 2000

Abstract. In this paper, by proving a new comparison result, we present a result on the existence of extremal solutions for nonlinear impulsive delay differential equations.

Keywords. Contraction mapping theorem; extremal solutions; impulsive delay differential equations.

1. Introduction

In this paper, we discuss the impulsive retarded functional differential equation (IRFDE)

$$\begin{cases} x' = f(t, x_t), & t \in [0, T], t \neq t_k; \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \ldots, m; \\ x_0 = \Phi, \end{cases} \tag{1.1}$$

where $\Phi \in PC([-\tau, 0], R) = \{x, x$ is a mapping from $[\tau, 0]$ into $R, x(t^-) = x(t)$ for all $t \in (-\tau, 0], x(t^+)$ exists for all $t \in [-\tau, 0)$, and $x(t^+) = x(t)$ for all but at most a finite number of points $t \in [-\tau, 0)\}$ and $M([-\tau, 0], R) = \{x, x$ is a bounded and measurable function from $[-\tau, 0]$ into $R\}$ with norm $\|x\| = \sup_{t\in[-\tau,0]} |x(t)|, \tau > 0, x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0], 0 = t_0 < t_1 < t_2 < \cdots < t_m < T, J = [0, T], J' = J - \{t_i\}_{i=1}^m$. It is easy to see that $PC_0([-\tau, 0], R) \subseteq M([-\tau, 0], R)$ and $M([-\tau, 0], R)$ is a Banach space. Now we suppose that $f \in C(J \times M([-\tau, 0], R), R), I_k \in C(R, R) (k = 1, 2, \ldots, m)$ throughout this paper.

In [1] and [2], some existence and uniqueness results were obtained for eq. (1.1) by the Tonelli's method or fixed point theorems. And it is well-known that the method of upper and lower solutions and its associated monotone iteration is powerful technique for establishing existence-comparison for differential equations (see [4, 5, 6]). But to impulsive differential equations with delay as eq. (1.1), this method has not been used yet as far as we know. In this paper, we discuss eq. (1.1) by the method and we can find that the delay and impulses make the discussions more difficult.

2. Main results

Assume $M([-\tau, T], R) = \{x, x$ is a bounded and measurable function from $[-\tau, T]$ into $R\}$ with norm $\|x\| = \sup_{t\in[-\tau,T]} |x(t)|$, $PC_0([-\tau, T], R) = \{x, x$ is a mapping from $[\tau, 0]$ into R, $x(t^-) = x(t)$ for all $t \in (-\tau, 0], x(t^+)$ exists for all $t \in [-\tau, 0), x(t^+) = x(t)$

for all but at most a finite number of points $t \in [-\tau, 0)$, and $x(t)$ is continuous at $t \in [0, T] - \{t_i\}_{i=1}^m$ left continuous at $t = t_k$, and $x(t_k^+)$ exists $(k = 1, 2, \ldots, m)\}$.

DEFINITION 2.1

A function $x \in PC_0([-\tau, T], R)$ is said to be a solution of (1.1) if x satisfies the first expression of eq. (1.1) for all $t \in J$ except on a set of Lebesgue measure zero (the exceptional points will generally include but may not be limited to impulse times t_k) and satisfies the second one of eq. (1.1) for all $t \in \{t_k\}_{k=1}^m$, and x is piecewise absolutely continuous on $[0, T]$ with $x_0 = \Phi$.

DEFINITION 2.2

A function $G : M([-\tau, 0], R) \to R$ is said to be weakly continuous at $\phi_0 \in M([-\tau, 0], R)$ if for any $\{\phi_n\} \subseteq M([-\tau, 0], R)$ with $\lim_{n\to+\infty} \phi_n(s) = \phi_0(s)$, a.e. $s \in [-\tau, 0]$, then

$$\lim_{n\to+\infty} G(\phi_n) = G(\phi_0).$$

And G is said to be weakly continuous on $M([-\tau, 0], R)$ if G is weakly continuous at ϕ for any $\phi \in M([-\tau, 0], R)$.

Remark 2.1. This condition is more direct than that in [1] and is different from that in [2], which need that $f(t, \psi)$ is continuous at each $(t, \psi_0) \in (0, T] \times L^1([-r, 0], R^n)$.

Lemma 2.1. *Assume that a function $g : J \times M([-\tau, 0], R) \to R$ is continuous at every $t \in J$ for each fixed $\phi \in M([-\tau, 0], R)$ and is weakly continuous at every $\phi \in M([-\tau, 0], R)$ for each fixed $t \in J$. Then for every $x \in PC([-\tau, T], R)$, $g(t, x_t)$ is measurable on $[0, T]$.*

Proof. Choose a continuous function sequence $\{x_n\}$ such that

$$\lim_{n\to+\infty} x_n(t) = x(t), \quad \text{for all } t \in [-\tau, T].$$

By Lemma 4 in [3], x_{nt} is continuous at $t \in [0, T]$. So $g(t, x_{nt})$ is measurable on $[0, T]$. Since $\lim_{n\to+\infty} x_{nt}(s) = x_t(s)$, for all $s \in [-\tau, 0]$, then

$$\lim_{n\to+\infty} g(t, x_{nt}) = g(t, x_t), \quad \text{for all } t \in [0, T].$$

So $g(t, x_t)$ is measurable on $[0, T]$. □

Set $B \in M([-\tau, 0], R)^*$. Moreover, suppose that there exists a $\gamma \in L^1([-\tau, 0], R)$ with $\gamma(t) \geq 0$ for all most $t \in [-\tau, 0]$ such that

$$B\psi = \int_{-\tau}^0 \psi(t)\gamma(t)\mathrm{d}t$$

for all $\psi \in M([-\tau, 0], R)$ and $\|B\| = \int_{-\tau}^0 \gamma(t)\mathrm{d}t$.

