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Periodic and boundary value problems for second order differential equations

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Abstract. In this paper we study second order scalar differential equations with Sturm–Liouville and periodic boundary conditions. The vector field $f(t,x,y)$ is Caratheodory and in some instances the continuity condition on x or y is replaced by a monotonicity type hypothesis. Using the method of upper and lower solutions as well as truncation and penalization techniques, we show the existence of solutions and extremal solutions in the order interval determined by the upper and lower solutions. Also we establish some properties of the solutions and of the set they form.

Keywords. Upper solution; lower solution; order interval; truncation map; penalty function; Caratheodory function; Sobolev space; compact embedding; Dunford–Pettis theorem; Arzela–Ascoli theorem; extremal solution; periodic problem; Sturm–Liouville boundary conditions.

1. Introduction

The method of upper and lower solutions offers a powerful tool to establish the existence of multiple solutions for initial and boundary value problems of the first and second order. This method generates solutions of the problem, located in an order interval with the upper and lower solutions serving as bounds. In fact the method is often coupled with a monotone iterative technique which provides a constructive way (amenable to numerical treatment) to generate the extremal solutions within the order interval determined by the upper and lower solutions.

In this paper we employ this technique to study scalar nonlinear periodic and boundary value problems. The overwhelming majority of the works in this direction, assume that the vector field is continuous in all variables and they look for solutions in the space $C^2(0,b)$. We refer to the books by Bernfeld–Lakshmikantham [2] and Gaines–Mawhin [6] and the references therein. The corresponding theory for discontinuous (at least in the time variable t) nonlinear differential equations is lagging behind. It is the aim of this paper to contribute in the development of the theory in this direction. Dealing with discontinuous problems, leads to Caratheodory or monotonicity conditions and to Sobolev spaces of functions of one variable. It is within such a framework that we will conduct our investigation in this paper. We should mention that an analogous study for first order problems can be found in Nkashama [18].

2. Sturm–Liouville problems

Let $T = [0, b]$. We start by considering the following second order boundary value problem:

$$\left\{ \begin{array}{l} -x''(t) = f(t, x(t), x'(t)) \text{ a.e. on } T \\ (B_0x)(0) = \nu_0,\ (B_1x)(b) = \nu_1 \end{array} \right\}. \tag{1}$$

Here $(B_0x)(0) = a_0x(0) - c_0x'(0)$ and $(B_1x)(b) = a_1x(b) + c_1x'(b)$, with $a_0, c_0, a_1, c_1 \geq 0$ and $a_0(a_1b + c_1) + c_0a_1 \neq 0$. Note that if $c_0 = c_1 = \nu_0 = \nu_1 = 0$, then we have the Dirichlet (or Picard in the terminology of Gaines–Mawhin [6]) problem. The vector field $f(t, x, y)$ is not continuous, but only a Caratheodory function; i.e. it is measurable in $t \in T$ and continuous in $(x, y) \in \mathbb{R} \times \mathbb{R}$ (later the continuity in y will be replaced by a monotonicity condition). Hence $x''(\cdot)$ is not continuous, but only an $L^1(T)$-function. Recently Nieto–Cabada [17] considered a special case of (1) with f independent of y. Also there is the work of Omari [19] where f is continuous.

We will be using the Sobolev spaces $W^{1,1}(T)$ and $W^{2,1}(T)$. It is well known (see for example Brezis [3], p. 125), that $W^{1,1}(T)$ is the space of absolutely continuous functions and $W^{2,1}(T)$ is the space of absolutely continuous function whose derivative is absolutely continuous too.

DEFINITION

A function $\psi \in W^{2,1}(T)$ is said to be a 'lower solution' for problem (1) if

$$\left\{ \begin{array}{l} -\psi''(t) \leq f(t, \psi(t), \psi'(t)) \text{ a.e. on } T \\ (B_0\psi)(0) \leq \nu_0,\ (B_1\psi)(b) \leq \nu_1 \end{array} \right\}. \tag{2}$$

A function $\phi \in W^{2,1}(T)$ is said to be an 'upper solution' for problem (1) if the inequalities in (2) are reversed.

For the first existence theorem we will need the following hypotheses:

$\mathrm{H(f)}_1$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x, y \in \mathbb{R}$, $t \to f(t, x, y)$ is measurable;
(ii) for every $t \in T$, $(x, y) \to f(t, x, y)$ is continuous;
(iii) for every $r > 0$ there exists $\gamma_r \in L^1(T)$ such that $|f(t, x, y)| \leq \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$ with $|x|, |y| \leq r$.

H_0: There exists an upper solution ϕ and a lower solution ψ such that $\psi(t) \leq \phi(t)$ for every $t \in T$ and there exists $h \in C(\mathbb{R}_+, (0, \infty))$ such that $|f(t, x, y)| \leq h(|y|)$ for all $t \in T$ and all $x, y \in \mathbb{R}$ with $\psi(t) \leq x \leq \phi(t)$ and $\int_\lambda^\infty \frac{r\mathrm{d}r}{h(r)} > \max_{t\in T} \phi(t) - \min_{t\in T} \psi(t)$, with $\lambda = \frac{\max[|\psi(0)-\phi(b)|, |\psi(b)-\phi(0)|]}{b}$.

Remark. The second part of hypothesis H_0 (the growth condition on f), is known as the 'Nagumo growth condition' and guarantees an *a priori* L^∞-bound for $x'(\cdot)$. More precisely, if H_0 holds, then there exists $N_1 > 0$ (depending only on ϕ, ψ, h) such that for every $x \in W^{2,1}(T)$ solution of $-x''(t) = f(t, x(t), x'(t))$ a.e. on T with $\psi(t) \leq x(t) \leq \phi(t)$ for all $t \in T$, we have $|x'(t)| \leq N_1$ for all $t \in T$ (the proof of this, is the same (with minor modifications) with that of Lemma 1.4.1, p. 26 of Bernfeld–Lakshmikantham [2]).

We introduce the order interval $K = [\psi, \phi] = \{x \in W^{1,2}(T) : \psi(t) \leq x(t) \leq \phi(t)$ for all $t \in T\}$ and we want to know if there exists a solution of (1) within the order interval K. Also we are interested on the existence of the least and the greatest solutions of (1) within K ('extremal solutions'). The next two theorems solve these problems. In theorem 1 we prove the existence of a solution in K and in theorem 2 we prove the existence of extremal solutions within K. Although the hypotheses in both theorems are the same, we decided to present them separately for reasons of clarity, since otherwise the proof would have been too long.

Theorem 1. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (1) *has a solution* $x \in W^{2,1}(T)$ *within the order interval* $K = [\psi, \phi]$.

Proof. As we already mentioned in a previous remark, the Nagumo growth condition (see H_0) implies the existence of $N_1 > 0$ (depending only on ψ, ϕ, h) such that $|x'(t)| \leq N_1$ for all $f \in T$, for every $x \in W^{2,1}(T)$ solution of (1) belonging in K. Set $N = 1 + \max\{N_1, ||\psi'||_\infty, ||\phi'||_\infty\}$. Also define the truncation operator τ: $W^{1,1}(T) \to W^{1,1}(T)$ by

$$\tau(x)(t) = \begin{cases} \phi(t) & \text{if} \quad \phi(t) \leq x(t) \\ x(t) & \text{if} \quad \psi(t) \leq x(t) \leq \phi(t) \\ \psi(t) & \text{if} \quad x(t) \leq \psi(t) \end{cases}.$$

The fact that $\tau(x) \in W^{1,1}(T)$ can be found in Gilbarg–Trudinger [8] (p. 145) and we know that

$$\tau(x)'(t) = \begin{cases} \phi'(t) & \text{if} \quad \phi(t) \leq x(t) \\ x'(t) & \text{if} \quad \psi(t) \leq x(t) \leq \phi(t) \\ \psi'(t) & \text{if} \quad x(t) \leq \psi(t) \end{cases}.$$

Also we define the truncation at N function $q_N \in C(\mathbb{R})$ by

$$q_N(x) = \begin{cases} N & \text{if} \quad N \leq x \\ x & \text{if} \quad -N \leq x \leq N \\ -N & \text{if} \quad x \leq -N \end{cases}$$

and the penalty function $u : T \times \mathbb{R} \to \mathbb{R}$ by

$$u(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \leq x \\ 0 & \text{if} \quad \psi(t) \leq x \leq \phi(t) \\ x - \psi(t) & \text{if} \quad x \leq \psi(t) \end{cases}.$$

Then we consider the following Sturm–Liouville problem

$$\left\{ \begin{array}{l} -x''(t) = f(t, \tau(x)(t), q_N(\tau(x)'(t))) - u(t, x(t)) \text{ a.e. on } T \\ (B_0 x)(0) = \nu_0, (B_1 x)(b) = \nu_1 \end{array} \right\}. \tag{3}$$

Denote by S the solution set of (3).

