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Completely monotone multisequences, symmetric probabilities and a normal limit theorem

J C GUPTA

Indian Statistical Institute, 7, SJS Sansanwal Marg, New Delhi 110 016, India
Corresponding Address: 32, Mirdha Tola, Budaun 243 601, India

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Abstract. Let $\mathbf{G}_{n,k}$ be the set of all partial completely monotone multisequences of order n and degree k, i.e., multisequences $c_n(\beta_1, \beta_2, \ldots, \beta_k), \beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots,$ $\beta_1 + \beta_2 + \cdots + \beta_k \leq n, c_n(0, 0, \ldots, 0) = 1$ and $(-1)^{\beta_0}\Delta^{\beta_0} c_n(\beta_1, \beta_2, \ldots, \beta_k) \geq 0$ whenever $\beta_0 \leq n - (\beta_1 + \beta_2 + \cdots + \beta_k)$ where $\Delta c_n(\beta_1, \beta_2, \ldots, \beta_k) = c_n(\beta_1 + 1, \beta_2, \ldots, \beta_k) + c_n(\beta_1, \beta_2 + 1, \ldots, \beta_k) + \cdots + c_n(\beta_1, \beta_2, \ldots, \beta_k + 1) - c_n(\beta_1, \beta_2, \ldots, \beta_k)$. Further, let $\prod_{n,k}$ be the set of all symmetric probabilities on $\{0, 1, 2, \ldots, k\}^n$. We establish a one-to-one correspondence between the sets $\mathbf{G}_{n,k}$ and $\prod_{n,k}$ and use it to formulate and answer interesting questions about both. Assigning to $\mathbf{G}_{n,k}$ the uniform probability measure, we show that, as $n \to \infty$, any fixed section $\{c_n(\beta_1, \beta_2, \ldots, \beta_k), 1 \leq \sum \beta_i \leq m\}$, properly centered and normalized, is asymptotically multivariate normal. That is, $\left\{\sqrt{\binom{n+k}{k}}(c_n(\beta_1, \beta_2, \ldots, \beta_k) - c_0(\beta_1, \beta_2, \ldots, \beta_k),\ 1 \leq \beta_1 + \beta_2 + \cdots + \beta_k \leq m\right\}$ converges weakly to MVN$[0, \Sigma_m]$; the centering constants $c_0(\beta_1, \beta_2, \ldots, \beta_k)$ and the asymptotic covariances depend on the moments of the Dirichlet $(1, 1, \ldots, 1; 1)$ distribution on the standard simplex in R^k.

Keywords. Moment spaces; completely monotone multisequences; normal limit.

1. Introduction

The notion of a *completely monotone multisequence* was introduced in [2] in the context of solving the moment problem on the standard k-simplex. We recall the definition.

DEFINITION

A multisequence of real numbers

$$c(\beta_1, \beta_2, \ldots, \beta_k), \beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots, c(0, 0, \ldots, 0) = 1 \tag{1.1}$$

is said to be *completely monotone* if

$$(-1)^{\beta_0}\Delta^{\beta_0} c(\beta_1, \beta_2, \ldots, \beta_k) \geq 0 \ \forall\ \beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots, \tag{1.2}$$

where

$$\begin{aligned}\Delta c(\beta_1, \beta_2, \ldots, \beta_k) = {} & c(\beta_1 + 1, \beta_2, \ldots, \beta_k) + c(\beta_1, \beta_2 + 1, \ldots, \beta_k) + \cdots \\ & + c(\beta_1, \beta_2, \ldots, \beta_k + 1) - c(\beta_1, \beta_2, \ldots, \beta_k)\end{aligned} \tag{1.3}$$

and Δ^{β_0} stands for β_0 iterates of the operator Δ.

The following theorem solves the moment problem on the standard k-simplex

$$S_k := \{(x_1, x_2, \ldots, x_k) : x_i \geq 0, x_1 + x_2 + \cdots + x_k \leq 1\}. \tag{1.4}$$

Theorem 1.1. [2]. *There exists a probability measure P on S_k such that*

$$\int_{S_k} x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k} \mathrm{d}P = c(\beta_1, \beta_2, \ldots, \beta_k) \tag{1.5}$$

for all $\beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots$ if and only if the multisequence $\{c(\beta_1, \beta_2, \ldots, \beta_k) : \beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots\}$ is completely monotone.

For $k = 1, S_k = [0, 1]$, a completely monotone multisequence is just a completely monotone sequence and the above theorem reduces to Hausdorff's solution [4] to the moment problem on $[0, 1]$. In the sequel, for the sake of brevity, we call a completely monotone sequence an H-sequence and a competely monotone multisequence a G-multisequence.

In [3] we explored the interplay between partial H-sequences and symmetric probabilities on products of $\{0, 1\}$. In response to a query raised by a referee of [2] and [3], in this paper we carry out an analysis of the interplay between G-multisequences on the one hand and symmetric probabilities on products of $\{0, 1, 2, \ldots, k\}$ on the other. This paper is organized as follows. In § 2 we establish an affine bijection between the set of all partial G-multisequences of order n and degree k (see below for definition) and the set of all symmetric probabilities on $\{0, 1, 2, \ldots, k\}^n$ and use it to answer questions about both these sets. In § 3 we formulate and prove a normal limit theorem for partial G-multisequences. In § 4 we make some remarks.

2. *G*-multisequences and symmetric probabilities

We introduce the notion of a partial G-multisequence of order n and degree k.

DEFINITION

A multisequence

$$\begin{aligned} &c_n(\beta_1, \beta_2, \ldots, \beta_k),\ \beta_1, \beta_2, \ldots, \beta_k = 0, 1, 2, \ldots \\ &c_n(0, 0, \ldots, 0) = 1,\ \beta_1 + \beta_2 + \cdots + \beta_k \leq n \end{aligned} \tag{2.1}$$

such that

$$(-1)^{\beta_0} \Delta^{\beta_0} c_n(\beta_1, \beta_2, \ldots, \beta_k) \geq 0 \ \forall\ \beta_0 \leq n - \beta_1 - \beta_2 - \cdots - \beta_k \tag{2.2}$$

is called a partial G-multisequence of order n and degree k. The set

$$\begin{aligned} \mathbf{G}_{n,k} :=& \{\{c_n(\beta_1, \beta_2, \ldots, \beta_k), 1 \leq \beta_1 + \beta_2 + \cdots + \beta_k \leq n\} : \\ &(-1)^{\beta_0} \Delta^{\beta_0} c_n(\beta_1, \beta_2, \ldots, \beta_k) \geq 0 \\ &\text{for } \beta_0 = 0, 1, \ldots, n - \beta_1 - \beta_2 - \cdots - \beta_k\} \end{aligned} \tag{2.3}$$

with the understanding that $c_n(0, 0, \ldots, 0) \equiv 1$ denotes the set of all partial G-multisequences of order n and degree k.