Now we list a main lemma.

Lemma 2.2 (*Comparison result*). *Assume that* $p \in PC([-\tau, T], R) \cap C^1(J', R)$ *satisfies*

$$\begin{cases} p' \le -Mp(t) - Bp_t, & t \in J, t \ne t_k \\ \Delta p|_{t=t_k} \le -L_k p(t_k), & (k = 1, 2, \ldots, m) \end{cases}, \tag{2.1}$$

where constants $M \ge 0$, $0 \le L_k \le 1$ $(k = 1, 2, \ldots, m)$ *and* $M_0 = \int_{-\tau}^{0} \mathrm{e}^{-Mt}\gamma(t)\mathrm{d}t$. *And suppose further that*

(a) *either* $p(0) \le p_0(s) \le 0$, $s \in [-\tau, 0]$ *and*

$$M_0\Delta_1 \le \frac{\Pi_{k=1}^{m}(1 - L_k)}{1 + \sum_{j=1}^{m} \Pi_{k=1}^{j}(1 - L_k)}, \tag{2.2}$$

where $\Delta_1 = \max\{t_1, t_2 - t_1, \ldots, T - t_m\}$; *or*

(b) $p(0) \ge -\lambda$, $p_0 \in PC_0([-\tau, 0], R) \cap C^1(I', R)$ *where* $I' = [-\tau, 0] - \{t_l\}_{l=-r}^{-1}$, $\{t_l\}_{l=-r}^{-1}$ *is the set of the discontinuous points of* P_0, $p'(t) \le M_0\lambda$,

$$p(t_{-i}^{+}) - p(t_{-i}) \le -L_{-i}\, p(t_{-i}), \tag{2.3}$$

$\inf_{s\in[-\tau,0]} p(s) = -\lambda < 0$ *and*

$$M_0\Delta_2 \le \frac{\Pi_{k=-r}^{m}(1 - L_k)}{1 + \sum_{j=-r}^{m} \Pi_{k=j}^{m}(1 - L_k)}, \tag{2.4}$$

where $\Delta_2 = \max\{t_{-r} + \tau, t_{-r+1} - t_r, \ldots, -t_{-1}, t_1, t_2 - t_1, \ldots, T - t_m\}$. *Then* $p(t) \le 0$ *for a.e.* $t \in J$.

Proof. Now let $v(t) = \mathrm{e}^{Mt}u(t)$, $t \in [-\tau, 0]$. By the definition of B, the eq. (2.1) can be listed as

$$\begin{cases} v'(t) \le -\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)}v(s)\gamma(s - t)\mathrm{d}s, & t \in J, t \ne t_k, \\ \Delta v|_{t=t_k} \le -L_k v(t_k), & (k = 1, 2, \ldots, m). \end{cases} \tag{2.5}$$

Now we will prove $v(t) \le \theta$, $t \in [-\tau, T]$.

In fact, if there exists a $0 < t^*$ with $v(t^*) > 0$, we might well suppose $t^* \ne t_1, t_2, \ldots, t_m$ (otherwise, we can choose a $\bar{t}$ nearing t^* enough with $v(\bar{t}) > 0$), let

$$\inf_{-\tau\le t\le t^*} v(t) = -b. \tag{2.6}$$

First we consider the case (a).

(A) *In case of* $b = 0$: $v(t) \ge 0$, $t \in [0, t^*]$. Then $v'(t) \le 0$, $t \in [0, t^*]$. So $v'(t^*) \le 0$. This is a contradiction.

(B) *In case of* $b > 0$: Assume $t^* \in (t_i, t_{i+1}]$. It is clear that there exists a $0 \le t_* < t^*$ with $v(t_*) = -b$, where t_* in some $J_j (j \le i)$ or $v(t_j^+) = -b$. We may assume that $v(t_*) = -b$ (in case of $v(t_j^+) = -b$, the proof is similar). By mean value theorem, we have

$$\begin{cases} v(t^*) - v(t_i^+) = v'(\zeta_i)(t^* - t_i), & t_i < \zeta_i < t^*; \\ v(t_i) - v(t_{i-1}^+) = v'(\zeta_{i-1})(t_i - t_{i-1}), & t_{i-1} < \zeta_{i-1} < t_i; \\ \dots, & \dots \\ v(t_{j+2}) - v(t_{j+1}^+) = v'(\zeta_{j+1})(t_{j+2} - t_{j+1}), & t_{j+1} < \zeta_{j+1} < t_{j+2}; \\ v(t_{j+1}) - v(t_*) = v'(\zeta_*)(t_{j+1} - t_*), & t_* < \zeta_* < t_{j+1}. \end{cases}$$