Claim # 1. $S \subseteq K = [\psi, \phi]$. Let $x \in S$. Then we have

$$-x''(t) = f(t, \tau(x)(t), q_N(\tau(x)'(t))) - u(t, x(t)) \text{ a.e. on } T. \tag{4}$$

Also since $\psi \in W^{2,1}(T)$ is a lower solution of (1), we have

$$\psi''(t) \geq -f(t, \psi(t), \psi'(t)) \qquad \text{a.e. on } T. \tag{5}$$

Adding (4) and (5), we obtain

$$\begin{aligned}\psi''(t) - x''(t) &\geq f(t, \tau(x)(t), q_N(\tau(x)'(t))) \\ &\quad - f(t, \psi(t), \psi'(t)) - u(t, x(t)) \qquad \text{a.e. on } T.\end{aligned}$$

Multiplying with $(\psi - x)_+(t)$ and integrating over $T = [0, b]$, we have

$$\begin{aligned}&\int_0^b (\psi''(t) - x''(t))(\psi - x)_+(t)\mathrm{d}t \\ &\quad \geq \int_0^b [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))] \\ &\quad (\psi - x)_+(t)\mathrm{d}t - \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t.\end{aligned} \tag{6}$$

From the integration by parts formula (Green's identity), we have

$$\begin{aligned}&\int_0^b (\psi''(t) - x''(t))(\psi - x)_+(t)\mathrm{d}t \\ &\quad = (\psi' - x')(b)(\psi - x)_+(b) - (\psi' - x')(0)(\psi - x)_+(0) \\ &\quad - \int_0^b (\psi' - x')(t)(\psi - x)'_+(t)\mathrm{d}t.\end{aligned} \tag{7}$$

Using the boundary conditions for x and ψ at $t = 0$, we have

$$\begin{aligned}&a_0\psi(0) - c_0\psi'(0) \leq \nu_0 = a_0x(0) - c_0x'(0) \\ &\quad \Rightarrow -c_0(\psi'(0) - x'(0)) \leq -a_0(\psi(0) - x(0)).\end{aligned}$$

If $c_0 = 0$, then $\psi(0) \leq x(0)$ and so $(\psi - x)_+(0) = 0$. Therefore $-(\psi' - x')(0)(\psi - x)_+(0) = 0$.

If $c_0 > 0$, then $-(\psi'(0) - x'(0)) \leq -\frac{a_0}{c_0}(\psi(0) - x(0)) \Rightarrow -(\psi'(0) - x'(0))(\psi - x)_+(0) \leq -\frac{a_0}{c_0}(\psi(0) - x(0))(\psi - x)_+(0)$. Thus if $(\psi(0) - x(0)) \geq 0$, we have $-(\psi' - x')(0)(\psi - x)_+(0) \leq 0$ and if $(\psi(0) - x(0)) < 0$, we have $(\psi - x)_+(0) = 0$ and so $-(\psi' - x')(0)(\psi - x)_+(0) = 0$. Therefore we always have

$$-(\psi' - x')(0)(\psi - x)_+(0) \leq 0. \tag{8}$$

From the boundary condition at $t = b$, we have

$$\begin{aligned}&a_1\psi(b) + c_1\psi'(b) \leq \nu_1 = a_1x(b) + c_1x'(b) \\ &\quad \Rightarrow c_1(\psi'(b) - x'(b)) \leq -a_1(\psi(b) - x(b)).\end{aligned}$$

Then arguing as above, we infer that

$$(\psi' - x')(b)(\psi - x)_+(b) \leq 0. \tag{9}$$

Finally recall that

$$(\psi - x)'(t) = \begin{cases} (\psi - x)'(t) & \text{if} \quad x(t) \le \psi(t) \\ 0 & \text{if} \quad x(t) \ge \psi(t) \end{cases}$$

(see Gilbarg–Trudinger [8], p. 145). Hence it follows that

$$\int_0^b (\psi' - x')(t)(\psi - x)'_+(t)\mathrm{d}t = \int_{\{x \le \psi\}} [(\psi - x)'(t)]^2 \mathrm{d}t \ge 0. \tag{10}$$

Using (8), (9), (10) in (7), we deduce that

$$\int_0^b (\psi'' - x'')(t)(\psi - x)_+(t)\mathrm{d}t \le 0. \tag{11}$$

Also note that

$$\begin{aligned} &\int_0^b [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))](\psi - x)_+(t)\mathrm{d}t \\ &\quad = \int_{\{x \le \psi\}} [f(t, \tau(x)(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t))](\psi - x)(t)\mathrm{d}t \\ &\quad = \int_{\{x \le \psi\}} [f(t, \psi(t), \psi'(t)) - f(t, \psi(t), \psi'(t))](\psi - x)(t)\mathrm{d}t = 0 \end{aligned} \tag{12}$$

since on the set $\{t \in T : x(t) \le \psi(t)\}$, we have $\tau(x)(t) = \psi(t)$ and $\tau(x)'(t) = \psi'(t)$. Using (11) and (12) in (6), we have that

$$\begin{aligned} 0 &\le \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t = \int_{\{x \le \psi\}} u(t, x(t))(\psi - x)(t)\mathrm{d}t \\ &= \int_0^b -(\psi - x)_+^2(t)\mathrm{d}t \le 0 \end{aligned}$$

(recall the definition of $u(t, x)$). So $\psi(t) \le x(t)$ for all $t \in T$. In a similar way we show that $x(t) \le \phi(t)$ for all $t \in T$. Therefore $S \subseteq K$ as claimed.

Claim # 2. S is nonempty. This will be proved by means of Schauder's fixed point theorem. To this end let $D = \{x \in W^{2,1}(T) : (B_0 x)(0) = \nu_0,\ (B_1 x)(b) = \nu_1\}$ and let $\hat{L} : D \subseteq L^1(T) \to L^1(T)$ be defined by $\hat{L}x = -x''$ for every $x \in D$. First note that for every $h \in L^1(T)$ the boundary value problem

$$\left\{ \begin{array}{l} -x''(t) + x(t) = h(t) \text{ a.e. on } T \\ (B_0 x)(0) = \nu_0,\ (B_1 x)(b) = \nu_1 \end{array} \right\} \tag{13}$$

has a unique solution $x \in W^{2,1}(T)$. Indeed uniqueness of the solution is clear. For the existence, note that if $h \in C(T)$, then it follows from corollary 3.1 of Mönch [15]. In the general case, let $h \in L^1(T)$ and take $h_n \in C(T)$ such that $h_n \to h$ in $L^1(T)$ as $n \to \infty$. For each h_n, $n \ge 1$, the solution $x_n(\cdot)$ of (13) is given by $x_n(t) = u(t) + \int_0^b G(t, s)(x_n(s) - h_n(s))\mathrm{d}s$, where $u \in C^2(T)$ is the unique solution of $x''(t) = 0$ $t \in T$, $(B_0 x)(0) = \nu_0$, $(B_1 x)(b) = \nu_1$ and $G(t, s)$ is the Green's function for the problem $x'' = g(t)$ $t \in T$, $(B_0 x)(0) = 0$, $(B_1 x)(b) = 0$ for $g \in C(T)$ given. From the proof of corollary 3.1 (b) of Mönch [15], we know that $\sup_{n \ge 1} ||x_n||_\infty \le \sup_{n \ge 1} ||\eta_n||_\infty$, where $\eta_n \in C^2(T)$ is the unique solution of $\eta''(t) = -h_n(t)$ $t \in T$, $(B_0 \eta)(0) = |\nu_0|$, $(B_1 \eta)(b) = |\nu_1|$. We know that

$\eta_n(t) = u(t) - \int_0^b G(t,s)h_n(s)\mathrm{d}s$ and so it follows that $\sup_{n\geq 1} ||\eta_n||_\infty < \infty$. Hence $\{x_n\}_{n\geq 1}$ is bounded in $C(T)$. Since $-x_n''(t) = h_n(t) - x_n(t)$, $t \in T$, it follows that $\{x_n''\}_{n\geq 1}$ is uniformly integrable. From Brezis [3] (p. 132) we know that the norm $||\cdot||_{W^{2,1}(T)}$ is equivalent to the norm $||x|| = ||x||_1 + ||x''||_1$. Therefore $\{x_n''\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Since $W^{2,1}(T)$ embeds continuously in $C^1(T)$ and compactly in $L^1(T)$ and by the Dunford–Pettis compactness criterion, by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $C^1(T)$ (hence $x_n'(t) \to x'(t)$ for all $t \in T$), $x_n \to x$ in $L^1(T)$ and $x_n'' \xrightarrow{w} y$ in $L^1(T)$ as $n \to \infty$. Evidently $y = x''$. So in the limit as $n \to \infty$, we have $-x''(t) + x(t) = h(t)$ a.e. on T, $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Therefore we have proved that $R(I + \hat{L}) = L^1(T)$.

