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On a Tauberian theorem of Hardy and Littlewood

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Abstract. In this paper, we give a simple alternative proof of a Tauberian theorem of Hardy and Littlewood (Theorem E stated below, [3]).

Keywords. Abel's theorem; Tauberian theorem; Hardy-Littlewood Tauberian theorem; divergent series.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be an infinite series of real terms. Let

$$0 \leq \lambda_0 < \lambda_1 <, \ldots, \lambda_n \to \infty$$

and let $\sum a_n \mathrm{e}^{-\lambda_n x}$ be convergent for all $x > 0$. If

$$f(x) = \sum a_n \mathrm{e}^{-\lambda_n x} \to s$$

as $x \to 0$, then we say that $\sum a_n$ is summable (A, λ_n) to s. When $\lambda_n = n$, the method (A, λ_n) reduces to the classical method summability (A), named after Abel.

It is a famous result due to Abel that if $\sum a_n$ is convergent to s, then $\sum a_n$ is summable (A) to s. That the converse is not necessarily true is evident from the example of the series

$$1 - 1 + 1 - 1 \cdots$$

which is summable (A) to $\frac{1}{2}$, but not convergent. The question naturally arises as to whether one can determine a suitable restriction or restrictions on the general term a_n so that $\sum a_n$ will be convergent to s whenever it is summable (A). The first answer to this question was given by Tauber in 1897 in the form of the following theorem.

Theorem A [7]. *If $\sum a_n$ is summable* (A) *to s and $na_n = o(1)$, then $\sum a_n$ is convergent to s.*

A generalization of Theorem A to the set-up of summability (A, λ_n) was proved by Landau [4].

Another significant generalization of Theorem A was obtained by Littlewood in 1910 in the form of

Theorem B. *If $\sum a_n$ is summable* (A) *to s, and $na_n = O(1)$, then $\sum a_n$ is convergent to s.*

In fact Littlewood proved the following more general theorem.

Theorem C [5]. *If* $\sum \mu_n$ *is a series of positive terms such that, as* $n \to \infty$,

$$\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \to \infty, \quad \mu_n/\lambda_n \to 0,$$

$\sum a_n \mathrm{e}^{-\lambda_n x} \to s$ *as* $x \to 0$, *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then $\sum a_n$ *is convergent to* s.

Littlewood had stated that Theorem C is true even without the restriction: $\mu_n/\lambda_n \to 0$. This result is stated below as Theorem C*. It was proved in 1928 by Ananda-Rau [1]. A simple alternative proof was supplied by Bosanquet (see Hardy [2]).

Theorem C*. *If* $\sum \mu_n$ *is a series of positive terms such that* $\lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n \to \infty$ *as* $n \to \infty$, $\sum a_n \mathrm{e}^{-\lambda_n x} \to s$ *as* $x \to 0$ *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \tag{1.1}$$

then $\sum a_n$ *is convergent to* s.

Littlewood also conjectured [5] that the following theorem is true.

Theorem D. *If* $\lambda_1 > 0, \ \lambda_{n+1}/\lambda_n \geq \theta > 1 \ (n = 1, 2, ...)$, *and*

$$f(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n x} \to s \ \text{ as } \ x \to 0,$$

then $\sum a_n$ *converges to* s.

The truth of this conjecture was proved by Hardy and Littlewood [3][1]. Theorems of this kind are called 'high indices' theorems, as distinguished from 'Tauberian' theorems, since in such theorems no restriction is needed to be imposed upon the general term a_n of the series in question, excepting, of course, that $\sum a_n \mathrm{e}^{-\lambda_n x}$ is convergent for every $x > 0$. Such a theorem shows that the method (A, λ_n) with the type of λ_n involved does not sum any series which is not convergent, and therefore shows the 'ineffectiveness' of the method (A, λ_n).

Hardy and Littlewood first established Theorem D in the special case in which

$$a_n = O(1)$$

and then, by further analysis, derived Theorem D itself. This is an instance of a Tauberian theorem leading to a high indices theorem. Thus Hardy and Littlewood first established the following Tauberian theorem.