On the other hand, for $t \in [0, t^*]$

$$v'(t) \le -\int_{t-\tau}^{t} \mathrm{e}^{M(t-s)} v(s)\gamma(s-t)\mathrm{d}s \le bM_0. \tag{2.7}$$

Now from (2.1), we get

$$v(t_k^+) \le (1 - L_k)v(t_k), \ \ (k = 1, 2 \dots, m),$$

and

$$\begin{cases} v(t^*) - (1 - L_i)v(t_i) \le bM_0\Delta_1, \\ v(t_i) - (1 - L_{i-1})v(t_{i-1}) \le bM_0\Delta_1, \\ \dots, \dots \\ v(t_{j+2}) - (1 - L_{j+1})v(t_{j+1}) \le bM_0\Delta_1, \\ v(t_{j+1}) + b \le bM_0\Delta_1, \end{cases} \tag{2.8}$$

which implies

$$0 < v(t^*) \le -b\Pi_{k=j+1}^{i}(1 - L_k) + bM_0\Delta_1 \left\{ 1 + \sum_{l=j+1}^{i} \Pi_{k=l}^{i}(1 - L_k) \right\}.$$

Moreover,

$$\begin{aligned} M_0\Delta_1 &> \frac{\Pi_{k=j+1}^{i}(1 - L_k)}{1 + \sum_{l=j+1}^{i} \Pi_{k=l}^{i}(1 - L_k)} \\ &\ge \frac{\Pi_{k=j+1}^{m}(1 - L_k)}{\Pi_{k=i+1}^{m} + \sum_{l=j+1}^{i} \Pi_{k=l}^{m}(1 - L_k)} \\ &\ge \frac{\Pi_{k=1}^{m}(1 - L_k)}{1 + \sum_{l=1}^{m} \Pi_{k=l}^{m}(1 - L_k)}, \end{aligned}$$

which contradicts (2.2).

By virtue of (A) and (B), $v(t) \le 0$, $t \in J$.

Next we consider the case (b).

(A′) If $-b = \inf_{t\in[0,t^*]} v(t)$, we can obtain a contraction similarly as (a).

(B′) If $-b < \inf_{t\in[0,t^*]} v(t)$, then $b = \lambda$ and there exists a $t_* \in (t_{-j-1}, t_{-j}]$ with $v(t_*) = -b$ (or $v(t_{-j-1}^+) = -b$, the proof is similar). So

$$\begin{cases} v(t^*) - v(t_i^+) = v'(\zeta_i)(t^* - t_i), & t^i < \zeta_i < t^*; \\ v(t_i) - v(t_{i-1}^+) = v'(\zeta_{i-1})(t_i - t_{i-1}), & t_{i-1} < \zeta_{i-1} < t_i; \\ \dots, & \dots \\ v(t_1) - v(t_{-1}^+) = v'(\zeta_{-1})(t_1 - t_{-1}), & t_{-1} < \zeta_{-1} < t_1, \\ v(t_{-1}) - v(t_{-2}^+) = v'(\zeta_{-2})(t_{-1} - t_{-2}), & t_{-2} < \zeta_{-2} < t_{-1}, \\ \dots, & \dots; \\ v(t_{-j+1}) - v(t_{-j}^+) = v'(\zeta_{-j})(t_{-j+1} - t_{-j}), & t_{-j} < \zeta_{-j} < t_{-j+1}, \\ v(t_{-j}^1) - v(t_*) = v'(\zeta_*)(t_{-j} - t_*), & t_* < \zeta_* < t_{-j}. \end{cases} \tag{2.9}$$

By (2.9) and (2.3), one has

$$\begin{cases} v(t^*) - (1 - L_i)v(t_i) \le bM_0\Delta_2, \\ v(t_i) - (1 - L_{i-1})v(t_{i-1}) \le bM_0\Delta_2, \\ \dots, \dots \\ v(t_1) - (1 - L_{-1})v(t_{-1}) \le bM_0\Delta_2, \\ v(t_{-1}) - (1 - L_{-2})v(t_{-2}) \le bM_0\Delta_2, \\ \dots, \dots \\ v(t_{-j+1}) - (1 - L_{-j})v(t_{-j}) \le bM_0\Delta_2, \\ v(t_{-j}) + b \le bM_0\Delta_2, \end{cases}$$

which implies

$$\begin{aligned} 0 &< v(t^*) \\ &< -b\Pi_{k=-j}^{i}(1 - L_k) + bM_0\Delta_2 \left\{ 1 + \sum_{l=-j}^{i} \Pi_{k=l}^{i}(1 - L_k) \right\}. \end{aligned}$$

Similarly we get

$$\begin{aligned} M_0\Delta_2 &> \frac{\Pi_{k=-j}^{i}(1 - L_k)}{1 + \sum_{l=-j}^{i} \Pi_{k=l}^{i}(1 - L_k)} \\ &\ge \frac{\Pi_{k=-j}^{m}(1 - L_k)}{\Pi_{k=-j}^{m} + \sum_{l=-j}^{i} \Pi_{k=l}^{m}(1 - L_k)} \\ &\ge \frac{\Pi_{k=-r}^{m}(1 - L_k)}{1 + \sum_{l=-r}^{m} \Pi_{k=l}^{m}(1 - L_k)}, \end{aligned}$$

which contradicts (2.4).

