

On oscillation and asymptotic behaviour of solutions of forced first order neutral differential equations

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Abstract. In this paper, sufficient conditions have been obtained under which every solution of

$$\left[y(t) \pm y(t - \tau) \right]' \pm Q(t) G(y(t - \sigma)) = f(t), \quad t \geq 0,$$

oscillates or tends to zero or to $\pm\infty$ as $t \rightarrow \infty$. Usually these conditions are stronger than

$$\int_0^{\infty} Q(t) dt = \infty. \quad (*)$$

An example is given to show that the condition (*) is not enough to arrive at the above conclusion. Existence of a positive (or negative) solution of

$$\left[y(t) - y(t - \tau) \right]' + Q(t) G(y(t - \sigma)) = f(t)$$

is considered.

Keywords. Oscillation; nonoscillation; neutral equations; asymptotic behaviour.

1. Introduction

In a recent paper [8], the authors have obtained necessary and sufficient conditions so that every solution of

$$\left[y(t) - p y(t - \tau) \right]' + Q(t) G(y(t - \sigma)) = f(t)$$

oscillates or tends to zero as $t \rightarrow \infty$ on various ranges of p , where $G \in C(R, R)$, $Q \in C([0, \infty), [0, \infty))$, $f \in C([0, \infty), R)$, $\tau \geq 0$ and $\sigma \geq 0$. They have studied the similar problem in [9] for equations of the form

$$\left[y(t) - p(t) y(t - \tau) \right]' \pm Q(t) G(y(t - \sigma)) = f(t)$$

for different ranges of $p \in C([0, \infty), R)$, where f, G, Q, τ and σ are same as above. In these results, the primary assumption is

$$\int_0^{\infty} Q(t) dt = \infty. \quad (1)$$

However, these results don't hold good for the critical case $p(t) \equiv 1$ or $p(t) \equiv -1$. In this paper, an attempt is made to study oscillatory and asymptotic behaviour of solutions of equations of the form

$$[y(t) \pm y(t - \tau)]' \pm Q(t) G(y(t - \sigma)) = f(t), \quad (2)$$

where $xG(x) > 0$ for $x \neq 0$ and G is nondecreasing. We assume that

$$\int_0^{\infty} |f(t)| dt < \infty.$$

In most of our results, the assumptions are stronger than (1). It seems that it is possible to obtain an example of a neutral differential equation in the critical case such that (1) holds but the equation admits a nonoscillatory solution which does not tend to zero as $t \rightarrow \infty$. A similar example is obtained in the discrete case by Yu and Wang [13].

Several open problems are stated in [2] (see 6.12.9 and 6.12.10, pp. 161) for equations of the type

$$[y(t) \pm y(t - \tau)]' + Q(t) y(t - \sigma) = 0.$$

In a recent paper [10], Piao has solved one open problem with an extra condition. Indeed, he showed that every nonoscillatory solution of

$$[y(t) + y(t - \tau)]' + Q(t) y(t - \sigma) = 0$$

tends to zero as $t \rightarrow \infty$ if (1) holds and $Q(t + \tau/n) \leq Q(t)$ for $t \in [0, \infty)$ where n is any fixed positive integer. However, Ladas and Sficas [6] have shown that every solution of

$$[y(t) - y(t - \tau)]' + Q(t) y(t - \sigma) = 0 \quad (3)$$

oscillates if (1) holds. Chuanxi and Ladas [1] posed the open problem that whether (1) is a necessary condition for the oscillation of all solutions of (3). In other words, whether

$$\int_0^{\infty} Q(t) dt < \infty$$

implies that (3) admits a nonoscillatory solution. Liu *et al* [7] (see also [11]) gave an example to show that the open problem is not true. They have shown that a stronger condition, viz,

$$\int_0^{\infty} t Q(t) dt < \infty$$

implies that (3) admits a bounded nonoscillatory solution.

By a solution of eq. (2) on $[T, \infty)$, $T \geq 0$, we mean a function $y \in C([T - r, \infty), R)$ such that $y(t) \pm y(t - \tau)$ is continuously differentiable and (2) is satisfied identically for $t \geq T$, where $r = \max\{\tau, \sigma\}$ and T is depending on y . Such a solution of (2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Sufficient conditions

In this section we obtain sufficient conditions so that every solution of (2) oscillates or tends to zero or to $\pm \infty$ as $t \rightarrow \infty$.