Next let $x_1, x_2 \in D$ and $x = x_1 - x_2$. Define

$$T_+ = \{t \in T : x(t) > 0\} \quad \text{and} \quad T_- = \{t \in T : x(t) < 0\}$$

both open sets in T. For $\lambda > 0$ we have

$$\begin{aligned}
&\int_0^b |x(t) - \lambda x''(t)|\mathrm{d}t \geq \int_{T_+} |x(t) - \lambda x''(t)|\mathrm{d}t + \int_{T_-} |x(t) - \lambda x''(t)|\mathrm{d}t \\
&\geq \int_{T_+} (x(t) - \lambda x''(t))\mathrm{d}t - \int_{T_-} (x(t) - \lambda x''(t))\mathrm{d}t \\
&= \int_{T_+} x(t)\mathrm{d}t - \int_{T_-} x(t)\mathrm{d}t - \lambda \int_{T_+} x''(t)\mathrm{d}t + \lambda \int_{T_-} x''(t)\mathrm{d}t \\
&= \int_0^b |x(t)|\mathrm{d}t - \lambda\left[\int_{T_+} x''(t)\mathrm{d}t - \int_{T_-} x''(t)\mathrm{d}t\right].
\end{aligned}$$

Let (a,c) be a connected component of T_+. Then $x(a) = x(c) = 0$ and $x(t) > 0$ for all $t \in (a,c)$. Thus $x'(a) \geq 0$ and $x'(c) \leq 0$ and from this it follows that $\int_a^c x''(t)\mathrm{d}t = x'(c) - x'(a) \leq 0$. Therefore we deduce that $\int_{T_+} x''(t)\mathrm{d}t \leq 0$. Similarly we show that $\int_{T_-} x''(t)\mathrm{d}t \geq 0$. So finally we have $-\lambda[\int_{T_+} x''(t)\mathrm{d}t - \int_{T_-} x''(t)\mathrm{d}t] \geq 0$ and thus we obtain

$$\begin{aligned}
&\int_0^b |x(t) - \lambda x''(t)|\mathrm{d}t \geq \int_0^b |x(t)|\mathrm{d}t \\
&\qquad \Rightarrow ||x_1 + \lambda\hat{L}x_1 - (x_2 + \lambda\hat{L}x_2)||_1 \geq ||x_1 - x_2||_1.
\end{aligned}$$

This last inequality together with the fact that $R(I + \hat{L}) = L^1(T)$, implies that $(I + \hat{L})^{-1}$: $L^1(T) \to D \subseteq L^1(T)$ is well-defined and nonexpansive (is the resolvent of the m-accretive operator $\hat{L}$; see Vrabie [21], Lemma 1.1.5, p. 20). For $k > 0$ consider the set

$$\Gamma_k = \{x \in D : ||x||_1 + ||x''||_1 \leq k\}.$$

Recalling that $||x||_1 + ||x''||_1$ is an equivalent norm on $W^{2,1}(T)$ see Brezis [3], p. 132), it follows that Γ_k is bounded in $W^{2,1}(T)$ and since the latter embeds compactly in $L^1(T)$, we conclude that Γ_k is relatively compact in $L^1(T)$. So from Vrabie [21] (Proposition 2.2.1, p. 56), we have that $(I + \hat{L})^{-1}$ is a compact operator. If $C \subseteq L^1(T)$ is bounded and $u \in C$, let $x = (I + \hat{L})^{-1}(u)$. Then $-x'' + x = u$ and from what we proved we have

$$||x||_1 \leq ||-x'' + x||_1 \leq \sup[||u||_1 : u \in C] = |C| < \infty.$$

So $||x''||_1 \leq 2|C|$ and thus we conclude that $(I + \hat{L})^{-1}(C)$ is bounded in $W^{2,1}(T)$. Since the latter embeds compactly in $W^{1,1}(T)$, we infer that $(I + \hat{L})^{-1}(C)$ is relatively compact in $W^{1,1}(T)$. Moreover, if $u_n \to u$ in $L^1(T)$ as $n \to \infty$ and $x_n = (I + \hat{L})^{-1}(u_n)$, then

$x_n \to x = (I+\hat{L})^{-1}(u)$ in $L^1(T)$ as $n \to \infty$ (recall that $(I+\hat{L})^{-1}$ is continuous on $L^1(T)$) and $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$. Exploiting the compact embedding of $W^{2,1}(T)$ in $W^{1,1}(T)$, we have that $x_n \to x$ in $W^{1,1}(T)$, i.e. $(I+\hat{L})^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is continuous, hence a compact operator.

Now let $H : W^{1,1}(T) \to L^1(T)$ be defined by

$$H(x)(\cdot) = f(\cdot, \tau(x)(\cdot), q_N(\tau(x)'(\cdot))) - u(\cdot, x(\cdot)) + x(\cdot).$$

We will show that $H(\cdot)$ is bounded and continuous. Boundedness is a straightforward consequence of hypothesis $\mathrm{H(f)}_1$ (iii) and of the definition of the penalty function $u(t,x)$. So we need to show that $H(\cdot)$ is continuous. To this end let $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ and $x_n'(t) \to x'(t)$ a.e. on T as $n \to \infty$. Hence we have $\tau(x_n)(t) \to \tau(x)(t)$ for every $t \in T$ and $q_N(\tau(x_n)'(t)) \to q_N(\tau(x)'(t))$ a.e. on T as $n \to \infty$. Note that $\{x_n\}_{n\geq 1}$ is bounded in $C(T)$ (since $W^{1,1}(T)$ embeds continuously in $C(T)$) and so by virtue of hypotheses $H(f)_1$, the continuity of $u(t,\cdot)$ and the dominated convergence theorem, we have that $H(x_n) \to H(x)$ in $L^1(T)$ as $n \to \infty$ and so we have proved the continuity of $H : W^{1,1}(T) \to L^1(T)$.

Then consider the operator $(I+\hat{L})^{-1}H : W^{1,1}(T) \to W^{1,1}(T)$. Evidently this operator is continuous (in fact compact), $(I+\hat{L})^{-1}H(D) \subseteq D$ and $\overline{(I+\hat{L})^{-1}H(D)}$ is compact in $W^{1,1}(T)$ (since for every $x \in W^{1,1}(T)$, $||H(x)||_1 \leq k^*$ with $k^* = ||\gamma_r||_1 + b\max\{||\phi||_\infty, ||\psi||_\infty\}$ and $r = \max\{||\phi||_\infty, ||\psi||_\infty, N\}$). Since $D \subseteq W^{1,1}(T)$ is closed, convex, we can apply Schauder's fixed point theorem (see Gilbarg–Trudinger [8], Corollary 10.2, p. 222), to obtain $x = (I+\hat{L})^{-1}H(x)$. Then $-x'' + x = H(x)$, $x \in D$; i.e. $x \in W^{2,1}(T)$ is a solution of (3). This proves the nonemptiness of S.

To conclude the proof of the theorem, note that if $x \in S$, then from claim # 1 we have $\psi(t) \leq x(t) \leq \phi(t)$ for all $t \in T$. So we have $\tau(x)(t) = x(t)$, $\tau(x)'(t) = x'(t)$ and $u(t, x(t)) = 0$. Also recalling that $|x'(t)| \leq N$ for all $t \in T$, we also have that $q_N(x'(t)) = x'(t)$. Therefore finally

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x'(t)) & \text{a.e. on } T \\ (B_0x)(0) = \nu_0, \ (B_1x)(b) = \nu_1 & \end{array} \right\}$$

i.e., $x \in W^{2,1}(T)$ solves problem (1) and $x \in [\psi, \phi]$.

Now we will improve the conclusion of theorem 1, by showing that problem (1) has extremal solutions in the order interval $K = [\psi, \phi]$; i.e. there exist a least solution $x_* \in K$ and a greatest solution $x^* \in K$ of (1), such that if $x \in W^{2,1}(T)$ is any other solution of (1) in K, we have $x_*(t) \leq x(t) \leq x^*(t)$ for all $t \in T$.

Theorem 2. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (1) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

Proof. Let S_1 be the set of solutions of (1) contained in the order interval $K = [\psi, \phi]$. From theorem 1 we have that $S_1 \neq \phi$. First we will show that S_1 is a directed set (i.e. if $x_1, x_2 \in S_1$, then there exists $x \in S_1$ such that $x_1(t) \leq x(t)$ and $x_2(t) \leq x(t)$ for all $t \in T$). To this end let $x_1, x_2 \in S_1$ and let $x_3 = \max\{x_1, x_2\}$. Since $x_1, x_2 \in W^{2,1}(T)$, we have that $x_3 \in W^{1,1}(T)$ (see Gilbarg–Trudinger [8], Lemma 7.6, p. 145). Let $\tau_k : W^{1,1}(T) \to W^{1,1}(T)$ be defined by

$$\tau_k(x)(t) = \begin{cases} \phi(t) & \text{if} \quad \phi(t) \leq x(t) \\ x(t) & \text{if} \quad x_k(t) \leq x(t) \leq \phi(t) \\ x_k(t) & \text{if} \quad x(t) \leq x_k(t) \end{cases} \qquad k = 1, 2, 3\,.$$

Also we introduce the penalty function $u_3 : T \times \mathbb{R} \to \mathbb{R}$ and the truncation function $q_N : \mathbb{R} \to \mathbb{R}$ ($N = 1 + \max\{N_1, ||\psi'||_\infty, ||\phi'||_\infty\}$) defined by

$$u_3(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \le x \\ 0 & \text{if} \quad x_3(t) \le x \le \phi(t) \\ x - x_3(t) & \text{if} \quad x \le x_3(t) \end{cases}$$

and

$$q_N(x) = \begin{cases} N & \text{if} \quad N \le x \\ x & \text{if} \quad -N \le x \le N\,. \\ -N & \text{if} \quad x \le -N \end{cases}$$