By virtue of (A′) and (B′), $v'(t) \le 0$, a.e. $t \in J$. And the proof is complete. □

Lemma 2.3. *Let $\sigma, \eta \in M([-\tau, T], R)$. Then $x \in PC_0([-\tau, T], R)$ is a solution of the equation*

$$\begin{cases} x' + Mx + Bx_t = \sigma(t), & t \in J, t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(\eta_k) - L_k[x(t_k) - \eta(t_k)], & (k = 1, 2, \dots, m), \\ x_{t_0} = \Phi \end{cases} \tag{2.10}$$

if and only if $x \in PC_0([-\tau, T], R)$ is a solution of the following integral equation

$$\begin{aligned} x(t) = \Phi(0)\mathrm{e}^{-Mt} &+ \int_0^t \mathrm{e}^{-M(t-s)}[\sigma(s) - Bx_s]\mathrm{d}s \\ &+ \sum_{0<t_k<t} \mathrm{e}^{-M(t-t_k)}\{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}, t \in J, \end{aligned} \tag{2.11}$$

where $x_t(s) = x(t + s) = \Phi(t + s)$ if $t + s \le 0$.

Proof. Assume that $x \in PC_0([-\tau, T], R)$ is a solution of IRFDE (2.10). Let $z(t) = x(t)e^{-Mt}$. Then $z \in PC([-\tau, T], R)$ and

$$z'(t) = [\sigma(t) - Bx_t)]e^{-Mt},\ t \in [0, T],\ t \neq t_k\ (k = 1, 2, \ldots, m).$$

Since $(\sigma(t) - Bx_t)e^{-Mt}$ is measurable on $[0, T]$, it is easy to establish the following formula:

$$z(t) = z(0) + \int_0^t z'(s)\mathrm{d}s + \sum_{0<t_k<t}[z(t_k^+) - z(t_k)],\ t \in [0, T].$$

And from the second expression of (2.10), we have

$$z(t_k^+) - z(t_k) = \{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}e^{Mt_k}.$$

Consequently,

$$\begin{aligned} x(t)e^{Mt} &= \Phi(0) + \int_0^t [\sigma(t) - Bx_s]\mathrm{d}s \\ &\quad + \sum_{0<t_k<t}\{I_k(\eta(t_k)) - L_k[x(t_k) - \eta(t_k)]\}e^{Mt_k}, t \in [0, T], \end{aligned}$$

i.e., $x(t)$ satisfies (2.11).

Conversely, if $x \in PC([-\tau, T])$ is a solution of eq. (2.11), by direct differentiation, it is easy to see the first expression of (2.10) is true for all $t \in [0, T] - \{t_k\}_{k=1}^m$ except on a set of Lebesgue measure zero and the second one and the third one of (2.10) are true. The proof is complete. □

Lemma 2.4. *Equation* (2.11) *has a unique solution in* $PC_0([-\tau, T], R)$ *with* $x_0 = \Phi$.

Proof. For $x \in C([0, t_1], R)$, let $\|x\| = \max\{e^{-M_1 t}|x(t)|, t \in [0, t_1]\}$ and

$$(A_1x)(t) = \Phi(0)e^{-Mt} + \int_0^t e^{-M(t-s)}[\sigma(s) - (Bx_s)]\mathrm{d}s,\ t \in J,$$

where $x(t+s) = \Phi(t+s)$ if $t+s \leq 0$ and $M_1 = \|B\|+1$. Obviously $A_1 : C([0, t_1], R) \to C([0, t_1], R)$ is a continuous operator. For $x, y \in C([0, t_1], R)$,

$$\begin{aligned} &|(A_1x)(t) - (A_1y)(t)| \\ = \ &\int_0^t [(Bx_s) - (By_s)]\mathrm{d}s \\ = \ &\int_0^t \int_{-\tau}^0 |x_s(r) - y_s(r)|\gamma(r)\mathrm{d}r\mathrm{d}s \\ = \ &\int_{-\tau}^0 \int_0^t |x_s(r) - y_s(r)|\gamma(r)\mathrm{d}s\mathrm{d}r \\ = \ &\int_{-\tau}^0 \int_0^t |x_s(r) - y_s(r)|\mathrm{d}s\gamma(r)\mathrm{d}r \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\tau}^{0}\int_{r}^{t+r} |x(s)-y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&= \int_{-\tau}^{0}\int_{0}^{t+r} |x(s)-y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&\leq \int_{-\tau}^{0}\int_{0}^{t} |x(s)-y(s)|\mathrm{d}s\gamma(r)\mathrm{d}r \\
&= \int_{0}^{t} |x(s)-y(s)|\mathrm{d}s \int_{-\tau}^{0}\gamma(r)\mathrm{d}r \\
&= \|B\|\int_{0}^{t} \mathrm{e}^{M_1 s}\mathrm{e}^{-M_1 s}\|x(s)-y(s)\|\mathrm{d}s \\
&\leq \frac{\|B\|}{M_1}\mathrm{e}^{M_1 t}\|x-y\|.
\end{aligned}
$$