Theorem 2.1. *Suppose that*

$$G(x) + G(y) \geq \alpha G(x + y), \quad x > 0, y > 0$$

and

$$G(x) + G(y) \leq \beta G(x + y), \quad x < 0, y < 0, \quad (\text{H}_1)$$

where $\alpha > 0$ and $\beta > 0$ are constants. If

$$\int_{\tau}^{\infty} Q^*(t) dt = \infty, \quad (\text{H}_2)$$

where $Q^*(t) = \min\{Q(t), Q(t - \tau)\}$, then every solution of

$$[y(t) + y(t - \tau)]' + Q(t) G(y(t - \sigma)) = f(t) \quad (4)$$

oscillates or tends to zero as $t \rightarrow \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (4) on $[T_y, \infty)$, $T_y \geq 0$. Hence there exists a $t_0 > T_y$ such that $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + y(t - \tau)$$

and

$$w(t) = z(t) - F(t), \quad F(t) = \int_0^t f(s) ds, \quad (5)$$

for $t \geq t_1 > t_0 + r$, we obtain $z(t) > 0$ and

$$w'(t) = -Q(t)G(y(t - \sigma)) \leq 0 \quad (6)$$

for $t \geq t_1$. Hence $w(t) > 0$ or < 0 for $t \geq t_2 > t_1$. If $w(t) > 0$ for $t \geq t_2$, then $\lim_{t \rightarrow \infty} w(t)$ exists. If $w(t) < 0$ for $t \geq t_2$, then $0 < y(t) < z(t) < F(t)$ implies that $y(t)$ is bounded and hence $w(t)$ is bounded. Thus $\lim_{t \rightarrow \infty} w(t)$ exists. In either case $\lim_{t \rightarrow \infty} z(t)$ exists. We claim that $\lim_{t \rightarrow \infty} z(t) = 0$. If not, then $z(t) > \lambda > 0$ for $t \geq t_3 > t_2$. From (4) we obtain, for $t \geq t_4 > t_3 + \sigma + \tau$,

$$\begin{aligned} f(t) + f(t - \tau) &= z'(t) + z'(t - \tau) + Q(t)G(y(t - \sigma)) + Q(t - \tau)G(y(t - \tau - \sigma)) \\ &\geq z'(t) + z'(t - \tau) + Q^*(t)(G(y(t - \sigma)) + G(y(t - \tau - \sigma))) \\ &\geq z'(t) + z'(t - \tau) + \alpha Q^*(t)G(y(t - \sigma) + y(t - \tau - \sigma)) \\ &= z'(t) + z'(t - \tau) + \alpha Q^*(t)G(z(t - \sigma)) \\ &\geq z'(t) + z'(t - \tau) + \alpha Q^*(t)G(\lambda). \end{aligned}$$

Hence,

$$z(t) < z(t) + z(t - \tau) \leq z(t_4) + z(t_4 - \tau) \\ -\alpha G(\lambda) \int_{t_4}^t Q^*(s) ds + \int_{t_4}^t f(s) ds + \int_{t_4}^t f(s - \tau) ds$$

implies that $z(t) < 0$ for large t , a contradiction. Hence the claim holds. Consequently, $\lim_{t \rightarrow \infty} y(t) = 0$. Similarly, when $y(t) < 0$ for $t \geq t_0$, we obtain $\lim_{t \rightarrow \infty} y(t) = 0$. Thus the theorem is proved.

Remark. Clearly, (H₂) implies (1).

Remark. If $G(u) = u^\gamma$, where $\gamma > 0$ is a ratio of odd integers, then (H₁) is satisfied due to well-known inequalities

$$(|a| + |b|)^p \leq |a|^p + |b|^p, 0 < p \leq 1, \\ (|a| + |b|)^p \leq 2^{p-1} (|a|^p + |b|^p), p \geq 1,$$

where a and b are any two real numbers. If $G(u) = |u|^\gamma \operatorname{sgn} u$, where $\gamma > 0$, then (H₁) is also satisfied.

Remark. Clearly, (1) and $Q\left(t + \frac{\tau}{n}\right) \leq Q(t)$ for $t \in [0, \infty)$, where n is any fixed positive integer, imply (H₂) because $Q(t) \geq Q(t + \tau)$, $t \in [0, \infty)$. Hence Theorem 2.1 may be regarded as an improvement and generalization of the work in [10].