Notation. k is a natural number; $\alpha_i, \beta_i, \gamma_i, \delta_i, i = 1, 2, \ldots, k$, are non-negative integers and $x_1, x_2, \ldots, x_k$ are real numbers.

For $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$,

$$
\begin{aligned}
|\boldsymbol{\beta}| &:= \beta_1 + \beta_2 + \cdots + \beta_k;\\
\mathbf{x}^{\boldsymbol{\beta}} &:= \prod_{i=1}^{k} x_i^{\beta_i} \text{ where } \mathbf{x} = (x_1, x_2, \ldots, x_k);\\
L_{m,k} &:= \{(\beta_1, \beta_2, \ldots, \beta_k) : |\boldsymbol{\beta}| = m\}, \quad m = 0, 1, 2, \ldots;\\
\mathcal{L}_{j,k} &:= \bigcup_{m=0}^{j} L_{m,k} = \{\boldsymbol{\beta} : |\boldsymbol{\beta}| \le j\};\\
\tilde{\mathcal{L}}_{j,k} &:= \bigcup_{m=1}^{j} L_{m,k} = \{\boldsymbol{\beta} : 1 \le |\boldsymbol{\beta}| \le j\};\\
\boldsymbol{\beta}! &:= \prod_{i=1}^{k} \beta_i!;\\
\binom{n}{\boldsymbol{\beta}} &:= \frac{n!}{\prod_{i=1}^{k} \beta_i!(n - |\boldsymbol{\beta}|)!}.
\end{aligned}
\tag{2.4}
$$

We say $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k) \le \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ if $\beta_i \le \gamma_i, i = 1, 2, \ldots, k$ and in that case define

$$
\binom{\boldsymbol{\gamma}}{\boldsymbol{\beta}} := \frac{\boldsymbol{\gamma}!}{\boldsymbol{\beta}!(\boldsymbol{\gamma} - \boldsymbol{\beta})!}.
$$

We define $\boldsymbol{\alpha} \vee \boldsymbol{\beta} := (\alpha_1 \vee \beta_1, \alpha_2 \vee \beta_2, \ldots, \alpha_k \vee \beta_k)$. For $\boldsymbol{\delta} = \boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}$, we define

$$
\binom{\boldsymbol{\delta}}{\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\gamma}} := \prod_{i=1}^{k} \binom{\delta_i}{\alpha_i\ \beta_i\ \gamma_i}.
$$

For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_k)$,

$$
\sum_{\boldsymbol{\lambda}=0}^{\boldsymbol{\gamma}} \text{ stands for } \sum_{\lambda_1=0}^{\gamma_1} \sum_{\lambda_2=0}^{\gamma_2} \cdots \sum_{\lambda_k=0}^{\gamma_k}.
$$

For a partial multisequence $c_n(\boldsymbol{\beta})$ of order n, we define, for $|\boldsymbol{\beta}| \le m \le n$,

$$
q_m(\boldsymbol{\beta}) = (-1)^{m-|\beta|} \Delta^{m-|\beta|} c_n(\boldsymbol{\beta}) \tag{2.5}
$$

and observe that, by (2.2),

$$
q_m(\boldsymbol{\beta}) \ge 0, \quad |\boldsymbol{\beta}| \le m \le n. \tag{2.6}
$$

For $m \ge |\boldsymbol{\beta}| + 1$, we define

$$
\nabla q_m(\boldsymbol{\beta}) = q_m(\boldsymbol{\beta}) + q_m(\beta_1 + 1, \beta_2, \ldots, \beta_k) + \cdots + q_m(\beta_1, \beta_2, \ldots, \beta_k + 1). \tag{2.7}
$$

By (2.5) and (2.7), it easily follows that

$$
\nabla q_m(\boldsymbol{\beta}) = q_{m-1}(\boldsymbol{\beta}), \quad |\boldsymbol{\beta}| + 1 \le m \le n. \tag{2.8}
$$

Consequently, for $|\boldsymbol{\beta}| \le m \le n$,

$$q_m(\boldsymbol{\beta}) = \nabla^{n-m} q_n(\boldsymbol{\beta}) = \sum_{j=0}^{n-m} \binom{n-m}{j} \sum_{\delta \in L_{j,k}} \binom{j}{\boldsymbol{\delta}} q_n(\boldsymbol{\beta}+\boldsymbol{\delta})$$
$$= \sum_{\delta \in \mathcal{L}_{n-m,k}} \binom{n-m}{\boldsymbol{\delta}} q_n(\boldsymbol{\beta}+\boldsymbol{\delta}). \tag{2.9}$$

By (2.5), for $\boldsymbol{\beta} \in \mathcal{L}_{n,k}$,

$$c_n(\boldsymbol{\beta}) = q_{|\beta|}(\boldsymbol{\beta}) = \sum_{\delta \in \mathcal{L}_{n-|\beta|,k}} \binom{n-|\boldsymbol{\beta}|}{\boldsymbol{\delta}} q_n(\boldsymbol{\beta}+\boldsymbol{\delta}). \tag{2.10}$$

By (2.5), for $\boldsymbol{\beta} \in \mathcal{L}_{n,k}$,

$$q_n(\boldsymbol{\beta}) = (-1)^{n-|\beta|} \Delta^{n-|\beta|} c_n(\boldsymbol{\beta}) = \sum_{\delta \in \mathcal{L}_{n-|\beta|,k}} (-1)^{|\delta|} \binom{n-|\boldsymbol{\beta}|}{\boldsymbol{\delta}} c_n(\boldsymbol{\beta}+\boldsymbol{\delta}). \tag{2.11}$$

By (2.10),

$$\sum_{\beta \in \mathcal{L}_{n,k}} \binom{n}{\boldsymbol{\beta}} q_n(\boldsymbol{\beta}) = \nabla^n q_n(0) = c_n(0) = 1. \tag{2.12}$$

Finally, by (2.9), the non-negativity of $q_n(\boldsymbol{\beta})$'s implies that conditions (2.2) hold and consequently,

$$\mathbf{G}_{n,k} = \{\{c_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : q_n(\boldsymbol{\beta}) \ge 0 \ \forall \ \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}, \tag{2.13}$$

where $q_n(\boldsymbol{\beta})$'s are given by (2.11). Of course $c_n(0) = 1$.