Theorem E. *If* $\lambda_1 > 0, \ \lambda_{n+1}/\lambda_n \geq \theta > 1 \ (n = 1, 2, \ldots)$,

$$f(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n x} \to s \ \text{ as } \ x \to 0,$$

[1] For a proof of Theorem D due to A E Ingham, see [2], proof of Theorem 114, where too, the result has been obtained via a Tauberian theorem.

and $a_n = O(1)$, *then* $\sum a_n$ *is convergent to* s.

It should be observed that Theorem E is included in the theorem of Ananda-Rau, in which no extra restriction is imposed on λ_n, in view of the fact that whenever $\lambda_{n+1}/\lambda_n \geq \theta > 1$, and $a_n = O(1)$, $a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right)$. On the other hand, under the hypotheses of Theorem C*, (1.1) implies: $s_n = O(1)$, and hence $a_n = O(1)$ (see Lemma 2 in the sequel).

The object of the present paper is to give an alternative proof of Theorem E which is quite straightforward, not requiring Lemmas 1 and 2 of Hardy and Littlewood [3].

2. Lemmas

We shall need the following lemmas.

Lemma 1 [5]. *If, as* $y \to 0$, $\psi(y) \to s$, *and for every positive integer* r,

$$y^r \psi^{(r)}(y) = O(1),$$

then for every positive integer r, $y^r \psi^{(r)}(y) = o(1)$.

Lemma 2 [4].[2] *If* $0 < \lambda_1 < \lambda_2 <, \ldots, \lambda_n \to \infty$, *as* $n \to \infty$, $f(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n x} = O(1)$ *as* $x \to 0$, *and*

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),$$

then

$$s_n = a_1 + a_2 + \cdots + a_n = O(1).$$

Lemma 3 [3]. *If* $\lambda_1 > 0$ *and*

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \theta > 1 \quad (n = 1, 2, \ldots)$$

then, for $r = 1, 2, \ldots,$

$$\sum_{n=1}^{\infty} \lambda_n^r \mathrm{e}^{-\lambda_n x} = O(x^{-r}).$$

3. Proof of Theorem E

We may assume, without loss of generality, that $s = 0$. Thus $f(x) = o(1)$ as $x \to 0$. Also, since $a_n = O(1)$, for $r = 1, 2, \ldots,$

$$\begin{aligned} x^r f^{(r)}(x) &= (-1)^r x^r \sum_{n=1}^{\infty} a_n \lambda_n^r \mathrm{e}^{-\lambda_n x} \\ &= O\left(x^r \sum_{n=1}^{\infty} \lambda_n^r \mathrm{e}^{-\lambda_n x}\right) \\ &= O(1), \end{aligned}$$

[2] As remarked by Ananda-Rau in [1], the argument in Landau [4], pp. 13–14, has only to be slightly modified to yield the result of Lemma 2.

by Lemma 3. Hence, by Lemma 1,

$$x^r f^{(r)}(x) = o(1).$$

Since

$$f(x) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-\lambda_n x} = \sum_{n=1}^{\infty} s_n \left(\mathrm{e}^{-\lambda_n x} - \mathrm{e}^{-\lambda_{n+1} x}\right) = x \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} \mathrm{e}^{-xu} \mathrm{d}u,$$

we have

$$\begin{aligned}(-1)^r x^r f^{(r)}(x) &= x^{r+1} \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^r \mathrm{e}^{-xu} \mathrm{d}u - r x^r \sum_{n=1}^{\infty} s_n \int_{\lambda_n}^{\lambda_{n+1}} u^{r-1} \mathrm{e}^{-xu} \mathrm{d}u \\ &= V_r - r V_{r-1}, \quad \text{say.}\end{aligned}$$