Theorem 2.2. *If (H₃) holds, then every solution of (4) oscillates or tends to zero as $t \rightarrow \infty$, where (H₃) is stated as follows:*

(H₃) *For every sequence $\langle \sigma_i \rangle \subset (0, \infty)$, $\sigma_i \rightarrow \infty$ as $i \rightarrow \infty$; and for every $\eta > 0$, such that the intervals $(\sigma_i - \eta, \sigma_i + \eta)$, $i = 1, 2, \dots$, and nonoverlapping,*

$$\sum_{i=0}^{\infty} \int_{\sigma_i - \eta}^{\sigma_i + \eta} Q(t) dt = \infty.$$

Proof. If $y(t)$ is a nonoscillatory solution of (4) on $[T_y, \infty)$, $T_y \geq 0$, then $y(t) > 0$ or < 0 for $t \geq T_0 > T_y$. Let $y(t) > 0$, $t \geq T_0$. Setting $z(t)$ and $w(t)$ as in (5) for $t \geq T_1 > T_0 + r$, we obtain (6). Hence $w(t) > 0$ or < 0 for $t \geq T_2 > T_1$. Proceeding as in Theorem 2.1, we show that $\lim_{t \rightarrow \infty} w(t)$ and $\lim_{t \rightarrow \infty} z(t)$ exist. Since $y(t) < z(t)$, then $\limsup_{t \rightarrow \infty} y(t)$ exists. We claim that $\limsup_{t \rightarrow \infty} y(t) = 0$. If not, then $\limsup_{t \rightarrow \infty} y(t) = \alpha$, $0 < \alpha < \infty$. Hence there exists a sequence $\langle t_n \rangle \subset [T, \infty)$, $T \geq T_2$, such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $y(t_n) \rightarrow \alpha$ as $n \rightarrow \infty$. Thus, for large $N_1 > 0$, $y(t_n) > \beta > 0$ if $n \geq N_1$. Since $y(t)$ is continuous at t_n , then there exists $\delta_n > 0$ such that $y(t) > \beta$ for $t \in (t_n - \delta_n, t_n + \delta_n)$ and $\liminf_{n \rightarrow \infty} \delta_n > 0$. Hence $\delta_n > \delta > 0$ for $n \geq N_2$. Choosing $N = \max \{N_1, N_2\}$, we obtain

$$\begin{aligned}
& \int_T^\infty Q(t) G(y(t - \sigma)) dt \\
& \geq \sum_{n=N}^\infty \int_{t_n - \delta_n + \sigma}^{t_n + \delta_n + \sigma} Q(t) G(y(t - \sigma)) dt \\
& \geq G(\beta) \sum_{n=N}^\infty \int_{t_n - \delta_n + \sigma}^{t_n + \delta_n + \sigma} Q(t) dt \\
& \geq G(\beta) \sum_{n=N}^\infty \int_{t_n - \delta + \sigma}^{t_n + \delta + \sigma} Q(t) dt
\end{aligned}$$

which implies that

$$\int_T^\infty Q(t) G(y(t - \sigma)) dt = \infty$$

by (H₃). However, integrating (6) we obtain

$$\int_T^t Q(s) G(y(s - \sigma)) ds = w(T) - w(t).$$

Thus

$$\int_T^\infty Q(t) G(y(t - \sigma)) dt < \infty,$$

a contradiction. Hence our claim holds. Consequently, $\lim_{t \rightarrow \infty} y(t) = 0$. The proof is similar for $y(t) < 0$, $t \geq T_0$. This completes the proof of the theorem.

Remark. Clearly, (H₃) implies (1). From the following example it is clear that (1) does not imply (H₃).

Remark. Theorem 2.2 holds if we assume that

$$-\infty < \liminf_{t \rightarrow \infty} F(t) < \limsup_{t \rightarrow \infty} F(t) < \infty$$

instead of

$$\int_0^\infty |f(t)| < \infty,$$

where $F(t)$ is given by (5). In the following we give an example to show that the condition (1) is not enough to arrive at the conclusion of Theorem 2.