Let $E_k := \{0, 1, 2, \ldots, k\}$. Given an element $\{c_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}$ of $\mathbf{G}_{n,k}$, let Q_n be the symmetric measure on the n-fold product space E_k^n which assigns mass $q_n(\boldsymbol{\beta})$ to each point $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ of E_k^n for which $\#\{i : \omega_i = j\} = \beta_j$, $j = 1, 2, \ldots, k$ and $\#\{i : \omega_i = 0\} = n - |\boldsymbol{\beta}|$. By (2.6) and (2.12), Q_n is a probability. Conversely, given $\{q_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}$, eqs (2.10) determine an element of $\mathbf{G}_{n,k}$. This establishes a one-to-one correspondence between $\mathbf{G}_{n,k}$ and

$$\prod_{n,k} := \left\{ \{q_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \mathcal{L}_{n,k}\} : q_n(\boldsymbol{\beta}) \ge 0, \sum_{\beta \in \tilde{\mathcal{L}}_{n,k}} \binom{n}{\boldsymbol{\beta}} q_n(\boldsymbol{\beta}) \le 1 \right\}, \tag{2.14}$$

the set of all symmetric probabilities on E_k^n. Of course

$$q_n(0) = 1 - \sum_{\beta \in \tilde{\mathcal{L}}_{n,k}} \binom{n}{\boldsymbol{\beta}} q_n(\boldsymbol{\beta}). \tag{2.15}$$

Equations (2.10) and (2.11) define maps

$$\phi_n : \mathbf{G}_{n,k} \longrightarrow \prod_{n,k}$$

and

$$\psi_n : \prod_{n,k} \longrightarrow \mathbf{G}_{n,k}. \tag{2.16}$$

Clearly, these maps establish an affine congruence between the convex sets $\mathbf{G}_{n,k}$ and $\prod_{n,k}$. Further, these maps are inverses of each other.

We define the projection map

$$\pi_n : \mathbf{G}_{n+1,k} \longrightarrow \mathbf{G}_{n,k}$$

by

$$\{c(\boldsymbol{\beta}) : \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n+1,k}\} \longmapsto \{c(\boldsymbol{\beta}) : \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}. \tag{2.17}$$

Likewise, we define

$$\tilde{\pi}_n : \prod_{n+1,k} \longmapsto \prod_{n,k}$$

by

$$q^* \longmapsto q, \tag{2.18}$$

where q is the n-dimensional marginal of q^* in $\prod_{n+1,k}$. We note that both these projection maps are affine. We briefly illustrate the use of these maps to answer questions about $\mathbf{G}_{n,k}$ and $\prod_{n,k}$.

(a) *Extreme points of* $\mathbf{G}_{n,k}$ *and* $\prod_{n,k}$. Clearly, the extreme points of $\prod_{n,k}$ correspond to probabilities $Q_n^{\gamma}, \boldsymbol{\gamma} \in \mathcal{L}_{n,k}$, where Q_n^{γ} is the uniform probability distribution on the set of those elements of E_k^n for which $\#\{i : w_i = j\} = \gamma_j, j = 1, 2, \ldots, k$, i.e.,

$$q_n^{\gamma}(\boldsymbol{\beta}) = \begin{cases} \dfrac{1}{\binom{n}{\gamma}} & \text{if} \quad \boldsymbol{\beta} = \boldsymbol{\gamma} \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

The extreme points of $\mathbf{G}_{n,k}$, which are otherwise not so apparent, can easily be obtained by using the map ψ_n which, being an affine congruence, maps $\partial\prod_{n,k}$ onto $\partial\mathbf{G}_{n,k}$. Simple calculations show that

$$\partial\mathbf{G}_{n,k} = \{\{c_n^{\gamma}(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : \boldsymbol{\gamma} \in \mathcal{L}_{n,k}\},$$

where

$$c_n^{\gamma}(\boldsymbol{\beta}) = \begin{cases} \dfrac{\binom{\gamma}{\beta}}{\binom{n}{\beta}} = \dfrac{\prod_{i=1}^{k}(\gamma_i(\gamma_i-1)\cdots(\gamma_i-\beta_i+1))}{n(n-1)\cdots(n-|\beta|+1)} & \text{if } \boldsymbol{\beta} \le \boldsymbol{\gamma} \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

(b) *Extendability of partial* $\mathbf{G}$*-multisequences.* We define

$$\mathbf{G}_{n,k}^{n+m,k} := \left\{ \begin{matrix} \{c(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k}\} : \exists\{c(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n+m,k} - \tilde{\mathcal{L}}_{n,k}\} \text{ s.t.} \\ \{c(\boldsymbol{\beta}) : \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n+m,k}\} \in \mathbf{G}_{n+m,k} \end{matrix} \right\}. \tag{2.21}$$

This is the set of those partial multisequences of order n which can be extended to ones of order $n+m$. Clearly,

$$\mathbf{G}_{n,k}^{n+m,k} = \pi_n \circ \pi_{n+1} \circ \cdots \circ \pi_{n+m-1}(\mathbf{G}_{n+m,k})$$

and consequently,

$$\mathbf{G}_{n,k}^{n+m,k} = \text{Convex Hull } \{\pi_n \circ \pi_{n+1} \circ \ldots \circ \pi_{n+m-1}(\partial \mathbf{G}_{n+m,k})\}.$$

Simple calculations show that

$$\partial \mathbf{G}_{n,k}^{n+m,k} = \{\{c_n^{\gamma}(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : \boldsymbol{\gamma} \in \mathcal{L}_{n+m,k}\},$$

where

$$c_n^{\gamma}(\boldsymbol{\beta}) = \begin{cases} \dfrac{\binom{\gamma}{\beta}}{\binom{n+m}{\beta}} = \dfrac{\prod_{i=1}^{k}(\gamma_i(\gamma_i-1)\cdots(\gamma_i-\beta_i+1))}{(n+m)(n+m-1)\cdots(n+m-|\beta|+1)} & \text{if } \boldsymbol{\beta} \le \boldsymbol{\gamma} \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

(c) *Extendability of symmetric probabilities.* We define

$$\prod_{n,k}^{n+m,k} := \left\{ \begin{array}{c} \{q(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : \exists \{q(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n+m,k} - \tilde{\mathcal{L}}_{n,k}\} \\ \text{s.t. } \{q(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n+m,k}\} \in \prod_{n+m,k} \end{array} \right\}. \tag{2.23}$$