Then we consider the following boundary value problem:

$$\left\{ \begin{array}{l} -x''(t) = f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t))) + \displaystyle\sum_{k=1}^{2} |f(t, \tau_k(x)(t), q_N(\tau_k(x)'(t))) \\ \qquad\qquad -f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t)))| - u_3(t, x(t)) \qquad \text{a.e. on } T \\ (B_0x)(0) = \nu_0, (B_1x)(b) = \nu_1 \end{array} \right\}. \tag{14}$$

Arguing as in the proof of theorem 1, we establish that problem (14) has a nonempty solution set. We will show that this solution set is in the order interval $[x_3, \phi]$. So let $x \in W^{2,1}(T)$ be a solution of (14). We have

$$\begin{aligned} x_1''(t) - x''(t) &= f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t))) - f(t, x_1(t), x_1'(t)) \\ &+ \sum_{k=1}^{2} |f(t, \tau_k(x)(t), q_N(\tau_k(x)'(t))) - f(t, \tau_3(x)(t), q_N(\tau_3(x)'(t)))| \\ &- u_3(t, x(t)) \quad \text{a.e on } T. \end{aligned}$$

Multiply with $(x_1 - x)_+(t)$ and then integrate over $T = [0, b]$. Using the definition of the truncation functions $r_k (k = 1, 2, 3), q_N$ and boundary conditions, we obtain

$$\begin{aligned} &\int_0^b u_3(t, x(t))(x_1 - x)_+(t)\mathrm{d}t \ge 0 \\ &\Rightarrow \int_0^b (x_1 - x)_+^2(t)\mathrm{d}t = 0 \qquad \text{(recall the definition of } u_3\text{)} \\ &\Rightarrow x_1(t) \le x(t) \quad \text{for all } t \in T. \qquad \text{a.e. on } I. \end{aligned}$$

In a similar way we show that $x_2(t) \le x(t)$ and $x(t) \le \phi(t)$ for all $t \in T$. Therefore we conclude that every solution $x(\cdot) \in W^{2,1}(T)$ of (14) is located in the order interval $[x_3, \phi]$. Hence $\tau_k(x)(t) = x(t)$ and $\tau_k(x)'(t) = x'(t)$ for all $t \in T$ and all $k \in \{1, 2, 3\}$ and $u_3(t, x(t)) = 0$. Thus

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), q_N(x'(t))) & \text{a.e. on } T \\ (B_0x)(0) = \nu_0, (B_1x)(b) = \nu_1 & \end{array} \right\}.$$

As we already mentioned the Nagumo growth condition (see (H_0)) guarantees that $|x'(t)| \le N$ for all $t \in T$ and so $q_N(x'(t)) = x'(t)$. Therefore $x \in S_1$ and we have proved that S_1 is a directed set.

Now let C be a chain in S_1. Then since $C \subseteq L^1(T)$, according to Dunford–Schwartz [5] (Corollary IV.II.7, p. 336), we can find $\{x_n\}_{n\geq 1} \subseteq C$ such that $\sup C = \sup_{n\geq 1} x_n$. Then by the monotone convergence theorem, we have that $x_n \to x$ in $L^1(T)$ as $n \to \infty$ and so $\psi(t) \leq x(t) \leq \phi(t)$ a.e. on T. For every $n \geq 1$ we know that $||x_n||_\infty \leq \max\{||\psi||_\infty, ||\phi||_\infty\} = r_0$ and $\sup_{n\geq 1} ||x_n'||_\infty \leq N_1$. So if $r = \max\{r_0, N_1\}$, by virtue of hypothesis $\mathrm{H(f)}_1$ (iv) we have that $||x_n''(t)|| \leq \gamma_r(t)$ a.e. on T. Thus $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$ and $\{x_n''\}_{n\geq 1}$ is uniformly integrable. So as before exploiting the compact embedding of $W^{2,1}(T)$ in $W^{1,1}(T)$, the continuous embedding of $W^{2,1}(T)$ in $C^1(T)$ and invoking the Dunford–Pettis theorem, we may assume that $x_n \to x$ in $W^{1,1}(T)$, $x_n(t) \to x(t)$, $x_n'(t) \to x'(t)$ for all $t \in T$ and $x_n'' \xrightarrow{w} y$ in $L^1(T)$ as $n \to \infty$. It is easy to see that $y = x''$ and $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Also from the dominated convergence theorem, we have that $-x''(\cdot) = f(\cdot, x(\cdot), x'(\cdot))$ in $L^1(T)$. Hence $-x''(t) = f(t, x(t), x'(t))$ a.e. on T, $(B_0x)(0) = \nu_0$, $(B_1x)(b) = \nu_1$. Thus $x = \sup C \in S_1$. Using Zorn's lemma, we infer that S_1 has a maximal element $x^* \in S_1$. Since S_1 is directed, it follows that x^* is unique and is the greatest element of S_1 in $[\psi, \phi]$. Similarly we can prove the existence of a least solution x_* of (1) in $[\psi, \phi]$. Therefore (1) has extremal solutions in $K = [\psi, \phi]$.

3. Periodic problems

In this section, we focus our attention on the 'periodic problem':

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x'(t)) & \text{a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b) & \end{array} \right\}. \tag{15}$$

This problem was studied using the method of upper and lower solutions by Gaines–Mawhin [6], Leela [14], Lakshmikantham–Leela [13], Nieto [16], Cabada–Nieto [4], Omari–Trombetta [20] and Gao–Wang [7]. From these works only Gaines–Mawhin, Cabada–Nieto, Omari–Trombetta and Gao–Wang had a vector field depending also on x' and moreover, among these papers only Cabada–Nieto and Gao–Wang used Caratheodory type conditions on $f(t, x, y)$ with Lipschitz continuity in the y-variable in Cabada–Nieto (see Theorem 2.2 in Cabada–Nieto [4]). Theorem 3 below extends all these results. A similar result using a different method of proof, was obtained by Gao–Wang [7].

DEFINITION

A function $\psi \in W^{2,1}(T)$ is said to be a 'lower solution' of (18) if

$$\left\{ \begin{array}{ll} -\psi''(t) \leq f(t, \psi(t), \psi'(t)) & \text{a.e. on } T \\ \psi(0) = \psi(b),\ \psi'(0) \geq \psi'(b) & \end{array} \right\}.$$

A function $\phi \in W^{2,1}(T)$ is said to be an 'upper solution' of (18) if it satisfies the reverse inequalities.

Theorem 3. *If hypotheses* $\mathrm{H(f)}_1$ *and* H_0 *hold, then problem* (18) *has a solution* $x \in W^{2,1}(T)$ *within the order interval* $K = [\psi, \phi]$.

Proof. The proof is the same as that of theorem 1, with some minor modifications. Note that in this case $D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and $\hat{L} : D \subseteq L^1(T) \to L^1(T)$ is defined by $\hat{L}x = -x''$ for all $x \in D$. The rest of the proof is identical and only in the applications of the integration by parts formula (Green's identity), we use the periodic conditions instead of the Sturm–Liouville boundary conditions.

Next we look for the extremal solutions in the order interval $[\psi, \phi]$ of the periodic problem (18). For this we introduce a different set of hypotheses on the vector field $f(t,x,y)$.

$\mathrm{H(f)}_2$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x, y \in \mathbb{R}$, $t \to f(t,x,y)$ is measurable;
(ii) there exists $M > 0$ such that for almost all $t \in T$ and all $y \in [-N, N]$, $x \to f(t,x,y) + Mx$ is strictly increasing (recall that $N = 1 + \max\{N_1, ||\psi'||_\infty, ||\phi'||_\infty\}$);
(iii) there exists $k \in L^1(T)$ such that $|f(t,x,y_1) - f(t,x,y_2)| \leq k(t)|y_1 - y_2|$ a.e. on T for all $x, y_1, y_2 \in \mathbb{R}$;
(iv) for every $r > 0$, there exists $\gamma_r \in L^1(T)$ such that $|f(t,x,y)| \leq \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}, |x|, |y| \leq r$.

Remark. Hypothesis $\mathrm{H(f)}_2$ (ii) allows for jump discontinuities (countably many) in the x–variable. However note that for every $x : T \to \mathbb{R}$ measurable, $t \to f(t, x(t), y)$ is measurable. This is an immediate consequence of Theorem 1.9, p. 32 of Appell–Zabrejko [1]. Moreover since $(t,y) \to f(t,x(t),y)$ is a Caratheodory function, is jointly measurable and so in particular superpositionally measurable; if $y : T \to \mathbb{R}$ is measurable, then so is $t \to f(t, x(t), y(t))$.

Theorem 4. *If hypotheses* $\mathrm{H(f)}_2$ *and* H_0 *hold, then problem* (18) *has extremal solutions in the order interval* $K = [\psi, \phi]$.