Example. Consider

$$[y(t) + y(t - 1)]' + Q(t)y(t - 1) = h'(t) + h'(t - 1), \quad t \geq 1,$$

where

$$Q(t) = (e^2 + e) \left[e^{t+1} h(t-1) + e^2 \right]^{-1} > 0, \quad t \geq 1,$$

and $h \in C^1([0, \infty), [0, \infty))$ defined by

$$h(t) = \begin{cases} 0, & t \in [0, 1] \\ (t-1)^2(2-t)^2, & t \in [1, 2] \end{cases}$$

and extended to ∞ by the periodicity $h(t) = h(t+2)$, $t \geq 0$. Clearly, $y(t) = h(t) + e^{-t}$ is a positive solution of the equation with $\limsup_{t \rightarrow \infty} y(t) = \limsup_{t \rightarrow \infty} h(t) = \frac{1}{16}$. Further,

$$\int_1^\infty Q(t) dt > \sum_{n=1}^\infty \int_{2n-1}^{2n} Q(t) dt = \sum_{n=1}^\infty \frac{(e^2 + e)}{e^2} = \infty.$$

Thus (1) holds but the equation admits a nonoscillatory solution which does not tend to zero as $t \rightarrow \infty$. This suggests that stronger conditions are needed to show that every nonoscillatory solution of (4) tends to zero as $t \rightarrow \infty$.

Example. Consider

$$[y(t) + y(t - \pi)]' + (t - \pi)^{-1/2} y(t - \pi) = f(t), \quad t \geq 2\pi$$

where

$$f(t) = \frac{\cos t}{t^2} + \frac{2 \sin t}{(t - \pi)^3} - \frac{2 \sin t}{t^3} - \frac{\cos t}{(t - \pi)^2} - \frac{\sin t}{(t - \pi)^{5/2}}.$$

Since

$$Q^*(t) = \min \{Q(t), Q(t - \pi)\} = \min \left\{ \frac{1}{\sqrt{t - \pi}}, \frac{1}{\sqrt{t - 2\pi}} \right\} = \frac{1}{\sqrt{t - \pi}},$$

then

$$\int_{2\pi}^\infty Q^*(t) dt = \infty.$$

From Theorem 2.1 it follows that every solution of the equation oscillates or tends to zero as $t \rightarrow \infty$. In particular, $y(t) = \sin t/t^2$ is such a solution of the equation. We may note that Theorem 2.2 fails to hold for this equation because

$$\begin{aligned} \sum_{i=0}^\infty \int_{\sigma_i - \eta}^{\sigma_i + \eta} Q(t) dt &= 2 \sum_{i=0}^\infty \left[(\sigma_i + \eta - \pi)^{1/2} - (\sigma_i - \eta - \pi)^{1/2} \right] \\ &= 4\eta \sum_{i=0}^\infty \left[(\sigma_i + \eta - \pi)^{1/2} + (\sigma_i - \eta - \pi)^{1/2} \right]^{-1} \\ &< \infty \end{aligned}$$

for a sequence $\langle \sigma_i \rangle \equiv \langle i^4 \rangle \subset [2\pi, \infty)$.

Example. Consider

$$[y(t) + y(t-1)]' + y(t-1) \exp(y(t-1)) = f(t), \quad t \geq 1,$$

where

$$f(t) = -e^{-t} - e^{-t+1} + e^{-t+1} e^{e^{-t+1}}, \quad G(u) = ue^u.$$

Since $Q(t) \equiv 1$, then (H_3) holds trivially. Thus every nonoscillatory solution of the equation tends to zero as $t \rightarrow \infty$ by Theorem 2.2. In particular, $y(t) = e^{-t}$ is such a solution. However, Theorem 2.1 cannot be applied to this equation because

$$G(u+v) = (u+v)e^{u+v} > ue^u + ve^v = G(u) + G(v)$$

for $u > 0$ and $v > 0$ and hence (H_1) fails to hold.

Theorem 2.3. *Every unbounded solution of (4) oscillates. In other words, every nonoscillatory solution of (4) is bounded.*

Proof. Let $y(t)$ be an unbounded nonoscillatory solution of (4). Let $y(t) > 0$ for $t \geq t_0 > 0$. The case $y(t) < 0$ for $t \geq t_0 > 0$ may be dealt with similarly. Setting $z(t)$ and $w(t)$ as in (5) we obtain (6). If $w(t) > 0$ for large t , then $z(t)$ is bounded and hence $y(t)$ is bounded, a contradiction. If $w(t) < 0$ for large t and is bounded, then $z(t)$ is bounded and hence $y(t)$ is bounded, a contradiction. Thus $w(t) < 0$ for large t is unbounded. Consequently, $\lim_{t \rightarrow \infty} w(t) = -\infty$ which implies that $z(t) < 0$ for large t , a contradiction. Hence the theorem is proved.

Theorem 2.4. *If (1) holds, then every solution of*

$$[y(t) - y(t-\tau)]' + Q(t)G(y(t-\sigma)) = 0 \tag{7}$$

oscillates.