This is the set of those symmetric probabilities on E_k^n which can be extended to symmetric probabilities on E_k^{n+m}. Simple calculations show that

$$\partial \prod_{n,k}^{n+m,k} = \{\{q_n^{\gamma}(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : \boldsymbol{\gamma} \in \mathcal{L}_{n+m,k}\},$$

where

$$q_n^{\gamma}(\boldsymbol{\beta}) = \begin{cases} \dfrac{\binom{m}{\gamma-\beta}}{\binom{n+m}{\gamma}} & \text{if } \boldsymbol{\beta} \le \boldsymbol{\gamma} \text{ and } |\boldsymbol{\beta}| \ge |\boldsymbol{\gamma}| - m \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

(d) *The moment space of the standard k-simplex.* We define

$$\mathbf{M}_{\mathbf{n},\mathbf{k}} = \bigcap_{m=1}^{\infty} \mathbf{G}_{n,k}^{n+m,k} = \bigcap_{m=1}^{\infty} [\text{Convex Hull } \{\{c_n^{\gamma}(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : \boldsymbol{\gamma} \in \mathcal{L}_{n+m,k}\}], \tag{2.25}$$

where $c_n^{\gamma}(\cdot)$ are given by (2.22). This is the set of those partial multisequences of order n which can be extended to ones of order $m+n$ for each $m = 1, 2, \ldots$. In view of Theorem 1.1, it follows that

$$\mathbf{M}_{\mathbf{n},\mathbf{k}} = \left\{ \left\{ c_n(\boldsymbol{\beta}) = \int_{S_k} x^{\beta} \mathrm{d}\lambda, \ \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k} \right\} : \lambda \in \Lambda \right\}, \tag{2.26}$$

where Λ is the set of all probability measures on the simplex S_k.

Equation (2.25) expresses the nth moment space, i.e., the space of all moments of total order n or less, of the simplex, as an intersection of a decreasing sequence of polytopes in $R^{|\tilde{\mathcal{L}}_{n,k}|}$. Each $\mathbf{G}_{n,k}^{n+m,k}$ can be written as an intersection of finitely many closed half-spaces

and consequently, $\mathbf{M_{n,k}}$ can be expressed as an intersection of countably many closed half-spaces of $R^{|\tilde{\mathcal{L}}_{n,k}|}$.

The above description of $\mathbf{M_{n,k}}$ is rather indirect and we do not consider it very useful. Kemperman and Skibinsky [6], among others, gave explicit description of $\mathbf{M_{n,k}}$ for $n = 2, 3$ and $k = 2$; their description involves some non-linear conditions.

3. A normal limit theorem for G-multisequences

To get a better insight into the shape and structure of the sets $\mathbf{G}_{n,k}$, we would like to look at a typical point of it. For this purpose, for each $n = 1, 2, \ldots$, we put uniform probability measure on $\mathbf{G}_{n,k}$, i.e., the Lebesgue measure normalized by volume $(\mathbf{G}_{n,k})$. The coordinate variables $c_n(\boldsymbol{\beta})$ can now be viewed as random variables on the probability space $\mathbf{G}_{n,k}$. We fix an initial segment $\{c_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k}\}$, and prove that, after suitable centering and normalization, it converges in law, as $n \to \infty$, to a multivariate normal (briefly: MVN). This convergence to MVN is *controlled* by the Dirichlet distribution $D = D(1, 1, \ldots 1; 1)$ on the simplex S_k in the sense that the centering constants and the covariance matrix of the limiting MVN depend on the moments of D; see (3.12) and (3.13).

We find it convenient to introduce a suitable system of canonical coordinates in the space $\mathbf{G}_{n,k}$. Let

$$\mathcal{S}_{n,k} = \left\{ \{p_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\} : p_n(\boldsymbol{\beta}) \geq 0, \sum_{\beta \in \tilde{\mathcal{L}}_{n,k}} p_n(\boldsymbol{\beta}) \leq 1 \right\} \tag{3.1}$$

be the standard simplex in R^N, where

$$N = N(n,k) = |\tilde{\mathcal{L}}_{n,k}| = \binom{n+k}{k} - 1. \tag{3.2}$$

We put

$$p_n(0) = 1 - \sum_{\beta \in \tilde{\mathcal{L}}_{n,k}} p_n(\boldsymbol{\beta}) \tag{3.3}$$

and set up a bijection between $\mathcal{S}_{n,k}$ and $\prod_{n,k}$ by putting

$$p_n(\boldsymbol{\beta}) = \binom{n}{\boldsymbol{\beta}} q_n(\boldsymbol{\beta}), \quad \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}. \tag{3.4}$$

Of course, by (2.15) and (3.4),

$$p_n(0) = q_n(0). \tag{3.5}$$

This gives a bijection between $\mathbf{G}_{n,k}$ and $\mathcal{S}_{n,k}$. Explicitly, by (2.10), (2.11) and (3.4), we have

$$c_n(\boldsymbol{\beta}) = \frac{1}{\binom{n}{\beta}} \sum_{\delta \in \mathcal{L}_{n-|\beta|,k}} \binom{\boldsymbol{\beta}+\boldsymbol{\delta}}{\boldsymbol{\beta}} p_n(\boldsymbol{\beta}+\boldsymbol{\delta}) \tag{3.6}$$

and

$$p_n(\boldsymbol{\beta}) = \binom{n}{\boldsymbol{\beta}} \sum_{\delta \in \mathcal{L}_{n-|\beta|,k}} (-1)^{|\delta|} \binom{n-|\boldsymbol{\beta}|}{\boldsymbol{\delta}} c_n(\boldsymbol{\beta}+\boldsymbol{\delta}). \tag{3.7}$$

We shall employ $\{p_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}$ as the canonical coordinates of the space $\mathbf{G}_{n,k} \subset R^{N(n,k)}$. We take $\{e(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}$ as the standard basis in $R^{N(n,k)}$ and order the elements of this basis by assigning the natural order to $L_{j,k}$, $j = 1, 2, \ldots, n$ and within each $L_{j,k}$ choosing and fixing some linear order. Clearly, the matrices of linear transformations (3.6) and (3.7) with respect to the chosen ordered basis, are upper triangular. Further,

$$\frac{\partial(c_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k})}{\partial(p_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k})} = \left[\prod_{\beta \in \tilde{\mathcal{L}}_{n,k}} \binom{n}{\boldsymbol{\beta}} \right]^{-1}. \tag{3.8}$$

We need some combinatorial identities.