Hence, by Lemma 1,

$$\begin{aligned} V_r &= r V_{r-1} + o(1) \\ &= r(r-1) V_{r-2} + o(r) + o(1) \\ &= r(r-1)(r-2) V_{r-3} + o(r(r-1)) + o(r) + o(1) \\ &= \ldots. \\ &= r! f(x) + o(r(r-1)\ldots 2) + \cdots + o(1), \end{aligned}$$

so that[3]

$$\begin{aligned} \frac{V_r}{r!} &= f(x) + o\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!}\right) \\ &= f(x) + o(1) \\ &= o(1). \end{aligned} \tag{3.1}$$

This can be explicitly written as

$$\frac{1}{r!}|F(x)| = \frac{1}{r!} \left| x^{r+1} \sum s_n \int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt} \mathrm{d}t \right| \to 0 \quad (r = 1, 2, \ldots) \tag{3.2}$$

as $x \to 0$. If s_n does not converge to zero, there exists a positive constant h such that $|s_n| > h$ for an infinite number of values of n. Let m be any one of these values. We shall show that, when r exceeds a sufficiently large positive integer r_0,

$$\overline{\lim}_{x \to 0} \frac{1}{r!} |F(x)| \geq \delta > 0,$$

where δ is a positive constant. This will contradict (3.2), and hence we will conclude that $\sum a_n$ converges to zero, which is required to be proved.

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[3] In Hardy and Littlewood [3], (2.41) should be replaced by our (3.1) $\frac{V_r}{r!} = o(1)$; line 4 from the top on p. 225 should be replaced by: $r! \sum_{n=0}^{\infty} s_n w_n = V_r$ so that $\sum_{n=0}^{\infty} s_n w_n = \frac{V_r}{r!} = o(1)$. For similar alterations needed in the papers [1], [5] and [8], see Pati [6].

Now, by the hypotheses of Theorem E and Lemma 2, $s_n = O(1)$ and hence

$$\begin{aligned}|F(x)| \quad &\geq \quad |s_m|x^{r+1}\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t - \left|x^{r+1}\sum_{n=1}^{\infty}(s_n - s_m)\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t\right| \\ &> \quad |s_m|x^{r+1}\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t - Kx^{r+1}\sum_{m+1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t \\ &\qquad -Kx^{r+1}\sum_{n=1}^{m-1}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t, \end{aligned} \tag{3.3}$$

where K is a positive constant. We choose

$$x = \frac{2r}{\lambda_{m+1} + \lambda_m}.$$

Then, for fixed r, $x \to 0$ iff $m \to \infty$. Since ([5], p. 440)

$$\lim_{x\to 0} x^{r+1}\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t = r!,$$

$$\frac{1}{r!}\overline{\lim}_{x\to 0}\ |s_m|x^{r+1}\sum_{n=1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t \geq h. \tag{3.4}$$

We now use the transformation $u = xt$, so that, for $t = \lambda_{m+1}$,

$$u = r\frac{2\lambda_{m+1}}{\lambda_{m+1} + \lambda_m} = r(1+\eta),$$

where

$$\eta = \frac{\lambda_{m+1} - \lambda_m}{\lambda_{m+1} + \lambda_m}. \tag{3.5}$$

Thus the second term in (3.3) gives

$$\overline{\lim}_{x\to 0}\ x^{r+1}\sum_{m+1}^{\infty}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t \leq \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u}\mathrm{d}u. \tag{3.6}$$

The third term in (3.3) gives

$$\overline{\lim}_{x\to 0}\ x^{r+1}\sum_{1}^{m-1}\int_{\lambda_n}^{\lambda_{n+1}} t^r \mathrm{e}^{-xt}\mathrm{d}t \leq \int_{0}^{r(1-\eta)} u^r \mathrm{e}^{-u}\mathrm{d}u, \tag{3.7}$$

where η is as defined in (3.5).