Proof. If possible, let $y(t)$ be a nonoscillatory solution of (7) on $[T_y, \infty)$. Without any loss of generality, we may assume that $y(t) > 0$ for $t \geq t_0 > T_y$. Setting $z(t) = y(t) - y(t-\tau)$ for $t \geq t_1 > t_0 + r$, we obtain

$$z'(t) = -Q(t)G(y(t-\sigma)) \leq 0.$$

Hence $z(t) > 0$ or < 0 for $t \geq t_2 > t_1$. If $z(t) > 0$, $t \geq t_2$, then

$$\int_{t_2}^{\infty} Q(t)G(y(t-\sigma)) dt < z(t_2) < \infty.$$

On the other hand, $z(t) > 0$ for $t \geq t_2$ implies that $y(t) > y(t-\tau)$ and hence $\liminf_{t \rightarrow \infty} y(t) > 0$. Thus $y(t) > \alpha > 0$ for $t \geq t_3 > t_2$. Then

$$\int_{t_3+\sigma}^{\infty} Q(t)G(y(t-\sigma)) dt > G(\alpha) \int_{t_3+\sigma}^{\infty} Q(t) dt$$

implies that

$$\int_{t_3+\sigma}^{\infty} Q(t)G(y(t-\sigma)) dt = \infty,$$

a contradiction. Therefore, $z(t) < 0$ for $t \geq t_2$, that is, $y(t) < y(t-\tau)$, $t \geq t_2$. Then $y(t)$ is bounded and hence $\liminf_{t \rightarrow \infty} y(t)$ and $\lim_{t \rightarrow \infty} z(t)$ exist. From Lemma 1.5.1 of [2] it follows that $\lim_{t \rightarrow \infty} z(t) = 0$, a contradiction because $z(t) < 0$ and monotonic decreasing. Hence the theorem is proved.

Remark. Theorem 2.4 generalizes Theorem 6.4.1 due to Gyori and Ladas [2].

Remark. In [7], an example is given to show that the condition (1) is not necessary for oscillation of all solutions of (7). They have proved that every bounded solution of (7) oscillates if and only if

$$\int_0^{\infty} t Q(t) dt = \infty. \quad (\text{H}_4)$$

We may note that (1) is stronger than (H₄).

Theorem 2.5. *If (H₃) holds, then every solution of*

$$[y(t) - y(t-\tau)]' + Q(t)G(y(t-\sigma)) = f(t) \quad (8)$$

oscillates or tends to zero as $t \rightarrow \infty$.

Proof. Let $y(t)$ be a solution of (8) on $[T_y, \infty)$, $T_y \geq 0$. If $y(t)$ oscillates, then there is nothing to prove. Let $y(t)$ be nonoscillatory. Hence $y(t) > 0$ or < 0 for $t \geq T_0 > T_y$. Let $y(t) > 0$ for $t \geq T_0$. Setting

$$z(t) = y(t) - y(t-\tau)$$

and

$$w(t) = z(t) - F(t), \quad F(t) = \int_0^t f(s) ds,$$

for $t \geq T_1 > T_0 + r$, we obtain

$$w'(t) = -Q(t)G(y(t-\sigma)) \leq 0.$$

If $w(t) > 0$ for $t \geq T_2 > T_1$, then $\lim_{t \rightarrow \infty} w(t)$ exists. If $w(t) < 0$ for $t \geq T_2$ is unbounded, then $\lim_{t \rightarrow \infty} w(t) = -\infty$ and hence $z(t) < 0$ for large t , that is, $y(t) < y(t-\tau)$ for large t . Thus $y(t)$ is bounded, which implies that $w(t)$ is bounded, a contradiction. Hence $w(t) < 0$ for $t \geq T_2$ is bounded. Then $\lim_{t \rightarrow \infty} w(t)$ exists. We claim that $\limsup_{t \rightarrow \infty} y(t) = 0$. If not, then $\limsup_{t \rightarrow \infty} y(t) = \alpha$, $0 < \alpha \leq \infty$. There exists a sequence $\langle t_n \rangle \subset [T_2, \infty)$ such that