PROPOSITION 3.1

(a) $$\sum_{j=0}^{t} \binom{s+j}{s} = \binom{s+t+1}{s+1} \text{ for } s, t = 0, 1, 2, \ldots. \tag{3.9a}$$

(b) $$\sum_{j=0}^{t} \binom{-a}{t-j}\binom{-b}{j} = \binom{-(a+b)}{t} \text{ for } t = 0, 1, 2, \ldots \text{ and}$$

real a and b, or equivalently,

$$\sum_{j=0}^{t} \binom{a+t-j-1}{t-j}\binom{b+j-1}{j} = \binom{a+b+t-1}{t}. \tag{3.9b}$$

(c) $$\sum_{\gamma \in L_{j,k}} \binom{\beta+\gamma}{\beta} = \binom{|\beta|+k+j-1}{j} \tag{3.9c}$$

for $j = 0, 1, 2, \ldots, k = 1, 2, \ldots$ *and* $\beta = (\beta_1, \beta_2, \ldots, \beta_k), \beta_i = 0, 1, 2, \ldots$.

(d) $$\sum_{\gamma \in \mathcal{L}_{t,k}} \binom{\beta+\gamma}{\beta} = \binom{|\beta|+k+t}{t} \text{ for } k = 1, 2, \ldots, t = 0, 1, 2, \ldots. \tag{3.9d}$$

(e) $$\sum_{\delta \in L_{t,k}} \prod_{i=1}^{k} (1+\rho_i)^{\delta_i} = \sum_{r=0}^{t} \binom{t+k-1}{r+k-1} \sum_{\gamma \in \mathcal{L}_{r,k}} \prod_{i=1}^{k} \rho_i^{\gamma_i} \tag{3.9e}$$

for $t = 0, 1, 2, \ldots$ *and* $k = 1, 2, \ldots$.

(f) $$\prod_{\beta \in \mathcal{L}_{n,k}} \beta! = \prod_{r=0}^{n} (r!)^{k\binom{n-r+k-1}{k-1}} \tag{3.9f}$$

for $n = 0, 1, 2, \ldots, k = 1, 2, \ldots$.

(g) $$\sum_{\delta \in L_{t,k}} \binom{\gamma+\delta}{\alpha}\binom{\gamma+\delta}{\beta}$$

$$= \sum_{r=o}^{t \wedge (|\alpha|+|\beta|)} \sum_{\lambda=0}^{\gamma} \binom{\gamma}{\lambda} \sum_{\substack{\phi \in L_{r,k} \\ \gamma \le \phi+\lambda \le \alpha+\beta}} \binom{t+k-1}{r+k-1} \binom{\phi+\lambda}{\phi+\lambda-\beta \; \phi+\lambda-\alpha \; \alpha+\beta-\phi-\lambda} \tag{3.9g}$$

for $\alpha, \beta \in \mathcal{L}_{m,k}$ *and* $\gamma = \alpha \vee \beta$.

Proof. (a) This is easily proved by induction. □

(b) This follows by comparing the coefficients of x^t in the identity $(1+x)^{-a}(1+x)^{-b} = (1+x)^{-(a+b)}$. □

(c) The case $k = 1$ is trivial. Fix $k \geq 2$; we use induction. Assume that the identity holds for all j, all $\boldsymbol{\beta}$ and k replaced by $k-1$. Then

$$\begin{aligned}\sum_{\gamma \in L_{j,k}} \binom{\boldsymbol{\beta}+\boldsymbol{\gamma}}{\boldsymbol{\beta}} &= \sum_{i=0}^{j} \binom{\beta_1+i}{\beta_1} \sum_{\gamma_2+\gamma_3+\cdots+\gamma_k=j-i} \prod_{\lambda=2}^{k} \binom{\beta_\lambda+\gamma_\lambda}{\beta_\lambda} \\ &= \sum_{i=0}^{j} \binom{\beta_1+i}{\beta_1} \binom{|\boldsymbol{\beta}|-\beta_1+k-1+j-i-1}{j-i} \\ &\qquad \text{by induction hypothesis} \\ &= \binom{|\boldsymbol{\beta}|+k+j-1}{j} \text{ by (b).}\end{aligned}$$

□

(d) This follows from (c) and (a). □

(e) The identity trivially holds for $t = 0, k = 1, 2 \ldots$ and for $k = 1, t = 1, 2, \ldots$. We give the lexicographic ordering to the index set $I = \{(s,\lambda) : s = 0, 1, 2, \ldots, \lambda = 1, 2, \ldots\}$. Fix $(t,k), t \geq 1$ and $k \geq 2$ and assume the induction hypothesis that the equality holds for all elements of I strictly preceding (t,k). We prove the equality for (t,k). The term $\rho_1^{\gamma_1} \prod_{i=2}^{k} \rho_i^{\gamma_i}$, with $\boldsymbol{\gamma} \in L_{r,k}$, occurs in

$$\sum_{\delta \in L_{t,k}} \prod_{i=1}^{k} (1+\rho_i)^{\delta_i} = \sum_{\delta \in L_{t,k}} (1+\rho_1)^{\delta_1} \prod_{i=2}^{k} (1+\rho_i)^{\delta_i}$$

with coefficient

$$\begin{aligned}\sum_{\delta_1=\gamma_1}^{t-r+\gamma_1} \binom{\delta_1}{\gamma_1} \binom{t-\delta_1+k-2}{r-\gamma_1+k-2} &= \sum_{m=0}^{t-r} \binom{\gamma_1+m}{\gamma_1} \binom{t-\gamma_1-m+k-2}{r-\gamma_1+k-2} \\ &= \sum_{m=0}^{t-r} \binom{\gamma_1+1+(m-1)}{m} \\ &\quad \cdot \binom{r-\gamma_1+k-1+(t-r-m-1)}{t-r-m} \\ &= \binom{t+k-1}{t-r} \text{ by (3.9b).} \\ &= \binom{t+k-1}{r+k-1}.\end{aligned}$$

Thus

$$\sum_{\delta \in L_{t,k}} \prod_{i=1}^{k} (1+\rho_i)^{\delta_i} = \sum_{r=0}^{t} \binom{t+k-1}{r+k-1} \sum_{\gamma \in L_{r,k}} \prod_{i=1}^{k} \rho_i^{\gamma_i}.$$

□

(f) The identity trivially holds for $n = 0, k = 1, 2, \ldots$ and $k = 1, n = 1, 2, \ldots$. We give the lexicographic ordering to the index set $I = \{(m,\lambda) : m = 0, 1, 2, \ldots, \lambda = 1, 2, \ldots\}$.