Combining (3.4), (3.6) and (3.7) we have

$$\frac{1}{r!}\overline{\lim}_{x\to 0}|F(x)| \geq h - \frac{K}{r!}\left[\int_{0}^{r(1-\eta)} u^r \mathrm{e}^{-u}\mathrm{d}u + \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u}\mathrm{d}u\right]. \tag{3.8}$$

We show below that

$$I_1 \equiv \int_{0}^{r(1-\eta)} u^r \mathrm{e}^{-u}\mathrm{d}u < K_1 r^r \mathrm{e}^{-r} \tag{3.9}$$

and

$$I_2 \equiv \int_{r(1+\eta)}^{\infty} u^r \mathrm{e}^{-u} \mathrm{d}u < K_2 r^r \mathrm{e}^{-r}, \tag{3.10}$$

where the K in each inequality denotes a positive constant, independent of r.

Proof of (3.9). We have

$$I_1 = -u^r \mathrm{e}^{-u}]_0^{r(1-\eta)} + r \int_0^{r(1-\eta)} u^{r-1} \mathrm{e}^{-u} \mathrm{d}u.$$

Hence

$$\int_0^{r(1-\eta)} \left(\frac{r}{u} - 1\right) u^r \mathrm{e}^{-u} \mathrm{d}u = r^r (1-\eta)^r \mathrm{e}^{-r(1-\eta)}.$$

Now, since $0 < \eta < 1$, $u \leq r(1-\eta)$ implies:

$$\frac{r}{u} - 1 \geq \frac{\eta}{1-\eta},$$

so that

$$\frac{\eta}{1-\eta} I_1 \leq r^r \mathrm{e}^{-r} \left[(1-\eta)\mathrm{e}^{\eta}\right]^r < r^r \mathrm{e}^{-r},$$

since

$$\mathrm{e}^{\eta} < 1 + \eta + \eta^2 + \cdots = \frac{1}{1-\eta}.$$

Thus

$$I_1 < K_1 r^r \mathrm{e}^{-r},$$

where

$$K_1 = \frac{1-\eta}{\eta} = \frac{2\lambda_m}{\lambda_{m+1} - \lambda_m} \leq \frac{2}{\theta - 1} \quad (\theta > 1).$$

Proof of (3.10). We have

$$I_2 = -u^r \mathrm{e}^{-u}]_{r(1+\eta)}^{\infty} + r \int_{r(1+\eta)}^{\infty} u^{r-1} \mathrm{e}^{-u} \mathrm{d}u.$$

Hence

$$\begin{aligned} \int_{r(1+\eta)}^{\infty} \left(1 - \frac{r}{u}\right) u^r \mathrm{e}^{-u} \mathrm{d}u &= r^r \mathrm{e}^{-r} \left(\frac{1+\eta}{\mathrm{e}^{\eta}}\right)^r \\ &< r^r \mathrm{e}^{-r}; \end{aligned}$$

since

$$u \geq r(1+\eta) \quad \text{implies:} \quad 1 - \frac{r}{u} \geq \frac{\eta}{1+\eta},$$

we have

$$\frac{\eta}{1+\eta} I_2 < r^r \mathrm{e}^{-r},$$

so that

$$I_2 < K_2 r^r \mathrm{e}^{-r},$$

where

$$K_2 = \frac{1+\eta}{\eta} = \frac{2\lambda_{m+1}}{\lambda_{m+1} - \lambda_m} = \frac{2}{1 - \frac{\lambda_m}{\lambda_{m+1}}} \le \frac{2\theta}{\theta - 1} \quad (\theta > 1).$$

Hence, from (3.8), (3.9) and (3.10), we have

$$\overline{\lim}_{x\to 0} \frac{1}{r!}|F(x)| \ge h - 2K\frac{\theta+1}{\theta-1}\frac{r^r \mathrm{e}^{-r}}{r!}.$$

Since by Stirling's theorem,

$$\frac{r^r \mathrm{e}^{-r}}{r!} \sim \frac{1}{\sqrt{2\pi}} r^{-\frac{1}{2}},$$

taking $r > r_0$, a sufficiently large positive integer, we have

$$\overline{\lim}_{x\to 0} \frac{1}{r!}|F(x)| \ge \delta > 0,$$

which contradicts (3.2). Hence our assumption that $\{s_n\}$ does not converge to 0 is false.

This completes the proof of Theorem E.

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