$t_n \rightarrow \infty$ and $y(t_n) \rightarrow \alpha$ as $n \rightarrow \infty$. Hence $y(t_n) > \beta > 0$ for $n \geq N_1 > 0$. Since $y(t)$ is continuous at t_n , there exists $\delta_n > 0$ such that $y(t) > \beta$ for $t \in (t_n - \delta_n, t_n + \delta_n)$ and $\liminf_{n \rightarrow \infty} \delta_n > 0$. Then $\delta_n > \delta > 0$ for $n \geq N_2 > 0$. Choosing $N = \max\{N_1, N_2\}$ and then proceeding as in the proof of Theorem 2.2, we arrive at a contradiction due to (H_3) . Hence our claim holds. Thus $\lim_{t \rightarrow \infty} y(t) = 0$. Similarly, we may show that $\lim_{t \rightarrow \infty} y(t) = 0$ when $y(t) < 0$ for $t \geq T_0$. This completes the proof of the theorem.

Theorem 2.6. Suppose that (H_1) and (H_2) hold. If $y(t)$ is a solution of

$$[y(t) + y(t - \tau)]' - Q(t)G(y(t - \sigma)) = f(t), \quad (9)$$

then $y(t)$ oscillates or tends to zero as $t \rightarrow \infty$ or $\limsup_{t \rightarrow \infty} |y(t)| = +\infty$.

Proof. If possible, let $y(t)$ be nonoscillatory. Hence there exists $t_0 > 0$ such that $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting $z(t)$ and $w(t)$ as in (5), we obtain $z(t) > 0$ and

$$w'(t) = Q(t)G(y(t - \sigma)) \geq 0$$

for $t \geq t_1 > t_0 + r$. If $w(t) < 0$ for $t \geq t_2 > t_1$, then $\lim_{t \rightarrow \infty} w(t)$ exists and hence $\lim_{t \rightarrow \infty} z(t)$ exists. If $w(t) > 0$ for $t \geq t_2$ is bounded, then $\lim_{t \rightarrow \infty} w(t)$ and $\lim_{t \rightarrow \infty} z(t)$ exist. We claim that $\lim_{t \rightarrow \infty} z(t) = 0$. If not, then $z(t) > \lambda > 0$ for $t \geq t_3 > t_2$. Using (9) and (H_1) we may write, for $t \geq t_4 > t_3 + r$,

$$\begin{aligned} f(t) + f(t - \tau) &\leq z'(t) + z'(t - \tau) - Q^*(t)(G(y(t - \sigma)) + G(y(t - \tau - \sigma))) \\ &\leq z'(t) + z'(t - \tau) - \alpha Q^*(t)G(z(t - \sigma)) \\ &\leq z'(t) + z'(t - \tau) - \alpha G(\lambda)Q^*(t). \end{aligned}$$

This implies, due to (H_2) , that $\lim_{t \rightarrow \infty} z(t) = \infty$, a contradiction. Hence our claim holds. Since $z(t) > y(t)$, then $\lim_{t \rightarrow \infty} y(t) = 0$. If $w(t) > 0$, $t \geq t_2$, is unbounded, then $\lim_{t \rightarrow \infty} w(t) = +\infty$. Hence $y(t)$ is unbounded. Similarly, if $y(t) < 0$ for $t \geq t_0$, then $\lim_{t \rightarrow \infty} y(t) = 0$ or $y(t)$ is unbounded. Thus $\lim_{t \rightarrow \infty} y(t) = 0$ or $\limsup_{t \rightarrow \infty} |y(t)| = +\infty$. This completes the proof of the theorem.

COROLLARY 2.7

If (H_1) and (H_2) hold, then every bounded solution of (9) oscillates or tends to zero as $t \rightarrow \infty$.

This follows from Theorem 2.6.

Theorem 2.8. Let (H_3) hold. If $y(t)$ is a solution of (9), then it oscillates or tends to zero as $t \rightarrow \infty$ or $\limsup_{t \rightarrow \infty} |y(t)| = +\infty$.

The proof is similar to that of Theorem 2.2.

COROLLARY 2.9

If (H_3) holds, then every bounded solution of (9) oscillates or tends to zero as $t \rightarrow \infty$.

Theorem 2.10. Suppose that (H_3) holds. If $y(t)$ is a solution of

$$[y(t) - y(t - \tau)]' - Q(t) G(y(t - \sigma)) = f(t), \quad (10)$$

then $y(t)$ oscillates or tends to zero or $|y(t)| \rightarrow +\infty$ as $t \rightarrow \infty$.

The proof is similar to that of Theorem 2.5.

COROLLARY 2.11

If (H_3) holds, then every bounded solution of (10) oscillates or tends to zero as $t \rightarrow \infty$.