So

$$
\mathrm{e}^{-M_1 t}|(A_1x)(t)-(A_1y)(t)| \leq \frac{\|B\|}{M_1}\|x-y\|,
$$

i.e.,

$$
\|(A_1x-A_1y)\| \leq \frac{\|B\|}{M_1}\|x-y\|. \tag{2.12}
$$

By contraction mapping theorem, A_1 has a unique fixed point $x_1 \in C([0,t_1],R)$. For $x \in C([t_1,t_2],R)$, let $\|x\| = \max\{\mathrm{e}^{-M_2 t}|x(t)|, t \in [t_1,t_2]\}$ and

$$
\begin{aligned}
(A_2x)(t) &= (x_1(t_1)) + [I_1(\eta(t_1)) - L_1(x_1(t_1)-\eta(t_1))]\mathrm{e}^{-M(t-t_1)} \\
&\quad + \int_{t_1}^{t} \mathrm{e}^{-M(t-s)}[\sigma(s)-Bx_s]\mathrm{d}s,\ t \in [t_1,t_2],
\end{aligned} \tag{2.13}
$$

where $x(t+s) = \Phi(t+s)$ if $t+s \leq 0, x(t+s) = x_1(t+s)$ if $t+s \in (0,t_1]$ and $M_2 = \|B\|+1$. Similarly, A_2 has a unique fixed point x_2 in $C([t_1,t_2],R)$. So forth and so on, for $x \in C([t_n,T],R)$, let $\|x\| = \max\{\mathrm{e}^{-M_{n+1}t}|x(t)|, t \in [t_n,T]\}$ and

$$
\begin{aligned}
(A_{n+1}x)(t) &= (x_n(t_n)) + [I_n(\eta(t_n)) - L_n(x_n(t_n)) - \eta(t_n)]\mathrm{e}^{-M(t-t_n)} \\
&\quad + \int_{0}^{t} \mathrm{e}^{-M(t-s)}[\sigma(s)-Bx_s]\mathrm{d}s,\ t \in [t_n,T],
\end{aligned} \tag{2.14}
$$

where $x(t+s) = \Phi(t+s)$ if $t+s \leq 0$, $x(t+s) = x_1(t+s)$ if $t+s \in (0,t_1], \ldots, x(t+s) = x_{n-1}(t+s)$ if $t+s \in (t_{n-2},t_n]$ and $M_{n+1} = \|B\|+1$.

Similarly A_{n+1} has a unique fixed point $x_{n+1} \in C([t_n,T],R)$. Let

$$
x^*(t) = \begin{cases}
\Phi(t), & t \in [-\tau,0]; \\
x_1(t), & t \in (0,t_1]; \\
x_2(t), & t \in (t_1,t_2]; \\
\ldots, & \ldots; \\
x_{n+1}(t), & t \in (t_n,T].
\end{cases}
$$

Then $x^* \in PC([-\tau,T],R)$ is a solution. If $y^* \in PC([-\tau,T],R)$ is another solution of equation, by $x^*(t) = y^*(t)$ for $t \in [-\tau,0]$, it is easy to verify $x^*(t) = y^*(t)$ for $t \in [0,t_1]$.

And so on, $x^*(t) = y^*(t)$ for $t \in (t_1, t_2]$. Continuing as before, we get $x^*(t) = y^*(t)$ for $t \in (t_n, T]$. Therefore $x^* = y^*$. The proof is complete. □

Now we list some independent conditions for convenience.

(A_1) There exist $u, v \in PC_0([-\tau, T], R)$ satisfying $u(t) \leq v(t)$ $(t \in J)$ and

$$\begin{cases} u'(t) \leq f(t, u_t), & t \in J, t \neq t_k; \\ \Delta u|_{t=t_k} \leq I_k(u(t_k)), & (k = 1, 2, \ldots, m), \\ u_0 \leq \Phi, \end{cases}$$

$$\begin{cases} v'(t) \geq f(t, v_t), & t \in J, t \neq t_k; \\ \Delta v|_{t=t_k} \geq I_k(v(t_k)), & (k = 1, 2, \ldots, m), \\ v_0 \geq \Phi. \end{cases}$$

Moreover, $\Phi - u_0$ and $v_0 - \Phi$ satisfy either the assumption (a) or (b) of Lemma 2.1.

(A_2) There exist constants $M \geq 0$ such that

$$f(t, \phi) - f(t, \psi) \geq -M(\phi(0) - \psi(0)) - B(\phi - \psi),$$

whenever $t \in J, \phi, \psi \in \{x_t, u(t) \leq x(t) \leq v(t), t \in J\}$ with $\phi \geq \psi$.

(A_3) There exist constants $0 \leq L_k \leq 1$ $(k = 1, 2, \ldots, m)$ such that

$$I_k(x) - I_k(y) \geq -L_k(x - y),$$

whenever $u(t_k) \leq y \leq x \leq v(t_k)$, $(k = 1, 2, \ldots, m)$.

(A_4) $f : J \times M([-\tau, 0], R) \to R$ is continuous at every $t \in J$ for each fixed $\phi \in M([-\tau, 0], R)$ and is weakly continuous at every $\phi \in M([-\tau, 0], R)$ for each fixed $t \in J$.