Proof. Without any loss of generality, we may assume that $M > 1$. Then for any $z \in K = [\psi, \phi]$, we consider the following periodic problem

$$\left\{ \begin{array}{l} -x''(t) = f(t, z(t), q_N(\tau(x)'(t))) - u(t, x(t)) + M(z(t) - x(t)) \quad \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array} \right\}. \tag{16}$$

We will establish the existence of solutions for problem (18). So let $D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and let $L : D \subseteq L^1(T) \to L^1(T)$ be defined by $Lx = -x'' + (M-1)x$. As in the proof of theorem 1, we can check that L is invertible and $L^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is a compact, linear operator. Also as before we define $H : W^{1,1}(T) \to L^1(T)$ by

$$H(x)(t) = f(t, z(t), q_N(\tau(x)'(t))) - u(t, x(t)) - x(t) + Mx(t).$$

This map is bounded and continuous. Note that $x \in D$ solves (19) if and only if $x = L^{-1}H(x)$. As in the proof of theorem 1, the existence of a fixed point of $L^{-1}H$ is implied by corollary 10.2, p. 222 of Gilbarg–Trudinger [8], since $L^{-1}(D) \subseteq D$ and $\overline{L^{-1}H(D)}$ is compact in $W^{1,1}(T)$. So problem (19) has solutions.

Now we will show that any solution of (19) is within $K = [\psi, \phi]$. Indeed we have:

$$\left\{ \begin{array}{l} \psi''(t) - x''(t) \geq f(t, z(t), q_N(\tau(x)'(t))) - f(t, \psi(t), \psi'(t)) \\ \qquad\qquad -u(t, x(t)) + M(z(t) - x(t)) \quad \text{a.e. on } T \\ (\psi - x)(0) = (\psi - x)(b), \ (\psi - x)'(0) \geq (\psi - x)'(b) \end{array} \right\}.$$

Multiplying the above inequality with $(\psi - x)_+(t)$ and integrating over $T = [0, b]$ as in the proof of theorem 2, using the definitions of r, q_N and the boundary conditions for ψ and x, we obtain that

$$
\begin{aligned}
0 &\le \int_0^b u(t, x(t))(\psi - x)_+(t)\mathrm{d}t = \int_0^b (x(t) \\
&\quad - \psi(t))(\psi - x)_+(t)\mathrm{d}t = -\int_0^b [(\psi - x)_+(t)]^2 \mathrm{d}t \\
&\Rightarrow 0 = \int_0^b [(\psi - x)_+(t)]^2 \mathrm{d}t \\
&\Rightarrow \psi(t) \le x(t) \qquad \text{for all } t \in T.
\end{aligned}
$$

In a similar fashion, we show that $x(t) \le \phi(t)$ for all $t \in T$. Therefore every solution $x \in W^{2,1}(T)$ of (19) is located in $K = [\psi, \phi]$. Thus recalling the definitions of $\tau(x), q_N$ and u, we see that $-x''(t) = f(t, z(t), x'(t)) + M(x(t) - x(t))$ a.e. on T, $x(0) = x(b)$, $x'(0) = x'(b)$. Now we will show that this solution is unique. To this end, on $L^1(T)$ we consider an equivalent norm $|\cdot|_1$ given by

$$|x|_1 = \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|x(t)|\mathrm{d}t, \quad \lambda > 0.$$

Similarly on $W^{2,1}(T)$ we consider the equivalent norm given by

$$|x|_{2,1} = |x|_1 + |x'|_1 + |x''|_1.$$

Suppose that $x_1, x_2 \in W^{2,1}(T)$ are two solutions of (19). Then

$$x_1 = L_M^{-1} H_0(x_1) \quad \text{and} \quad x_2 = L_M^{-1} H_0(x_1),$$

where $L_M^{-1} = (MI + \hat{L})^{-1}$ with $\hat{L}x = -x''$ for all $x \in D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and $H_0(x)(\cdot) = f(\cdot, z(\cdot), q_N(\tau(x)'(\cdot)))$. Recall that $L_M^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is linear compact. So $L_M^{-1} : (L^1(T), |\cdot|_1) \to (W^{2,1}(T), |\cdot|_{2,1})$ is linear continuous. Moreover, using hypotheses $\mathrm{H(f)}_2$ we can easily check as before that $H_0 : (W^{2,1}(T), |\cdot|_{2,1}) \to (L^1(T), |\cdot|_1)$ is continuous. Then we have

$$
\begin{aligned}
|x_1 - x_2|_{2,1} &\le ||L_M^{-1}||_{\mathcal{L}} |H_0(x_1) - H_0(x_2)|_1 \\
&= ||L_M^{-1}||_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|H_0(x_1)(t) - H_0(x_2)(t)|\mathrm{d}t \\
&\le ||L_M^{-1}||_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)k(t)|x_1'(t) - x_2'(t)|\mathrm{d}t \\
&= -\frac{1}{\lambda}||L_M^{-1}||_{\mathcal{L}} \int_0^b |x_1'(t) - x_2'(t)| d\left(\exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)\right) \\
&\le \frac{1}{\lambda}||L_M^{-1}||_{\mathcal{L}} \int_0^b \exp\left(-\lambda \int_0^t k(s)\mathrm{d}s\right)|x_1''(t) - x_2''(t)|\mathrm{d}t \\
&= \frac{1}{\lambda}||L_M^{-1}||_{\mathcal{L}} |x_1'' - x_2''|_1.
\end{aligned}
$$

So if $\lambda > ||L_M^{-1}||_{\mathcal{L}}$, we infer that $x_1''(t) = x_2''(t)$ a.e. on T. Hence $x_1'(t) - x_2'(t) = c_1$ for all $t \in T$, with $c_1 \in \mathbb{R}$. Since $x_1'(0) = x_1'(b)$ and $x_2'(0) = x_2'(b)$, from the mean value theorem, we deduce that there exists $\xi \in (0,b)$ such that $x_1'(\xi) = x_2'(\xi)$. Therefore $c_1 = 0$ and so $x_1'(t) = x_2'(t)$ for all $t \in T$, which implies that $x_1(t) - x_2(t) = c_2$ for all $t \in T$, with $c_2 \in \mathbb{R}$. But for almost all $t \in T$, we have

$$\begin{aligned} & f(t, z(t), q_N(x_1'(t))) + M(z(t) - x_1(t)) = f(t, z(t), q_N(x_2'(t))) \\ & \qquad + M(z(t) - x_2(t)) \\ & \Rightarrow x_1(t) = x_1(t) + c_2 \quad \text{a.e. on } T; \ \text{i.e. } c_2 = 0 \text{ and so } \ x_1 = x_2. \end{aligned}$$

Then define $R : [\psi, \phi] \to [\psi, \phi]$ where $R(z)(\cdot)$ is the unique solution of (19). We claim that $R(\cdot)$ is increasing. Indeed let $z_1, z_2 \in [\psi, \phi]$, $z_1 \le z_2$, $z_1 \ne z_2$ and set $x_1 = R(z_1)$, $x_2 = R(z_2)$. We have

$$-x_1''(t) = f(t, z_1(t), q_N(x_1'(t))) + M(z_1(t) - x_1(t)) \quad \text{a.e. on } T$$

and

$$-x_2''(t) = f(t, z_2(t), q_N(x_2'(t))) + M(z_2(t) - x_2(t)) \quad \text{a.e. on } T.$$

Suppose that $\max_{t \in T}[x_1(t) - x_2(t)] = \varepsilon > 0$ and suppose that this maximum is attained at $t_0 \in T$. First we assume that $0 < t_0 < b$. Then we have $x_1'(t_0) = x_2'(t_0) = \nu_0$ and we can find $\delta > 0$ such that for every $t \in T_\delta = [t_0, t_0 + \delta]$ we have $x_2(t) < x_1(t)$. So we obtain

$$\begin{aligned} -x_1''(t) &= f(t, z_1(t), q_N(x_1'(t))) + M(z_1(t) - x_1(t)) \\ &< f(t, z_2(t), q_N(x_1'(t))) + M(z_2(t) - x_2(t)) \\ &= f(t, z_2(t), q_N(x_1'(t))) + Mw(t) \\ & \qquad\qquad \text{a.e. on } T_\delta, \ \text{with } w(t) = z_2(t) - x_2(t), \end{aligned}$$

and

$$-x_2''(t) = f(t, z_2(t), q_N(x_2'(t))) + Mw(t) \quad \text{a.e. on } T_\delta.$$

Since $x_1'(t_0) = x_2'(t_0) = \nu_0$, from a well–known differential inequality (see for example Hale [9], theorem 6.1, p. 31), we obtain that $0 \le x_1'(t) - x_2'(t)$ for all $t \in T_\delta$. So after integration we see that $x_1(t_0) - x_2(t_0) \le x_1(t) - x_2(t)$ for every $t \in T_\delta$. Since $t_0 \in T$ is the point at which $(x_1 - x_2)(\cdot)$ attains its maximum on T, we have that $x_1(t) = x_2(t) + \varepsilon$ for every $t \in T_\delta$ and so $x_1'(t) = x_2'(t)$ for every $t \in T_\delta$. Thus we have

$$\begin{aligned} & 0 = x_1''(t) - x_2''(t) \ge f(t, z_2(t), q_N(x_2'(t))) - f(t, z_2(t), q_N(x_1'(t))) \\ & + M(x_1(t) - x_2(t)) > 0 \quad \text{a.e. on } T_\delta, \end{aligned}$$

a contradiction.

