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Common fixed points for weakly compatible maps

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Abstract. The purpose of this paper is to prove a common fixed point theorem, from the class of compatible continuous maps to a larger class of maps having weakly compatible maps without appeal to continuity, which generalizes the results of Jungck [3], Fisher [1], Kang and Kim [8], Jachymski [2], and Rhoades [9].

Keywords. Weakly compatible maps; fixed points.

1. Introduction

In 1976, Jungck [4] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem, which states that, 'let (X, d) be a complete metric space. If T satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y \in X$ where $0 \leq k < 1$, then T has a unique fixed point in X'. This theorem has many applications, but suffers from one drawback – the definition requires that T be continuous throughout X. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand Sessa [11] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further Jungck [5] introduced more generalized commutativity, the so-called compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors.

It has been known from the paper of Kannan [7] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. In 1998, Jungck and Rhoades [6] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but converse need not be true. In this paper, we prove a fixed point theorem for weakly compatible maps without appeal to continuity, which generalizes the result of Fisher [1], Jachymski [2], Kang and Kim [8] and Rhoades *et al* [9].

2. Preliminaries

DEFINITION 2.1 [6]

A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

Example 2.1. Let $X = [0, 3]$ be equipped with the usual metric space $d(x, y) = |x - y|$.

Define $f, g : [0, 3] \to [0, 3]$ by

$$f(x) = \begin{vmatrix} x & \text{if } x \in [0, 1) \\ \\ 3 & \text{if } x \in [1, 3] \end{vmatrix} \quad \text{and} \quad g(x) = \begin{vmatrix} 3 - x & \text{if } x \in [0, 1) \\ \\ 3 & \text{if } x \in [1, 3] \end{vmatrix} .$$

Then for any $x \in [1, 3]$, $fgx = gfx$, showing that f, g are weakly compatible maps on $[0, 3]$.

Example 2.2. Let $X = R$ and define $f, g : R \to R$ by $fx = x/3$, $x \in R$ and $gx = x^2$, $x \in R$. Here 0 and 1/3 are two coincidence points for the maps f and g. Note that f and g commute at 0, i.e. $fg(0) = gf(0) = 0$, but $fg(1/3) = f(1/9) = 1/27$ and $gf(1/3) = g(1/9) = 1/81$ and so f and g are not weakly compatible maps on R.

Remark 2.1. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $B, T : X \to X$ by $Bx = x$ if $x = 2$ or > 5, $Bx = 6$ if $2 < x \leq 5$, $Tx = x$ if $x = 2$, $Tx = 12$ if $2 < x \leq 5$, $Tx = x - 3$ if $x > 5$. The mappings B and T are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + (1/n)$, $n \geq 1$. Then $Tx_n \to 2$, $Bx_n = 2$, $TBx_n = 2$ and $BTx_n = 6$. But they are weakly compatible since they commute at coincidence point at $x = 2$.

3. Fixed point theorem

Let R^+ denote the set of non-negative real numbers and F a family of all mappings $\phi : (R^+)^5 \to R^+$ such that ϕ is upper semi-continuous, non-decreasing in each coordinate variable and, for any $t > 0$,

$$\phi(t, t, 0, \alpha t, 0) \leq \beta t, \phi(t, t, 0, 0, \alpha t) \leq \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \phi(t, t, \alpha_1 t, \alpha_2 t, \alpha_3 t) < t,$$

where $\gamma : R^+ \to R^+$ is a mapping and $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Lemma 3.1 [12]. *For every $t > 0$, $\gamma(t) < t$ if and only if* $\lim_{n\to\infty} \gamma^n(t) = 0$, *where γ^n denotes the n times composition of γ.*

Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions :

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{3.1}$$
$$d(Ax, By) \leq \phi(d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)) \tag{3.2}$$

for all $x, y \in X$, where $\phi \in F$. Then for arbitrary point x_0 in X, by (3.1), we choose a point x_1 such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, n = 0, 1, 2, 3, \ldots. \tag{3.3}$$

Lemma 3.2 $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, *where $\{y_n\}$ is the sequence in X defined by* (3.3).