Theorem 2.1. *Let the conditions* (A_1)–(A_4) *be satisfied and* $f \in C([0, T] \times M([-\tau, 0], R), R)$ *and* $[u, v] \subseteq PC_0([-\tau, 0], R)$. *Then there exist monotone sequence* $\{u_n\}, \{v_n\} \subseteq PC_0([-\tau, T], R)$ *which converge on* J *to the minimal and maximal solutions* $x_*, x^* \in PC_0([-\tau, T], R)$ *in* $[u, v]$ *respectively. That is, if* $x \in PC_0([-\tau, T], R)$ *is any solution satisfying* $x \in [u, v]$, *then*

$$u(t) \leq u_1(t) \leq \ldots \leq x_*(t) \leq x(t) \leq x^*(t) \leq \ldots \leq v_n(t) \leq \ldots \leq v_1(t) \leq v(t), \ t \in J.$$

Proof. For any $\eta \in [u, v]$, consider the linear eq. (2.10), where

$$\sigma(t) = f(t, \eta_t) + M\eta(t) + B\eta_t, \ t \in J.$$

By the condition (A_4) and Lemma 2.1, one has $\sigma \in M([-\tau, T], R)$. By Lemma 2.3, IRFDE (2.10) has a unique solution $x \in PC_0([-\tau, T], R)$ with $x_0 = \Phi$. Let

$$x(t) = (A\eta)(t), \ t \in J. \tag{2.15}$$

Then A is a continuous operator from $[u, v]$ into $PC_0([-\tau, T], R)$. Now we show

(a) $u \leq Au$, $Av \leq v$;

(b) A is nondecreasing in $[u, v]$.

To prove (a), we set $u_1 = Au$ and $p = u - u_1$. By Lemma 2.3, we have

$$\begin{cases} u_1'(t) + Mu_1(t) + Bu_{1t} = f(t, u_t) + Mu(t) + Bu_t, & t \in J, t \neq t_k, \\ \Delta u_1|_{t=t_k} = I_k(u(t_k)) - L_k[u_1(t_k) - u(t_k)], & k = 1, 2, \dots, m, \\ u_{10} = \Phi. \end{cases} \tag{2.16}$$

So

$$\begin{cases} p'(t) = u'(t) - u_1{}'(t) \leq -Mp(t) - Bp_t, & t \in J, t \neq t_k, \\ \Delta p|_{t=t_k} = \Delta u|_{t=t_k} - \Delta u_1|_{t=t_k} \leq -L_k p(t_k), & (k = 1, 2, \dots, m) \\ p_0 = u_0 - u_{10} \leq 0, \end{cases} \tag{2.17}$$

which implies by virtue of Lemma 2.2 that $p(t) \leq 0$ for $t \in J$, i.e. $u \leq u_1 = Au$. Similarly, we can show $v_1 = Av \leq v$.

To prove (b), for $\eta_1, \eta_2 \in [u, v]$ with $\eta_1 \leq \eta_2$, let $p = x_1 - x_2$, where $x_1 = A\eta_1, x_2 = A\eta_2$. From Lemma 2.2, we get

$$\begin{aligned} p' &= x_1' - x_2' \\ &= [f(t, \eta_{1t}) + M(\eta_1(t) - x_1(t)) + (B\eta_{1t} - Bx_{1t})] \\ &\quad -[f(t, \eta_{2t}) + M(\eta_2(t) - x_2(t)) + (B\eta_{2t} - Bx_{2t})] \\ &= -[f(t, \eta_{2t}) - f(t, \eta_{1t}) \\ &\quad +M(\eta_2(t) - \eta_1(t)) + (B\eta_{2t} - B\eta_{1t})] \\ &\quad -Mp - Bp_t \\ &\leq -Mp(t) - Bp_t, t \in J, t \neq t_k, \end{aligned}$$

$$\begin{aligned} \Delta p|_{t=t_k} &= \Delta x_1|_{t=t_k} - \Delta x_2|_{t=t_k} \\ &= \{I_k(\eta_1(t_k)) - L_k[x_1(t_k) - \eta_1(t_k)]\} - \{I_k(\eta_2(t_k)) - L_k[x_2(t_k) - \eta_2(t_k)]\} \\ &= -\{I_k(\eta_2(t_k)) - I_k(\eta_1(t_k)) + L_k[\eta_2(t_k) - \eta_1(t_k)]\} - L_k p(t_k) \\ &\leq -L_k p(t_k), (k = 1, 2, \dots, m), \end{aligned}$$

and

$$p_0 = x_{10} - x_{20} = 0.$$

Hence, by Lemma 2.2, $p(t) \leq 0$ for all $t \in J$, i.e., $A\eta_1 \leq A\eta_2$, and (b) is proved.