Next assume $t_0 = 0$. Then $\varepsilon = x_1(0) - x_2(0) \ge x_1(h) - x_2(h)$ for all $h \in [0, \delta]$ and $\varepsilon = x_1(b) - x_2(b) \ge x_1(h) - x_2(h)$ for all $h \in [b - \delta, b]$. From the first inequality we infer that $(x_1 - x_2)'(0) \le 0$ while from the second we have $(x_1 - x_2)'(b) \ge 0$ and so $(x_1 - x_2)'(0) \ge 0$. Therefore $x_1'(0) = x_2'(0) = \nu_0$ and so we can proceed as in the previous case and derive a contradiction. Similarly we treat the case $t_0 = b$. Therefore $x_1 \le x_2$ and so $R(\cdot)$ is increasing as claimed.

Now let $\{y_n\}_{n \ge 1}$ be an increasing sequence in $[\psi, \phi]$. Set $x_n = R(y_n)$, $n \ge 1$. The sequence $\{x_n\}_{n \ge 1} \subseteq [\psi, \phi]$ is increasing. From the monotone convergence theorem, we have that $y_n \to y$ and $x_n \to x$ in $L^1(T)$ as $n \to \infty$. Also by hypothesis $\mathrm{H(f)}_2$ (iii), $|x_n''(t)| \le \gamma_r(t)$ a.e. on T with $r = \max\{N, ||\phi||_\infty, ||\psi||_\infty\}$, with $\gamma_r \in L^1(T)$. So $\{x_n\}_{n \ge 1}$ is bounded in $W^{2,1}(T)$ and $\{x_n''\}_{n \ge 1}$ is uniformly integrable. From the compact embedding of

$W^{2,1}(T)$ in $W^{1,1}(T)$ and the Dunford–Pettis theorem, we have that $x_n \to x$ in $W^{1,1}(T)$ and at least for a subsequence we have $x_n'' \xrightarrow{w} g$ in $L^1(T)$ as $n \to \infty$. Clearly $x'' = g$ and so for the original sequence we have $x_n'' \xrightarrow{w} x''$ in $L^1(T)$ as $n \to \infty$. So finally $x_n \xrightarrow{w} x$ in $W^{2,1}(T)$. Invoking theorem 3.1 of Heikkila–Lakshmikantham–Sun [10], we deduce that $R(\cdot)$ has extremal fixed points in $K = [\psi, \phi]$. But note these extremal fixed points of $R(\cdot)$, are the extremal solutions in $K = [\psi, \phi]$ of the periodic problem (19).

Next we consider the situation where the vector field f is independent of x'. This is the case studied by Nieto [16]. However here we are more general than Nieto, since the dependence of f on x can be splitted into a continuous and a discontinuous part. So we will be studying the following periodic problem:

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x(t)) & \text{a.e. on } T \\ x(0) = x(b), \quad x'(0) = x'(b) & \end{array} \right\}. \tag{17}$$

H_0': There exist $\psi \in W^{2,1}(T)$ a lower solution and $\phi \in W^{2,1}(T)$ an upper solution such that $\psi(t) \le \phi(t)$ for all $t \in T$.

$\mathrm{H(f)}_3$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $y \in W^{2,1}(T)$ and every $x \in \mathbb{R}$, $t \to f(t, x, y(t))$ is measurable;
(ii) for almost all $t \in T$ and all $y \in \mathbb{R}$, $x \to f(t, x, y)$ is continuous;
(iii) there exists $M \in L^1(T)_+$ such that for almost all $t \in T$ and all $x \in [\psi(t), \phi(t)]$, $y \to f(t, x, y) + M(t)y$ is increasing;
(iv) for every $r > 0$ there exists $\gamma_r \in L^1(T)$ such that if $|f(t, x, y)| \le \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$ with $|x|, |y| \le r$.

Remark. The superpositional measurability hypothesis $\mathrm{H(f)}_3$ (i) is satisfied, if for every $x \in \mathbb{R}$, there exists $g_x : T \times \mathbb{R} \to \mathbb{R}$ a Borel measurable function such that $g_x(t, y) = f(t, x, y)$ for almost all $t \in T$ and all $y \in \mathbb{R}$. This follows from the monotonicity hypothesis $\mathrm{H(f)}_3$ (iii) and theorem 1.9 of Appell–Zabrejko [1].

Theorem 5. *If hypotheses* H_0' *and* $\mathrm{H(f)}_3$ *hold, then problem* (23) *has a solution* $x \in W^{2,1}(T)$ *in the order interval* $K = [\psi, \phi]$.

Proof. Let $y \in K = [\psi, \phi] = \{y \in W^{2,1}(T) : \psi(t) \le y(t) \le \phi(t) \text{ for all } t \in T\}$ and consider the following periodic problem

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x(t)) + M(t)(y(t) - x(t)) & \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) & \end{array} \right\}. \tag{18}$$

Problem (24) has at least one solution in K (see Nieto [16]). By $S(y)$ we denote the solutions of (24) in K. Let $y_1, y_2 \in K$, $y_1 \le y_2$, $x_1 \in S(y_1)$ and $y_1 \le x_1$. Consider the following problem:

$$\left\{ \begin{array}{l} -x''(t) = f(t, \tau_1(t, x(t)), y_2(t)) + M(t)(y_2(t) - \tau_1(t, x(t))) \\ \qquad\qquad -u_1(t, x(t)) \quad \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array} \right\}. \tag{19}$$

Here $\tau_1 : T \times \mathbb{R} \to \mathbb{R}$ (the truncation function) is defined by

$$\tau_1(t, x) = \begin{cases} \phi(t) & \text{if} \quad \phi(t) \le x \\ x & \text{if} \quad x_1(t) \le x \le \phi(t) \\ x_1(t) & \text{if} \quad x \le x_1(t) \end{cases}$$

and $u_1 : T \times \mathbb{R} \to \mathbb{R}$ (the penalty function) is defined by

$$u_1(t,x) = \begin{cases} x - \phi(t) & \text{if} \quad \phi(t) \leq x \\ 0 & \text{if} \quad x_1(t) \leq x \leq \phi(t) \\ x - x_1(t) & \text{if} \quad x \leq x_1(t) \end{cases}.$$

Both are Caratheodory functions. As before we let $D = \{x \in W^{2,1}(T) : x(0) = x(b), x'(0) = x'(b)\}$ and define $L : D \subseteq L^1(T) \to L^1(T)$ by $Lx = -x''$ for all $x \in D$. Again we can check that $\hat{L} = (I + L)$ is invertible and $\hat{L}^{-1} : L^1(T) \to D \subseteq W^{1,1}(T)$ is compact. Also $H : W^{1,1}(T) \to L^1(T)$ is given by

$$H(x)(\cdot) = f(\cdot, \tau_1(\cdot, x(\cdot)), y_2(\cdot)) + M(\cdot)(y_2(\cdot) - \tau_1(\cdot, x(\cdot))) - u_1(\cdot, x(\cdot)) + x(\cdot).$$

This map is continuous and there exists $k^* > 0$ such that $||H(x)||_1 \leq k^*$ for all $x \in W^{1,1}(T)$. So $\hat{L}^{-1}H(D)$ is relatively compact in $W^{1,1}(T)$ and thus we can apply corollary 10.2, p. 222, of Gilbarg–Trudinger [8] and obtain $x \in D$ such that $x = \hat{L}^{-1}H(x)$. Therefore problem (25) has a solution.

Note that by virtue of hypothesis $\mathrm{H(f)}_3$ (iii) and the fact that $\tau_1(t, x_1(t)) = x_1(t)$ and $u_1(t, x_1(t)) = 0$, we have

$$\left\{ \begin{array}{l} -x_1''(t) = f(t, x_1(t), y_1(t)) + M(y_1(t) - x_1(t)) \leq f(t, x_1(t), y_2(t)) \\ \qquad\qquad\qquad +M(y_2(t) - x_1(t)) \quad \text{a.e. on } T \\ x_1(0) = x_1(b), \ x_1'(0) = x_1'(b) \end{array} \right\}.$$

So $x_1 \in W^{2,1}(T)$ is a lower solution of (25). Similarly since $y_2 \leq \phi$, we have

$$\left\{ \begin{array}{l} -\phi''(t) \geq f(t, \phi(t), \phi(t)) \geq f(t, \phi(t), y_2(t)) + M(y_2(t) - \phi(t)) \quad \text{a.e. on } T \\ \phi(0) = \phi(b), \ \phi_1'(0) \leq \phi'(b) \end{array} \right\}$$

and so we see that $\phi \in W^{2,1}(T)$ is an upper solution of (25).

Now we will show that the solutions of (25) are within the order interval $K_1 = [x_1, \phi]$. Indeed we have

$$\begin{aligned} x_1''(t) - x''(t) = & f(t, \tau_1(t, x(t)), y_2(t)) + M(t)y_2(t) - f(t, x_1(t), y_1(t)) \\ & - M(t)y_1(t) + M(t)(x_1(t) - \tau_1(t, x(t))) - u_1(t, x(t)) \\ & \qquad \text{a.e. on } T. \end{aligned}$$

Multiply the above equation with $(x_1 - x)_+(\cdot)$ and then integrate over $T = [0, b]$. As in previous proofs we obtain

$$\begin{aligned} & \int_0^b u_1(t, x(t))(x_1 - x)_+(t) \mathrm{d}t \geq 0 \\ & \Rightarrow - \int_0^b [(x_1 - x)_+(t)]^2 \mathrm{d}t \geq 0; \quad \text{i.e. } x_1(t) \leq x(t) \quad \text{for all } t \in T. \end{aligned}$$

Similarly we show that $x(t) \leq \phi(t)$ for all $t \in T$. Therefore every solution of (25) is in the order interval $K_1 = [x_1, \phi]$. Because of this fact, equation (25) becomes

$$\left\{ \begin{array}{l} -x''(t) = f(t, x(t), y_2(t)) + M(t)(y_2(t) - x(t)) \quad \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) \end{array} \right\}$$

and so $x \in S(y_2)$ and $x_1 \leq x$.