Proof. Let $d_n = d(y_n, y_{n+1})$, $n = 0, 1, 2, \ldots$. Now, we shall prove the sequence $\{d_n\}$ is non-increasing in R^+, that is, $d_n \le d_{n-1}$ for $n = 1, 2, 3, \ldots$. From (3.2), we have

$$\begin{aligned}
d(Ax_{2n}, Bx_{2n+1}) &\le \phi(d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),\\
&\qquad d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n})).\\
d(y_{2n}, y_{2n+1}) &\le \phi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}),\\
&\qquad d(y_{2n+1}, y_{2n-1}))\\
&= \phi(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), 0, [d(y_{2n+1}, y_{2n})\\
&\qquad + d(y_{2n}, y_{2n-1})])\\
&= \phi(d_{2n-1}, d_{2n-1}, d_{2n}, 0, d_{2n} + d_{2n-1}). \qquad (3.4)
\end{aligned}$$

Suppose that $d_{n-1} < d_n$ for some n. Then, for some $\alpha < 2$, $d_{n-1} + d_n = \alpha d_n$. Since ϕ is non-increasing in each variable and $\beta < 1$ for some $\alpha < 2$. From (3.4), we have

$$d_{2n} \le \phi(d_{2n}, d_{2n}, d_{2n}, 0, \alpha d_{2n}) \le \beta d_{2n} < d_{2n}.$$

Similarly, we have $d_{2n+1} < d_{2n+1}$. Hence, for every $n, d_n \le \beta d_n < d_n$, which is a contradiction. Therefore, $\{d_n\}$ is a non-increasing sequence in R^+. Now, again by (3.2), we have

$$\begin{aligned}
d_1 = d(y_1, y_2) &= d(Ax_2, Bx_1)\\
&\le \phi(d(Sx_2, Tx_1), d(Ax_2, Sx_2), d(Bx_1, Tx_1),\\
&\quad d(Ax_2, Tx_1), d(Bx_1, Sx_2))\\
&= \phi(d(y_1, y_0), d(y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1))\\
&= \phi(d_0, d_1, d_0, d_0 + d_1, 0)\\
&\le \phi(d_0, d_0, d_0, 2d_0, d_0)\\
&= \gamma(d_0).
\end{aligned}$$

In general, we have $d_n \le \gamma^n(d_0)$, which implies that, if $d_0 > 0$, by Lemma 3.1,

$$\lim_{n\to\infty} d_n \le \lim_{n\to\infty} \gamma^n(d_0) = 0.$$

Therefore, we have $\lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, since $\{d_n\}$ is non-increasing, we have $\lim_{n\to\infty} d_n = 0$. This completes the proof.

Lemma 3.3. *The sequence* $\{y_n\}$ *defined by* (3.3) *is a Cauchy in* X.

Proof. By virtue of Lemma 3.2, it is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \ge 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \qquad (3.5)$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.5), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \qquad (3.6)$$

Then for each even integer $2k$, we have

$$\begin{aligned}
\epsilon &\le d(y_{2n(k)}, y_{2m(k)})\\
&\le d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).
\end{aligned}$$

By Lemma 3.2 and (3.6), it follows that

$$d(y_{2n(k)}, y_{2m(k)}) \to \epsilon \text{ as } k \to \infty. \tag{3.7}$$

By the triangle inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)})$$

and

$$\begin{aligned}&|d(y_{2n(k)}, y_{2m(k)-1} - d(y_{2n(k)}, y_{2m(k)})|\\ &\le d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1}).\end{aligned}$$

From Lemma 3.2 and eq. (3.7), as $k \to \infty$,

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \epsilon \text{ and } d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon. \tag{3.8}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}d(y_{2n(k)}, y_{2m(k)}) &\le d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)})\\ &= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1})\\ &\le d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(Sx_{2m(k)}, Tx_{2n(k)+1}),\\ &\quad d(Ax_{2m(k)}, Sx_{2m(k)}), d(Bx_{2n(k)+1}, Tx_{2n(k)+1}),\\ &\quad d(Ax_{2m(k)}, Tx_{2n(k)+1}), d(Bx_{2n(k)+1}, Sx_{2m(k)}))\\ &= d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(y_{2m(k)-1}, y_{2n(k)}),\\ &\quad d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)}),\\ &\quad d(y_{2n(k)+1}, y_{2m(k)-1})).\end{aligned} \tag{3.9}$$

Since ϕ is upper semi continuous, as $k \to \infty$ as in (3.8), by Lemma 3.2, eqs (3.7), (3.8) and (3.9) we have

$$\epsilon \le \phi(\epsilon, 0, 0, \epsilon, \epsilon) < \gamma(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X and so is $\{y_n\}$. This completes the proof.