Let $u_n = Au_{n-1}$, and $v_n = Av_{n-1}$ $(n = 1, 2, \dots, m)$. By (a) and (b), we get

$$u(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq v_n(t) \leq \dots \leq \dots \leq v_1(t) \leq v(t), \ t \in J, \tag{2.18}$$

and $u_n, v_n \in PC_0([-\tau, T], R)$ with $u_{n0} = v_{n0} = \Phi, n = 1, 2, \dots$. So there exist x_* and x^* such that

$$u_n(t) \to x_*(t), t \in [-\tau, T], \ n \to +\infty, \tag{2.19}$$

$$v_n(t) \to x^*(t), t \in [-\tau, T], \ n \to +\infty. \tag{2.20}$$

Therefore

$$\begin{aligned} u_{nt}(s) &\to x_{*t}(s), t \in J, s \in [-\tau, 0], \ n \to +\infty, \\ v_{nt}(s) &\to x_t^*(s), t \in J, s \in [-\tau, 0], \ n \to +\infty. \end{aligned}$$

So

$$\begin{aligned}&f(t,u_{nt})+Mu_{n-1}(t)-(Bu_{nt}-Bu_{n-1t})\\&\qquad\to f(t,x_{*t})+Mx_*(t),n\to+\infty.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}&\int_0^t \mathrm{e}^{-M(t-s)}[f(s,u_{ns})+Mu_{n-1}(s)-(Bu_{ns}-Bu_{n-1s})]\mathrm{d}s\\&\qquad\to\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ n\to+\infty.\end{aligned}\tag{2.21}$$

So

$$x_*(t)=\Phi(0)\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in[0,t_1],\tag{2.22}$$

where $x_{*0}=\Phi$. And by virtue of the continuity of I_1, we get

$$I_1(u_n(t_1))\to I_1(x_*(t_1)),\ n\to+\infty.\tag{2.23}$$

Similarly, one has

$$\begin{aligned}x_*(t)&=[x_*(t_1)+I_1(x_*(t_1))]\mathrm{e}^{-M(t-t_1)}\\&\quad+\int_{t_1}^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in(t_1,t_2],\end{aligned}\tag{2.24}$$

where $x_{*0}=\Phi$. So forth and so on,

$$\begin{aligned}x_*(t)&=[x_*(t_n)+I_n(x_*(t_n))]\mathrm{e}^{-M(t-t_n)}\\&\quad+\int_{t_n}^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s,\ t\in(t_n,T],\end{aligned}\tag{2.25}$$

where $x_{*0}=\Phi$. Then

$$\begin{aligned}x_*(t)&=\Phi\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x_{*s})+Mx_*(s)]\mathrm{d}s\\&\quad+\sum_{0<t_k<t}\mathrm{e}^{-M(t-t_k)}I_k(x_*(t_k)),\ t\in J.\end{aligned}\tag{2.26}$$

By the similar proof, we get

$$\begin{aligned}x^*(t)&=\Phi(0)\mathrm{e}^{-Mt}+\int_0^t \mathrm{e}^{-M(t-s)}[f(s,x^*_s)+Mx^*(s)]\mathrm{d}s\\&\quad+\sum_{0<t_k<t}\mathrm{e}^{-M(t-t_k)}I_k(x^*(t_k)),\end{aligned}\tag{2.27}$$

where $x^*_0=\Phi$.

Finally, if $x\in PC([-\tau,T],R)$ is a solution of eq. (1.1) in $[u,v]$, Now let $p=u_n-x$ and use mathematics induction. Obviously $u\le x$. Suppose $u_{n-1}\le x$. Then

$$\begin{aligned}p'&=u_n{}'-x'\\&=f(t,u_{n-1t})-M(u_n(t)-u_{n-1}(t))-(Bu_{nt}-B_{u_{n-1t}})-f(t,x_t)\\&=-Mp-Bp_t-[f(t,x_t)-f(t,u_{n-1t})]\\&\quad+M(-x(t)+u_{n-1}(t))+(-Bx_t+Bu_{n-1t})\\&\le-Mp-Bp_t,\ t\in J,\ t\ne t_k,\end{aligned}$$

$$\begin{aligned}\Delta p|_{t=t_k} &= \Delta u_n|_{t=t_k} - \Delta x|_{t=t_k}\\ &= I_k(u_{n-1}(t_k)) - L_k[u_n(t_k) - u_{n-1}(t_k)] - I_k(x(t_k))\\ &= -\{I_k(x(t_k)) - I_k(u_{n-1}(t_k) + L_k[x(t_k) - u_{n-1}(t_k)]\} - L_k p(t_k)\\ &\le -L_k p(t_k),\ (k = 1, 2, \ldots, m),\end{aligned}$$

and

$$p_0 = u_{n0} - x_0 = \theta.$$

Hence, by Lemma 2.2, $p(t) \le 0$ for all $t \in J$, i.e. $u_n(t) \le x(t), t \in J$. So $u_n(t) \le x(t)$, $t \in J, n = 1, 2, \ldots$. By the same proof, we can show $x(t) \le v^{(n)}(t), t \in J, n = 1, 2, \ldots$. Consequently, $x_*(t) \le x(t) \le x^*(t), t \in J$. The proof is complete. □

3. An example

We consider.

$$\begin{cases} x' = \dfrac{1}{72}(t - x(t))^3 + \dfrac{1}{40}(t^2 - x(t-1))^5 \\ \qquad + \dfrac{1}{144}\left(\sin^2 t - \displaystyle\int_{-1}^{0} x(t+s)\mathrm{d}s\right)^3, t \ne \dfrac{1}{2}, t \in (0, 1]; \\ \Delta x|_{t=\frac{1}{2}} = -\dfrac{1}{6}x\left(\dfrac{1}{2}\right), \\ x_0 = \phi, \end{cases} \tag{3.1}$$

where

$$\phi(t) = \begin{cases} 1, & t \in [-1, -\frac{1}{2}), \\ \dfrac{1}{2}, & t \in (-\frac{1}{2}, 0]. \end{cases}$$

Conclusion. IRFDE (3.1) admits minimal and maximal solutions.