Fix $(n,k), n \geq 1$ and $k \geq 2$ and assume the induction hypothesis that the equality holds for all elements of I strictly preceding (n,k). We prove the equality for (n,k). We have

$$\begin{aligned}
\prod_{\beta\in\mathcal{L}_{n,k}} \boldsymbol{\beta}! &= \prod_{r=0}^{n} \prod_{\beta_2+\beta_3+\cdots+\beta_k\leq n-r} \{r!(\beta_2!\beta_3!\ldots\beta_k!\} \\
&= \left\{\prod_{r=0}^{n}(r!)^{|\mathcal{L}(n-r,k-1)|}\right\}\left\{\prod_{r=0}^{n}\prod_{\beta\in\mathcal{L}_{n-r,k-1}}\boldsymbol{\beta}!\right\} \\
&= \left\{\prod_{r=0}^{n}(r!)^{|\mathcal{L}(n-r,k-1)|}\right\}\left\{\prod_{r=0}^{n}\prod_{s=0}^{n-r}(s!)^{(k-1)\binom{n-r-s+k-2}{k-2}}\right\} \\
&\qquad\qquad \text{by the induction hypothesis} \\
&= \left\{\prod_{r=0}^{n}(r!)^{|\mathcal{L}(n-r,k-1)|}\right\}\left\{\prod_{s=0}^{n}(s!)^{(k-1)\sum_{r=0}^{n-s}\binom{n-r-s+k-2}{k-2}}\right\} \\
&= \prod_{r=0}^{n}(r!)^{k\binom{n-r+k-1}{k-1}} \text{ by (3.9a).}
\end{aligned}$$

□

(g) Let $\mathbf{x} = (x_1, x_2, \ldots, x_k), \mathbf{y} = (y_1, y_2, \ldots, y_k)$ and $\mathbf{1}+\boldsymbol{\rho} := (1+\rho_1, 1+\rho_2, \ldots, 1+\rho_k)$ with $\rho_i = x_i + y_i + x_i y_i$. The LHS of the identity can be identified with the coefficient of $\mathbf{x}^{\alpha}\mathbf{y}^{\beta}$ in

$$\begin{aligned}
\sum_{\delta\in L_{t,k}}[(\mathbf{1}+\mathbf{x})(\mathbf{1}+\mathbf{y})]^{\gamma+\delta} &= (\mathbf{1}+\boldsymbol{\rho})^{\gamma}\sum_{\delta\in L_{t,k}}(\mathbf{1}+\boldsymbol{\rho})^{\delta} \\
&= (\mathbf{1}+\boldsymbol{\rho})^{\gamma}\sum_{r=0}^{t}\binom{t+k-1}{r+k-1}\sum_{\phi\in L_{r,k}}\boldsymbol{\rho}^{\phi} \text{ by (3.9e)} \\
&= \sum_{r=0}^{t}\sum_{\lambda=0}^{\gamma}\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}}\sum_{\phi\in L_{r,k}}\binom{t+k-1}{r+k-1}\boldsymbol{\rho}^{\phi+\lambda}.
\end{aligned}$$

We now observe that the coefficient of $u^a v^b$ in $(u+v+uv)^m$ equals $\binom{m}{m-b\ m-a\ a+b-m}$ if $a \vee b \leq m \leq a+b$ and zero otherwise. Thus

$$\text{LHS} = \sum_{r=0}^{t}\sum_{\lambda=0}^{\gamma}\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}}\sum_{\substack{\phi\in L_{r,k}\\ \gamma\leq\phi+\lambda\leq\alpha+\beta}}\binom{t+k-1}{r+k-1}\binom{\boldsymbol{\phi}+\boldsymbol{\lambda}}{\boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\beta}\ \boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\alpha}\ \boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\phi}-\boldsymbol{\lambda}}.$$

We now observe that if $r > |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$ then $|\boldsymbol{\phi}| + |\boldsymbol{\lambda}| > |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| + |\boldsymbol{\lambda}|$ and $\gamma_i \leq \phi_i + \lambda_i \leq \alpha_i + \beta_i$ is violated for some i. Thus LHS of the identity equals

$$\sum_{r=0}^{t\wedge(|\alpha|+|\beta|)}\sum_{\lambda=0}^{\gamma}\binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}}\sum_{\substack{\phi\in L_{r,k}\\ \gamma\leq\phi+\lambda\leq\alpha+\beta}}\binom{t+k-1}{r+k-1}\binom{\boldsymbol{\phi}+\boldsymbol{\lambda}}{\boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\beta}\ \boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\alpha}\ \boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\phi}-\boldsymbol{\lambda}}.$$

□

We put

$$V_{n,k} = \text{Volume}\,(\mathbf{G}_{n,k}). \tag{3.10}$$

PROPOSITION 3.2

As $n \to \infty$,

$$\frac{1}{n^{k+1}} \log\, V_{n,k} \to -A_k$$

where

$$A_k = \frac{k(k+1)}{k!} \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{(j+2)^2}. \tag{3.11}$$

The constants A_k *are positive with* $A_1 = \frac{1}{2}$, $A_2 = \frac{5}{12}$, $A_3 = \frac{13}{72}, \ldots$; *for* $k = 1$ *the above result agrees with the corresponding one in* [3].

Proof.

$$\begin{aligned}
V_{n,k} &= \int_{\mathbf{G}_{n,k}} \prod_{\beta \in \tilde{\mathcal{L}}_{n,k}} \mathrm{d}c_n(\boldsymbol{\beta}) \\
&= \left[\prod_{\beta \in \tilde{\mathcal{L}}_{n,k}} \binom{n}{\boldsymbol{\beta}} \right]^{-1} \int_{\mathcal{S}_{n,k}} \prod_{\beta \in \tilde{\mathcal{L}}_{n,k}} \mathrm{d}p_n(\boldsymbol{\beta}) \text{ by (3.8)} \\
&= \frac{\{\prod_{\beta \in \tilde{\mathcal{L}}_{n,k}} \boldsymbol{\beta}!\}\{\prod_{j=1}^{n} [(n-j)!]^{|L_{j,k}|}\}}{N(n,k)!(n!)^{|\tilde{\mathcal{L}}_{n,k}|}} \text{ by (2.4)} \\
&= \frac{\prod_{r=1}^{n} (r!)^{(k+1)\binom{n-r+k-1}{k-1}}}{N(n,k)!(n!)^{\binom{n+k}{k}}} \text{ by (3.2) and (3.9f)}.
\end{aligned}$$

Now,

$$\log N(n,k)! = o(n^{k+1}),$$

$$\binom{n+k}{k} \log n! = \frac{n^{k+1}}{k!} \log n - \frac{n^{k+1}}{k!} + o(n^{k+1})$$

and

$$\begin{aligned}
\log \left\{ \prod_{r=1}^{n} (r!)^{(k+1)\binom{n-r+k-1}{k-1}} \right\} &= (k+1)\, \frac{n^{k-1}}{k-1!} \sum_{r=1}^{n} \left(1 - \frac{r}{n}\right)^{k-1} (r \log r - r) + o(n^{k+1}) \\
&= \frac{(k+1)n^{k+1}}{k-1!} \sum_{r=1}^{n} \frac{1}{n} \left(1 - \frac{r}{n}\right)^{k-1} \left[\frac{r}{n} \log \frac{r}{n} + \frac{r}{n} \log n - \frac{r}{n}\right] + o(n^{k+1}) \\
&= \frac{(k+1)n^{k+1}}{k-1!} \left[\int_0^1 x(1-x)^{k-1} \log x \, \mathrm{d}x + \log n \int_0^1 x(1-x)^{k-1} \mathrm{d}x \right. \\
&\quad \left. - \int_0^1 x(1-x)^{k-1} \mathrm{d}x \right] + o(n^{k+1}).
\end{aligned}$$