Remark. Some of our results partially answer the open problems stated in 6.12.9 and 6.12.10 [2].

3. Existence of nonoscillatory solutions

In this section we obtain necessary and sufficient conditions for the existence of a bounded positive/negative solution of the eq. (8).

Theorem 3.1. Let $f(t) \geq 0$ with

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} f(t) dt < \infty. \quad (11)$$

Then eq. (8) admits a bounded negative solution if and only if

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t) dt < \infty. \quad (H_5)$$

Proof. Suppose that eq. (8) admits a bounded negative solution $y(t)$ on $[T_y, \infty)$, $T_y \geq 0$. Setting $z(t) = y(t) - y(t - \tau)$ and $w(t) = z(t) + F(t)$, where

$$F(t) = \int_t^{\infty} f(s) ds,$$

for $t \geq t_0 > T_y + r$, we obtain $w(t) > z(t)$, $w(t)$ is bounded, $F(t) \rightarrow 0$ as $t \rightarrow \infty$ and

$$w'(t) = -Q(t) G(y(t - \sigma)) \geq 0. \quad (12)$$

Hence $w(t) > 0$ or < 0 for $t \geq t_1 > t_0$. If $w(t) > 0$ for $t \geq t_1$, then $\lim_{t \rightarrow \infty} w(t)$ exists and

hence $\lim_{t \rightarrow \infty} z(t)$ exists. Since $\liminf_{t \rightarrow \infty} y(t)$ (or $\limsup_{t \rightarrow \infty} y(t)$) exists, then $\lim_{t \rightarrow \infty} z(t) = 0$ by Lemma 1.5.1 in [2]. Thus $\lim_{t \rightarrow \infty} w(t) = 0$, a contradiction to the fact that $w(t) > 0$ and nondecreasing. Hence $w(t) < 0$ for $t \geq t_1$. Consequently, $\lim_{t \rightarrow \infty} w(t)$ exists and $z(t) < 0$ for $t \geq t_1$. There exists $\alpha > 0$ such that $y(t) < -\alpha$ for $t \geq t_1$. Integrating (12) from s to t ($s > t \geq t_2 \geq t_1 + \sigma$) and then taking limit as $s \rightarrow \infty$ we obtain

$$w(t) \leq G(-\alpha) \int_t^{\infty} Q(u) du,$$

that is,

$$y(t - \tau) > y(t) + \int_t^\infty f(s) \, ds - G(-\alpha) \int_t^\infty Q(s) \, ds.$$

Putting the values of t successively one may obtain

$$y(t - \tau) > y(t + n\tau) + \sum_{k=0}^n \int_{t+k\tau}^\infty f(s) \, ds - G(-\alpha) \sum_{k=0}^n \int_{t+k\tau}^\infty Q(s) \, ds.$$

Since $y(t)$ is bounded, then using (11) we get

$$\sum_{k=0}^\infty \int_{t_2+k\tau}^\infty Q(s) \, ds < \infty.$$

From this (H₅) follows.

Next we assume that (H₅) holds. It is possible to choose $m > 0$ sufficiently large such that

$$\sum_{k=m}^\infty \int_{k\tau}^\infty Q(t) \, dt < -\frac{1}{2G(-1)}$$

and

$$\sum_{k=m}^\infty \int_{k\tau}^\infty f(t) \, dt < \frac{1}{2},$$

that is,

$$\sum_{k=0}^\infty \int_{T+k\tau}^\infty Q(t) \, dt < \frac{-1}{2G(-1)} \text{ and } \sum_{k=0}^\infty \int_{T+k\tau}^\infty f(t) \, dt < \frac{1}{2},$$

where $T = m\tau$. Define

$$L(t) = \begin{cases} 0, & 0 \leq t < T \\ G(-1) \int_t^\infty Q(s) \, ds - \int_t^\infty f(s) \, ds, & t \geq T. \end{cases}$$

Hence $L(t) < 0$ for $t \geq T$. Further, define

$$u(t) = \begin{cases} 0, & 0 \leq t < T \\ \sum_{i=0}^\infty L(t - i\tau), & t \geq T. \end{cases}$$