Theorem 3.1. *Let (A, S) and (B, T) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying* (3.1) *and* (3.2). *Then A, B, S and T have a unique common fixed point in X.*

Proof. By Lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there exists a point z in X such that $\lim_{n\to\infty} y_n = z$. $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and $\lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$ i.e.,

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Since $B(X) \subset S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (3.2),

$$\begin{aligned}d(Au, z) &\le d(Au, Bx_{2n-1}) + d(Bx_{2n-1}, z)\\ &\le \phi(d(Su, Tx_{2n-1}), d(Au, Su), d(Bx_{2n-1}, Tx_{2n-1}),\\ &\qquad d(Au, Tx_{2n-1})d(Bx_{2n-1}, Su)).\end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\begin{aligned} d(Au, z) &\le \phi(0, d(Au, Su), 0, d(Au, z), d(z, Su)) \\ &= \phi(0, d(Au, z), 0, d(Au, z), 0) \le \beta d(Au, z), \end{aligned}$$

where $\beta < 1$. Therefore $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists a point $v \in X$ such that $z = Tv$. Then, again using (3.2),

$$\begin{aligned} d(z, Bv) = d(Au, Bv) &\le \phi(d(Su, Tv), d(Au, Su), d(Bv, Tv), d(Au, Tv), d(Bv, Su)) \\ &= \phi(0, 0, d(Bv, z), 0, d(Bv, z)) \le \phi(t, t, t, t, t) < t, \end{aligned}$$

where $t = d(z, Bv)$. Therefore $z = Bv = Tv$. Thus $Au = Su = Bv = Tv = z$. Since pair of maps A and S are weakly compatible, then $ASu = SAu$ i.e, $Az = Sz$. Now we show that z is a fixed point of A. If $Az \neq z$, then by (3.2),

$$\begin{aligned} d(Az, z) = d(Az, Bv) &\le \phi(d(Sz, Tv), d(Az, Sz), d(Bv, Tv), \\ &\qquad d(Az, Tv), d(Bv, Sz)) \\ &= \phi(d(Az, z), 0, 0, d(Az, z), d(Az, z)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(Az, z). \end{aligned}$$

Therefore, $Az = z$. Hence $Az = Sz = z$.

Similarly, pair of maps B and T are weakly compatible, we have $Bz = Tz = z$, since

$$\begin{aligned} d(z, Bz) = d(Az, Bz) &\le \phi(d(Sz, Tz), d(Az, Sz), \\ &\quad d(Bz, Tz), d(Az, Tz), d(Bz, Sz)) \\ &= \phi(d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(z, Tz) = d(z, Bz). \end{aligned}$$

Thus $z = Az = Bz = Sz = Tz$, and z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that z and w, $z \neq w$, are common fixed points of A, B, S and T. Then by (3.2), we obtain

$$\begin{aligned} d(z, w) = d(Az, Bw) &\le \phi(d(Sz, Tw), d(Az, Sz), d(Bw, Tw), d(Az, Tw), d(Bw, Sz)) \\ &= \phi(d(z, w), 0, 0, d(z, w), d(z, w)) \\ &\le \phi(t, t, t, t, t) < t, \text{ where } t = d(z, w). \end{aligned}$$

Therefore, $z = w$. The following corollaries follow immediately from Theorem 3.1.

COROLLARY 3.1

Let (A, S) *and* (B, T) *be weakly compatible pairs of self maps of a complete metric space* (X, d) *satisfying* (3.1), (3.3) *and* (3.10)

$$d(Ax, By) \le hM(x, y), 0 \le h < 1, x, y \in X, \textit{ where}$$

$$M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Ax, Ty) + d(By, Sx)]/2\}. \tag{3.10}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\phi : [0, \infty)^5 \to [0, \infty)$ defined by

$$\phi(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, {}^1\!/_2(x_4 + x_5)\}.$$

Since $\phi \in F$, we can apply Theorem 3.1 and deduce the Corollary.

COROLLARY 3.2

Let (A, S) and (B, T) be weakly compatible pairs of self maps of a complete metric space (X, d) satisfying (3.1), (3.3) *and* (3.11).

$$\begin{aligned} d(Ax, By) \le h \max\{&d(Ax, Sx), d(By, Ty), {}^1\!/_2 d(Ax, Ty), \\ &{}^1\!/_2 d(By, Sx), d(Sx, Ty)\} \text{ for all } x, y \text{ in } X, \text{ where } 0 \le h < 1. \end{aligned} \tag{3.11}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. We consider the function $\phi : [0, \infty)^5 \to [0, \infty)$ defined by $\phi(x_1, x_2, x_3, x_4, x_5) = h \max\{x_1, x_2, x_3, {}^1\!/_2\, x_4, {}^1\!/_2\, x_5\}$. Since $\phi \in F$, we can apply Theorem 3.1 to obtain this Corollary.

Remark 3.2. Theorem 3.1 generalizes the result of Jungck [3] by using weakly compatible maps without continuity at S and T. Theorem 3.1 and Corollary 3.2 also generalize the result of Fisher [1] by employing weakly compatible maps instead of commutativity of maps. Further the results of Jachymski [2], Kang and Kim [8], Rhoades *et al* [9] are also generalized by using weakly compatible maps.

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