Thus

$$\log V_{n,k} = \frac{k(k+1)}{k!} \cdot n^{k+1} \int_0^1 x(1-x)^{k-1} \log x \, \mathrm{d}x + o(n^{k+1})$$

$$= -\frac{k(k+1)}{k!} \cdot n^{k+1} \cdot \sum_{j=0}^{k-1} \frac{(-1)^j \binom{k-1}{j}}{(j+2)^2} + o(n^{k+1}). \qquad \square$$

PROPOSITION 3.3

The uniform probability measure on the space $\mathbf{G}_{n,k}$ is equivalent to having the uniform probability measure on the space $\mathcal{S}_{n,k}$ of canonical coordinates.

Proof. This is an immediate consequence of (3.8) and the change of variables formula for an integral on $\mathbf{G}_{n,k}$ to $\mathcal{S}_{n,k}$ and vice-versa. $\square$

The main result of this section is the following.

Theorem 3.4. *For each $m = 1, 2, \ldots,$ as $n \to \infty$, the law of*

$$\left\{ \sqrt{\binom{n+k}{k}} (c_n(\boldsymbol{\beta}) - c_0(\boldsymbol{\beta})),\ \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k} \right\}$$

relative to uniform probability on $\mathbf{G}_{n,k}$ converges weakly to a multivariate normal distribution MVN $[0, \Sigma_m]$ on $R^{N(m,k)}$, where

$$c_0(\boldsymbol{\beta}) = \frac{k!\boldsymbol{\beta}!}{(k+|\boldsymbol{\beta}|)!}, \quad \Sigma_m = ((\sigma(\boldsymbol{\alpha}, \boldsymbol{\beta})))_{\alpha,\beta \in \tilde{\mathcal{L}}_{m,k}}$$

$$\sigma(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!k!}{(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+k)!}. \tag{3.12}$$

We note that the centering constants and the covariances are given by the moments of the Dirichlet distribution $D(1, 1, \ldots; 1)$ on S_k:

$$c_0(\boldsymbol{\beta}) = \mu(\boldsymbol{\beta}), \quad \sigma(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mu(\boldsymbol{\alpha} + \boldsymbol{\beta}),$$

where

$$\mu(\boldsymbol{\gamma}) = \int_{S_k} x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \cdots \mathrm{d}x_k. \tag{3.13}$$

Proof. The uniform probability on the simplex $\mathcal{S}_{n,k}$ is just the Dirichlet $(1, 1, \ldots, 1; 1)$ distribution on it. Let $Z_0, Z_1 \ldots$ be a sequence of i.i.d. standard exponential random variables defined on, say, the probability space $(\Omega, \mathcal{F}, P)$. We relabel these random variables as $Z_\alpha, \boldsymbol{\alpha} \in \bigcup_{n=0}^{\infty} L_{n,k}$. The law, under P, of $\left\{ \frac{Z(\beta)}{\sum_{\alpha \in \mathcal{L}_{n,k}} Z(\alpha)} : \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k} \right\}$ is Dirichlet $(1, 1, \ldots; 1)$. Hence, by (3.6) and Proposition 3.3, the law of $\{c_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{n,k}\}$, under uniform probability on $\mathbf{G}_{n,k}$, is same as the law, under P, of

$$Y_n(\boldsymbol{\beta}) = \left[\sum_{\delta\in\mathcal{L}_{n-|\beta|}} \binom{\boldsymbol{\beta}+\boldsymbol{\delta}}{\boldsymbol{\delta}} Z(\boldsymbol{\beta}+\boldsymbol{\delta})\right] \Big/ \left[\binom{n}{\boldsymbol{\beta}} \sum_{\alpha\in\mathcal{L}_{n,k}} Z(\boldsymbol{\alpha})\right], \quad \boldsymbol{\beta}\in\tilde{\mathcal{L}}_{n,k}. \tag{3.14}$$

Now choose and fix an integer $m \geq 1$ and consider $n \geq m$. We observe that, for $\boldsymbol{\beta} \in \widetilde{\mathcal{L}}_{m,k}$,

$$\begin{aligned} E(Y_n(\boldsymbol{\beta})) &= \left[\sum_{\delta\in\mathcal{L}_{n-|\beta|}} \binom{\boldsymbol{\beta}+\boldsymbol{\delta}}{\boldsymbol{\delta}}\right] \Big/ \left[\binom{n}{\boldsymbol{\beta}}\binom{n+k}{k}\right] = \binom{k+|\boldsymbol{\beta}|}{k}^{-1} \text{ by (3.9d)} \\ &= c_0(\boldsymbol{\beta}) \text{ by (3.12)}. \end{aligned}$$

We further observe that

$$\left[\sum_{\alpha\in\mathcal{L}_{n,k}} Z(\boldsymbol{\alpha})\right] \Big/ \left[\binom{n+k}{k}\right] \xrightarrow{P} 1. \tag{3.15}$$

Hence, by (3.14) and (3.15), to prove the stated weak convergence of $\left\{\sqrt{\binom{n+k}{k}}(c_n(\boldsymbol{\beta}) - c_0(\boldsymbol{\beta})), \boldsymbol{\beta} \in \widetilde{\mathcal{L}}_{m,k}\right\}$, it suffices to prove that

$$\{T_n(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k}\} \Longrightarrow \text{MVN}(0, \Sigma_m),$$

where

$$T_n(\boldsymbol{\beta}) := \left[\sum_{\delta\in\mathcal{L}_{n-|\beta|,k}} \binom{\boldsymbol{\beta}+\boldsymbol{\delta}}{\boldsymbol{\delta}}\{Z(\boldsymbol{\beta}+\boldsymbol{\delta}) - 1\}\right] \Big/ \left[\binom{n}{\boldsymbol{\beta}}\sqrt{\binom{n+k}{k}}\right]. \tag{3.16}$$