Next we will show that for every $y \in K = [\psi, \phi]$, the set $S(y)$ is compact in $L^1(T)$. To this end let $x \in S(y)$. Then $||x||_\infty \leq \max\{||\phi||_\infty, ||\psi||_\infty\} = r$. Hence $||x''(t)|| \leq \gamma_r(t) + 2M(t)r$ a.e. on T. Hence $S(y)$ is bounded in $W^{2,1}(T)$ and since the latter embeds compactly in $L^1(T)$, we have that $S(y)$ is relatively compact in $L^1(T)$. Then let $\{x_n\}_{n\geq 1} \subseteq S(y)$ and assume that $x_n \to x$ in $L^1(T)$ as $n \geq \infty$. Since $\{x''_n\}_{n\geq 1}$ is uniformly integrable, by passing to a subsequence if necessary we may assume that $x''_n \xrightarrow{w} g$ in $L^1(T)$ as $n \to \infty$. Because $W^{2,1}(T)$ embeds continuously in $C^1(T)$, $\{x'_n\}_{n\geq 1}$ is bounded in $C(T)$ and for all $0 \leq s \leq t \leq b$ and all $n \geq 1$, $|x'_n(t) - x'_n(s)| \leq \int_s^t (\gamma_r(\tau) + 2M(\tau)r)\mathrm{d}\tau$ from which it follows that $\{x'_n\}_{n\geq 1}$ is equicontinuous. So by the Arzela–Ascoli theorem we have that $x'_n \to x'$ in $C(T)$ as $n \to \infty$ and so $g = x''$. Then via the dominated convergence theorem, as before, we can check that

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), y(t)) + M(t)(y(t) - x(t)) & \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) & \end{array} \right\}.$$

Hence $x \in S(y)$ and this proves that $S(y)$ is closed, hence compact in $L^1(T)$. Since the positive cone $L^1(T)_+ = \{x \in L^1(T) : x(t) \geq 0 \text{ a.e. on } T\}$ is regular (in fact fully regular; see Krasnoselskii [12]), from proposition 2 of Heikkila–Hu [11], we infer that $S(\cdot)$ has a fixed point in K; i.e. there exists $x \in K = [\psi, \phi]$ such that $x \in S(x)$. Therefore

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t), x(t)) & \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b) & \end{array} \right\}$$

and so problem (23) has a solution in $K = [\psi, \phi]$.

4. Properties of the solutions

For problems linear in x', we can say something about the structure of the solution set of the periodic problem. Our result extends theorem 4.2 of Nieto [16].

The problem under consideration is the following:

$$\left\{ \begin{array}{ll} -x''(t) = f(t, x(t)) + Mx'(t) & \text{a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b). & \end{array} \right\}. \tag{20}$$

Our hypotheses on the vector field $f(t, x)$ are the following:

$\mathrm{H(f)}_4$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x \in \mathbb{R}$, $t \to f(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \to f(t, x)$ is continuous and decreasing;
(iii) for every $r > 0$ there exists $\gamma_r \in L^\infty(T)$ such that $|f(t, x)| \leq \gamma_r(t)$ a.e. on T for all $x \in \mathbb{R}$, $|x| \leq r$.

Remark. Under these hypotheses the Nagumo growth condition is automatically satisfied since for $k = \max\{||\phi||_\infty, ||\psi||_\infty\}$, we have $|f(t, x) + My| \leq \gamma_k(t) + M|y|$ a.e. on T for all $x \in [\psi(t), \phi(t)]$, and so if $h(r) = ||\gamma_k||_\infty + Mr$, we have for all

$$\lambda > 0 \int_\lambda^\infty \frac{r}{h(r)} \mathrm{d}r = \int_\lambda^\infty \frac{r}{||\gamma_k||_\infty + Mr} \mathrm{d}r = +\infty.$$

Theorem 6. *If hypotheses* H'_0 *and* $\mathrm{H(f)}_4$ *hold and* $M > 0$, *then the solution set* S *of* (30) *in* $K = [\psi, \phi]$ *is nonempty, w-compact and convex in* $W^{2,1}(T)$.

Proof. From theorem 1 we know that $S \neq \phi$. Let $x \in S$ and define $\hat{x}(t) = x(t) - \frac{1}{b}\int_0^b x(t)\mathrm{d}t$ $t \in T$. Let $T_0 = \{x \in \mathbb{R} : \hat{x} + c \in S\}$. Note that $T_0 \neq \phi$, since $c = \frac{1}{b}\int_0^b x(t)\mathrm{d}t \in T_0$. We claim that T_0 is an interval. Indeed let $c_1, c_2 \in T_0$, $c_1 < c_2$ and take $c \in (c_1, c_2)$. Set $y = x + c$. We have

$$\begin{aligned} -y''(t) &= -\hat{x}''(t) = f(t, (\hat{x} + c_1)(t)) + M(\hat{x} + c_1)'(t) \\ &= f(t, (\hat{x} + c_2)(t)) + M(\hat{x} + c_2)'(t) \text{ a.e. on } T. \end{aligned}$$

By hypothesis $\mathrm{H(f)}_4$ (ii), we have

$$\begin{aligned} & f(t, (\hat{x} + c_1)(t)) \geq f(t, y(t)) \geq f(t, (\hat{x} + c_2)(t)) \qquad \text{a.e. on } T \\ & \Rightarrow -y''(t) = f(t, y(t)) + My'(t) \qquad \text{a.e. on } T. \end{aligned}$$

Also it is clear that $y(0) = y(b)$ and $y'(0) = y'(b)$. Therefore $y \in S$ and so $c \in T_0$, which proves that T_0 is an interval.

Next we will show that $S = \{\hat{x} + c : c \in T_0\}$. Indeed if $\nu,\ x \in S$, then we have

$$\begin{aligned} & (x''(t) - \nu''(t))(x(t) - \nu(t)) \\ & \quad = (f(t, \nu(t)) + M\nu'(t) - f(t, x(t)) - Mx'(t))(x(t) - \nu(t)) \\ & \quad = (f(t, \nu(t)) - f(t, x(t)))(x(t) - \nu(t)) + M(\nu'(t) - x'(t))(x(t) - \nu(t)) \\ & \quad \geq M(\nu'(t) - x'(t))(x(t) - \nu(t)) \quad \text{a.e. on } T. \end{aligned}$$

Integrating over $T = [0, b]$, we obtain

$$\begin{aligned} & \int_0^b (x''(t) - \nu''(t))(x(t) - \nu(t))\mathrm{d}t = -\int_0^b (x'(t) - \nu'(t))^2 \mathrm{d}t \\ & \geq M \int_0^b (\nu'(t) - x'(t))(x(t) - \nu(t))\mathrm{d}t \\ & = -M \int_0^b (x(t) - \nu(t)) d(x - \nu)(t) = 0 \end{aligned}$$

$\Rightarrow x'(t) = \nu'(t)$ for every $t \in T$
$\Rightarrow (x - \nu)(\cdot) = \text{constant}$.

So indeed $S = \{\hat{x} + c : c \in T_0\}$ and since as we saw earlier T_0 is an interval, we deduce that S is convex.

Finally we will prove that S is w-compact in $W^{2,1}(T)$. To this end, let $y \in S$. Then there exists $k \in T_0$ such that $y = \hat{x} + k$, hence $||y||_{2,1} = ||\hat{x} + k||_{2,1}$. Since $y \in K = [\psi, \phi]$, we have $|k| \leq \max\{||\psi||_\infty + ||x_1||_\infty, ||\phi||_\infty + ||x_1||_\infty\} = \eta$. Therefore $||y||_{2,1} \leq ||\hat{x}||_1 + b|k|$ $+||\hat{x}'||_1 + ||\hat{x}''||_1 \leq ||\hat{x}||_{2,1} + b\eta$ and so S is bounded in $W^{2,1}(T)$. We will show that S is closed in $W^{2,1}(T)$. Let $\{y_n\}_{n\geq 1} \subseteq S$ and assume that $y_n \to y$ in $W^{2,1}(T)$. We have

$$-y_n''(t) = f(t, y_n(t)) + My_n'(t) \quad \text{a.e. on } T,\ n \geq 1. \tag{21}$$

Since $W^{2,1}(T)$ embeds continuously in $C^1(T)$, by passing to a subsequence if necessary, we may assume that $y_n''(t) \to y''(t)$ a.e. on T, $y_n'(t) \to y'(t)$ and $y_n(t) \to y(t)$ for all $t \in T$. So $f(t, y_n(t)) \to f(t, y(t))$ a.e. on T. Thus passing to the limit as $n \to \infty$ in (31), we obtain

$$\begin{aligned} & -y''(t) = f(t, y(t)) + My'(t) \quad \text{a.e. on } T,\ y(0) = y(b),\ y'(0) = y'(b) \\ & \Rightarrow y \in S. \end{aligned}$$

So S is closed, hence weakly closed since it is convex. To show that S is weakly compact in $W^{2,1}(T)$, we need to show that given $\{x_n\}_{n\geq 1} \subseteq S$, we can find a weakly convergent subsequence. Since $\{x_n\}_{n\geq 1}$ is bounded in $W^{2,1}(T)$ and the latter embeds compactly in $W^{1,1}(T)$, by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $W^{1,1}(T)$ as $n \to \infty$. Also $x_n'' = \hat{x}''$ and so $||x_n''(t)|| = ||\hat{x}''(t)||$ a.e. on T. Therefore by the Dunford–Pettis theorem, we may assume that $x_n'' \xrightarrow{w} g$ in $L^1(T)$ and $g = x''$. So $x \in W^{2,1}(T)$ and $x_n \xrightarrow{w} x$ in $W^{2,1}(T)$. Since S is weakly closed in $W^{2,1}(T)$, $x \in S$ and so S is weakly compact in $W^{2,1}(T)$.