Thus $u(t) < 0$ and $u(t) - u(t - \tau) = L(t)$, $t \geq T$. For $t \geq T$, there exists an integer $k \geq 0$ such that $T + k\tau \leq t < T + (k+1)\tau$. Hence $T \leq t - k\tau < T + \tau$ and $T - \tau \leq t - (k+1)\tau < T$. Then

$$\begin{aligned}
u(t) &= L(t) + L(t - \tau) + \cdots + L(t - k\tau) \\
&= G(-1) \int_t^\infty Q(s)ds - \int_t^\infty f(s)ds + \cdots + G(-1) \int_{t-k\tau}^\infty Q(s)ds - \int_{t-k\tau}^\infty f(s)ds \\
&\geq G(-1) \int_{T+k\tau}^\infty Q(s)ds - \int_{T+k\tau}^\infty f(s)ds + \cdots + G(-1) \int_T^\infty Q(s)ds - \int_T^\infty f(s)ds \\
&\geq G(-1) \sum_{k=0}^\infty \int_{T+k\tau}^\infty Q(t)dt - \sum_{k=0}^\infty \int_{T+k\tau}^\infty f(t)dt \\
&\geq -1.
\end{aligned}$$

Let $X = BC([T, \infty), R)$, the space of all real-valued, bounded continuous functions on $[T, \infty)$. It is a Banach space with respect to supremum norm. Let $K = \{x \in X : x(t) \geq 0, t \geq T\}$. For $u, v \in X$, we define $u \leq v$ if and only if $v - u \in K$. Thus X is a partially ordered Banach space (see pp. 30, [2]). Define

$$M = \{x \in X : u(t) \leq x(t) \leq 0\}.$$

Clearly, $u \in M$ and $u = \inf M$. If $\phi \subset A \subseteq M$, then $A = \{x \in M : u(t) \leq v(t) \leq x(t) \leq w(t) \leq 0, x \in A\}$. Setting $w_0(t) = \sup\{w(t) : x(t) \leq w(t) \leq 0, x \in A\}$, we notice that $w_0 = \sup A$ and $w_0 \in M$. Define $S : M \rightarrow X$ by

$$Sx(t) = \begin{cases} x(t - \tau) + \int_t^\infty Q(s)G(x(s - \sigma))ds - \int_t^\infty f(s)ds, & t \geq T_1 \\ \frac{tu(t)}{T_1 u(T_1)} Sx(T_1) + \left(1 - \frac{t}{T_1}\right) u(t), & T \leq t \leq T_1 \end{cases}, \quad (13)$$

where $T_1 = T + r$ and $r = \max\{\tau, \sigma\}$. Clearly, Sx is continuous on $[T, \infty)$ and $Sx(t) \leq 0$ for $t \geq T$. For $t \geq T_1$,

$$\begin{aligned}
Sx(t) &\geq x(t - \tau) + G(-1) \int_t^\infty Q(s)ds - \int_t^\infty f(s)ds \\
&\geq u(t - \tau) + L(t) = u(t).
\end{aligned}$$

For $T \leq t \leq T_1$,

$$Sx(t) \geq \frac{t}{T_1} u(t) + \left(1 - \frac{t}{T_1}\right) u(t) = u(t).$$

Thus $S : M \rightarrow M$. Moreover, $x_1 \geq x_2$ implies that $Sx_1 \geq Sx_2$. From the Knaster–Tarski fixed-point theorem (see pp. 30, [2]) it follows that S has a fixed point $y \in M$ which is a solution of (8) on $[T_1, \infty)$. Since $y(T_1 - \tau) \leq 0$, then from (13) it follows that

$$y(T_1) \leq y(T_1 - \tau) - \int_{T_1}^\infty f(s)ds < 0.$$

Thus $y(t) < 0$ for $t \in [T, T_1]$. For $t \in [T_1, T_1 + \tau]$, $y(t) < 0$. Consequently, $y(t) < 0$ for $t \geq T_1$. This completes the proof of the theorem.

Theorem 3.2. Let $f(t) \leq 0$ with

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} f(t) dt > -\infty.$$

Then eq. (8) admits a bounded positive solution if and only if (H_5) holds.

The proof is similar to that of Theorem 3.1.

Remark. Theorems 3.1 and 3.2 hold if $f(t) \equiv 0$. Hence, we have the following corollary.

COROLLARY 3.3

Every bounded solution of (7) oscillates if and only if

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t) dt = \infty.$$

This follows from Theorems 3.1 and 3.2.

Remark. We may note that

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} |f(t)| dt < \infty \text{ implies that } \int_0^{\infty} |f(t)| dt < \infty$$

and

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} Q(t) dt < \infty \text{ implies that } \int_0^{\infty} Q(t) dt < \infty.$$

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