Proof. Let

$$u(t) = 0,\ t \in [-1, 1]$$

and

$$v(t) = \begin{cases} 1, & t \in [-1, 0]; \\ 1 + t, & t \in (0, \frac{1}{2}]; \\ t + \dfrac{5}{6}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

It is easy to see that u, v are not solutions of eq. (3.1) and $u(t) \le v(t),\ t \in [-1, 1]$. Moreover,

$$
\begin{aligned}
&\Delta u|_{t=\frac{1}{2}} = -\frac{1}{6}u\left(\frac{1}{2}\right),\\
&\Delta v|_{t=\frac{1}{2}} = -\frac{1}{6} > -\frac{1}{2} = -\frac{1}{6}u\left(\frac{1}{2}\right)\\
&u'(t) = 0,\ t \in [0,1];\\
&v'(t) = 1,\ t \in \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right],\\
&f(t,u_t) = \frac{1}{72}t^3 + \frac{1}{40}t^{10} + \frac{1}{144}\sin^6 t,\ t \in [0,1],\\
&f(t,v_t) = \frac{1}{72}(t-(1+t))^3 + \frac{1}{40}(t^2-1)^5\\
&\qquad + \frac{1}{144}\left(\sin^2 t - \int_{-1}^{0} v(t+s)\mathrm{d}s\right)^3,\ t \in [0,1].
\end{aligned}
$$

Then

$$
\begin{cases}
u'(t) \le f(t,u_t), & t \in (0,1), t \ne \frac{1}{2};\\
\Delta u|_{t=\frac{1}{2}} \le -\frac{1}{6}u\left(\frac{1}{2}\right), &\\
u_0 \le \Phi, &
\end{cases}
$$

$$
\begin{cases}
v'(t) \ge f(t,v_t), & t \in J, t \ne t_k;\\
\Delta v|_{t=t_k} > -\frac{1}{6}v\left(\frac{1}{2}\right), &\\
v_0 \ge \Phi, &
\end{cases}
$$

i.e. the condition (A_3) is true.

By mean value theorem, we get

$$
\begin{aligned}
&\frac{1}{72}((t-x)^3 - (t-y)^3) = -\frac{1}{24}(t-\eta(x,y))^2(x-y),\\
&\frac{1}{40}((t^2-x)^5 - (t^2-y)^5) = -\frac{1}{8}(t-\zeta(x,y))^4(x-y)
\end{aligned}
$$

and

$$
\frac{1}{144}((\sin^2 t - x)^3 - (\sin^2 t - y)^3) = -\frac{1}{48}(\sin^2 t - \gamma(x,y))^2(x-y).
$$

For any $\psi \in M([-1,0],R)$, let

$$
B\psi = \frac{1}{8}\psi(-1) + \frac{1}{48}\int_{-1}^{0}\psi(s)\mathrm{d}s.
$$

Then

$$
f(t,\phi) - f(t,\psi) \ge -\frac{1}{24}(\phi(0)-\psi(0)) - (B\phi - B\psi)
$$

for all $\phi, \psi \in \{x_t, u(t) \le x(t) \le v(t), t \in [0,1]\}$ with $\phi \le \psi$.

So the condition (A_2) is true.

For $u\left(\frac{1}{2}\right) \le y \le x \le v\left(\frac{1}{2}\right)$,

$$
I(x) - I(y) = -\frac{1}{6}(x-y).
$$

So the condition (A_3) is true. So $M = \frac{1}{24}$, $L_1 = \frac{1}{6}$, $\Delta_1 = \frac{1}{2}$, $\Delta_2 = 1$,

$$M_0 < \frac{1}{24} + \frac{1}{8} = \frac{1}{6}.$$

For $p_1(t) = u(t) - \Phi(t)$, $t \in [-1, 0]$, we get

$$L_{-1} = \frac{1}{2},\ \Delta = \max\left\{\frac{1}{2}, 1, \frac{1}{2}\right\} = 1,\ \inf_{t\in[-1,0]} p_1(t) = -1 < p_1(0)$$

and

$$p_1'(t) = 0 < M_0,\ \ t \in \left[-1, -\frac{1}{2}\right) \cap \left(-\frac{1}{2}, 0\right].$$

Moreover,

$$M_0\Delta_1 < \frac{5}{23} = \frac{(1 - L_{-1})(1 - L_1)}{1 + (1 - L_{-1}) + (1 - L_{-1})(1 - L_1)}.$$

For $p_2 = \Phi(t) - v(t)$, we get

$$p_2(0) = -\frac{1}{2} \le p_2(t),\ \ t \in [-1, 0]$$

and

$$M_0 < \frac{5}{11} = \frac{(1 - L_1)}{1 + (1 - L_1)}.$$

And thus it is easy to see that (A_4) is true. By Theorem 2.1, eq. (3.1) has a maximal solution and a minimal solution. The proof is complete. □

Remark. Our result can be extended to impulsive delay differential equations in Banach spaces.

Acknowledgement

This project was supported by the National Natural Science Foundation of China (19771054) and YNF of Shandong Province (Q99A14).

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