In general if the vector field f is decreasing in the x-variable, then the upper and lower solutions of the problem, as well as the solutions exhibit some interesting properties.

First we consider the general periodic problem (18), with the following hypotheses on the vector field $f(t, x, y)$.

$\mathrm{H(f)}_5$: $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for every $x, y \in \mathbb{R}$, $t \to f(t, x, y)$ is measurable;
(ii) for almost all $t \in T$ and all $y \in \mathbb{R}$, $x \to f(t, x, y)$ is strictly decreasing;
(iii) for all $x, y, y' \in \mathbb{R}$ $|f(t, x, y) - f(t, x, y')| \leq k(t)|y - y'|$ a.e. on T with $k \in L^1(T)$;
(iv) for every $r > 0$ there exists $\gamma_r \in L^1(T)$ such that $|f(t, x, y)| \leq \gamma_r(t)$ a.e. on T for all $x, y \in \mathbb{R}$, $|x|, |y| \leq r$.

PROPOSITION 7

If $\mathrm{H(f)}_5$ *holds,* $\phi \in W^{2,1}(T)$ *is an upper solution and* $\psi \in W^{2,1}(T)$ *a lower solution for problem* (18), *then for all* $t \in T$, $\psi(t) \leq \phi(t)$.

Proof. Suppose not. Let $t_0 \in T$ be such that $\max_{t\in T}(\psi - \phi)(t) = (\psi - \phi)(t_0) = \varepsilon > 0$. First assume that $0 < t_0 < b$. Then $\psi'(t_0) = \phi'(t_0) = \nu_0$ and we can find $\delta > 0$ such that for all $t \in T_\delta = [t_0, t_0 + \delta]$, we have $\phi(t) < \psi(t)$. Then we have

$$\begin{aligned} -\psi''(t) &\leq f(t, \psi(t), \psi'(t)) < f(t, \phi(t), \psi'(t)) \quad \text{a.e. on } T_\delta \\ \text{and } -\phi''(t) &\geq f(t, \phi(t), \phi'(t)) \quad \text{a.e. on } T. \end{aligned}$$

Consider the following initial value problem

$$\left\{ \begin{array}{ll} -y'(t) = f(t, \phi(t), y(t)) & \text{a.e. on } T_\delta = [t_0, t_0 + \delta] \\ y(t_0) = \nu_0 & \end{array} \right\}. \tag{22}$$

Because of hypothesis $\mathrm{H(f)}_5$ (iii), problem (32) has a unique solution $y \in W^{1,1}(T_\delta)$. Moreover, from the definitions of upper and lower solutions and a well-known differential inequality (see Hale [9], p. 31), we infer that $\phi'(t) \leq y(t) \leq \psi'(t)$ for all $t \in T_\delta$ and so $(\psi - \phi)'(t) \geq 0$ for all $t \in T_\delta$. Integrating, we have $(\psi - \phi)(t_0) \leq (\psi - \phi)(t)$ for all $t \in T_\delta$. Recalling the choice of t_0, we see that $(\psi - \phi)(t) =$ constant for all $t \in T_\delta$, hence $\psi'(t) = \phi'(t)$ for all $t \in T_\delta$. Thus for almost all $t \in T_\delta$, we have

$$-\psi''(t) < f(t, \phi(t), \psi'(t)) = f(t, \phi(t), \phi'(t)) \leq -\phi''(t),$$

a contradiction to the fact that $(\psi - \phi)''(t) = 0$ for all $t \in T_\delta$.

If $t_0 = 0$, then since $(\psi - \phi)(0) = (\psi - \phi)(b)$, we can find $\delta > 0$ such that $(\psi - \phi)(0) \geq (\psi - \phi)(t) > 0$ for all $t \in [0, \delta]$ and $0 < (\psi - \phi)(t) \leq (\psi - \phi)(b)$ for all $t \in [b - \delta, b]$. From the first inequality we have that $(\psi - \phi)'(0) \leq 0$, while from the second it follows that $(\psi - \phi)'(b) \geq 0$. But from the definitions of the upper and lower solutions we have $(\psi - \phi)'(0) \geq (\psi - \phi)'(b) \geq 0$, therefore we conclude that $\psi'(0) = \phi'(0) = \nu_0$ and we can proceed as in the previous case.

The case $t_0 = b$ is treated in a similar fashion.

Our second observation concerning ϕ, ψ, refers to problem (30) where the vector field depends linearly in x'.

PROPOSITION 8

If $\mathrm{H(f)}_4$ *holds,* $\phi \in W^{2,1}(T)$ *is an upper solution of* (30), $\psi \in W^{2,1}(T)$ *is a lower solution of* (30) *and for all* $t \in T$ $\phi(t) \leq \psi(t)$, *then* $(\psi - \phi)(\cdot)$ *is constant.*

Proof. By definition we have

$$\begin{aligned} &-\psi''(t) \leq f(t, \psi(t)) + M\psi'(t) \qquad \text{a.e. on } T,\ \psi(0) = \psi(b),\ \psi'(0) \geq \psi'(b) \\ &-\phi''(t) \geq f(t, \phi(t)) + M\phi'(t) \qquad \text{a.e. on } T,\ \phi(0) = \phi(b),\ \phi'(0) \leq \phi'(b). \end{aligned}$$

Hence we have

$$\psi''(t) - \phi''(t) \geq f(t, \phi(t)) - f(t, \psi(t)) + M(\phi'(t) - \psi'(t)) \quad \text{a.e. on } T.$$

Multiplying with $(\psi - \phi)(t)$ and then integrating over $T = [0, b]$, we obtain

$$\begin{aligned} \int_0^b (\psi'' - \phi'')(t)(\psi - \phi)(t)\mathrm{d}t \geq &\int_0^b (f(t, \phi(t)) - f(t, \psi(t)))(\psi - \phi)(t)\mathrm{d}t \\ &+ M\int_0^b (\phi' - \psi')(t)(\psi - \phi)(t)\mathrm{d}t. \end{aligned} \tag{23}$$

By Green's formula, we have

$$\begin{aligned} &\int_0^b (\psi'' - \phi'')(t)(\psi - \phi)(t)\mathrm{d}t = (\psi - \phi)'(b) - (\psi - \phi)'()0) \\ &\quad - \int_0^b [(\psi' - \phi')(t)]^2\mathrm{d}t \leq -\int_0^b [(\psi' - \phi')(t)]^2\mathrm{d}t. \end{aligned} \tag{24}$$

Also from hypothesis $\mathrm{H(f)}_4$ (ii) it follows that

$$\int_0^b (f(t, \phi(t)) - f(t, \psi(t)))(\psi - \phi)(t)\mathrm{d}t \geq 0. \tag{25}$$

Finally note that

$$\begin{aligned} &\int_0^b M(\phi' - \psi')(t)(\psi - \phi)(t)\mathrm{d}t = -M\int_0^b (\psi' - \phi')(t)(\psi - \phi)(t)\mathrm{d}t \\ &\quad = -M\int_0^b (\psi - \phi)(t)d(\psi - \phi)(t) = -M(\psi - \phi)(b) + M(\psi - \phi)(0) = 0. \end{aligned} \tag{26}$$

Using (34), (35) and (36) in (33), we obtain

$$\begin{aligned} &\int_0^b [(\psi' - \phi')(t)]^2\mathrm{d}t \leq 0 \\ &\Rightarrow \psi'(t) = \phi'(t) \quad \text{for all } t \in T \quad \text{and so } (\psi - \phi)(\cdot) \text{ is constant.} \end{aligned}$$

An immediate consequence of proposition 8, is the following result:

COROLLARY 9

If $\mathrm{H(f)}_4$ *holds and* $x_1, x_2 \in W^{2,1}(T)$ *are two solutions of* (30) *such that* $x_1(t) \leq x_2(t)$ *for all* $t \in T$, *then* $(x_1 - x_2)(\cdot)$ *is constant.*

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