For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k}$, Let $\boldsymbol{\gamma} = \boldsymbol{\alpha} \vee \boldsymbol{\beta}$. Then, for $n \geq |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| + |\boldsymbol{\gamma}|$, we have

$$\text{Cov}\ (T_n(\boldsymbol{\alpha}), T_n(\boldsymbol{\beta})) = \left[\sum_{\delta\in\mathcal{L}_{n-|\gamma|,k}} \binom{\boldsymbol{\gamma}+\boldsymbol{\delta}}{\boldsymbol{\alpha}}\binom{\boldsymbol{\gamma}+\boldsymbol{\delta}}{\boldsymbol{\beta}}\right] \Big/ \left[\binom{n}{\boldsymbol{\alpha}}\binom{n}{\boldsymbol{\beta}}\binom{n+k}{k}\right]$$

and, by (3.9g),

$$\begin{aligned} \left[\binom{n}{\boldsymbol{\alpha}}\binom{n}{\boldsymbol{\beta}}\binom{n+k}{k}\right] \text{Cov}\ (T_n(\boldsymbol{\alpha}), T_n(\boldsymbol{\beta})) &= \sum_{t=0}^{n-|\gamma|} \sum_{r=0}^{t\wedge|\alpha+\beta|} \sum_{\lambda=0}^{\gamma} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} \\ &\times \sum_{\substack{\phi\in L_{r,k} \\ \gamma\leq\phi+\lambda\leq\alpha+\beta}} \binom{t+k-1}{r+k-1} \binom{\boldsymbol{\phi}+\boldsymbol{\lambda}}{\boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\beta}\ \boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\alpha}\ \boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\phi}-\boldsymbol{\lambda}} \\ &= \sum_{r=0}^{|\alpha+\beta|} \sum_{\lambda=0}^{\gamma} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} \sum_{\substack{\phi\in L_{r,k} \\ \gamma\leq\phi+\lambda\leq\alpha+\beta}} \binom{\boldsymbol{\phi}+\boldsymbol{\lambda}}{\boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\beta}\ \boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\alpha}\ \boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\phi}-\boldsymbol{\lambda}} \sum_{t=r}^{n-|\gamma|} \binom{t+k-1}{r+k-1} \\ &= \sum_{r=0}^{|\alpha+\beta|} \sum_{\lambda=0}^{\gamma} \binom{\boldsymbol{\gamma}}{\boldsymbol{\lambda}} \sum_{\substack{\phi\in L_{r,k} \\ \gamma\leq\phi+\lambda\leq\alpha+\beta}} \binom{\boldsymbol{\phi}+\boldsymbol{\lambda}}{\boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\beta}\ \boldsymbol{\phi}+\boldsymbol{\lambda}-\boldsymbol{\alpha}\ \boldsymbol{\alpha}+\boldsymbol{\beta}-\boldsymbol{\phi}-\boldsymbol{\lambda}} \binom{n-|\boldsymbol{\gamma}|+k}{r+k} \end{aligned}$$

by (3.9a).

As $n \to \infty$, the dominating term corresponds to $r = |\boldsymbol{\alpha} + \boldsymbol{\beta}|$ and the conditions $r = |\boldsymbol{\alpha} + \boldsymbol{\beta}|, \boldsymbol{\phi} \in L_{r,k}, \boldsymbol{\gamma} \leq \boldsymbol{\phi} + \boldsymbol{\lambda} \leq \boldsymbol{\alpha} + \boldsymbol{\beta}$ imply that $\boldsymbol{\phi} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ and $\boldsymbol{\lambda} = 0$. Thus

$$\begin{aligned} \operatorname{Cov}(T_n(\boldsymbol{\alpha}), T_n(\boldsymbol{\beta})) &\sim \left[\binom{n}{\boldsymbol{\alpha}}\binom{n}{\boldsymbol{\beta}}\binom{n+k}{k}\right]^{-1}\binom{n-|\boldsymbol{\gamma}|+k}{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+k}\binom{\boldsymbol{\alpha}+\boldsymbol{\beta}}{\boldsymbol{\alpha}} \\ &\sim \frac{(\boldsymbol{\alpha}+\boldsymbol{\beta})!k!}{(|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|+k)!} = \sigma(\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ by (3.12).} \end{aligned} \tag{3.17}$$

By the Cramér–Wold theorem it suffices to prove that

$$\sum_{\beta \in \tilde{\mathcal{L}}_{m,k}} a(\boldsymbol{\beta}) T_n(\boldsymbol{\beta}) \Longrightarrow N\Big[0, \sum\sum a(\boldsymbol{\alpha}) a(\boldsymbol{\beta}) \sigma(\boldsymbol{\alpha}, \boldsymbol{\beta})\Big]$$

for all $\{a(\boldsymbol{\beta}), \boldsymbol{\beta} \in \tilde{\mathcal{L}}_{m,k}\}$. We have

$$\begin{aligned} &\sum_{\beta \in \tilde{\mathcal{L}}_{m,k}} a(\boldsymbol{\beta}) T_n(\boldsymbol{\beta}) \\ &\quad = \binom{n+k}{k}^{-1/2}\left[\sum_{\beta \in \tilde{\mathcal{L}}_{m,k}} a(\boldsymbol{\beta})\binom{n}{\boldsymbol{\beta}}^{-1} \sum_{\delta \in \mathcal{L}_{n-|\beta|,k}} \binom{\boldsymbol{\beta}+\boldsymbol{\delta}}{\boldsymbol{\delta}}(Z(\boldsymbol{\beta}+\boldsymbol{\delta}) - 1)\right] \\ &\quad = \binom{n+k}{k}^{-1/2} \sum_{\gamma \in \mathcal{L}_{n,k}} b_n(\boldsymbol{\gamma})(Z(\boldsymbol{\gamma}) - 1), \end{aligned} \tag{3.18}$$

where

$$b_n(\boldsymbol{\gamma}) = \sum_{\substack{\beta \leq \gamma \\ \beta \in \tilde{\mathcal{L}}_{m,k}}} \frac{a(\boldsymbol{\beta})\binom{\gamma}{\beta}}{\binom{n}{\beta}}. \tag{3.19}$$

We write

$$U_n := \sum_{\gamma \in \mathcal{L}_{n,k}} X_n(\boldsymbol{\gamma}) \text{ with } X_n(\boldsymbol{\gamma}) = b_n(\boldsymbol{\gamma})(Z(\boldsymbol{\gamma}) - 1)$$

and observe that U_n is a sum of independent random variables centered at their expectations. Further, we have the following:

(i) $$\begin{aligned} s_n^2 = \operatorname{Var}(U_n) &= \binom{n+k}{k}\operatorname{Var}\left(\sum a(\boldsymbol{\beta}) T_n(\boldsymbol{\beta})\right) \\ &\sim \binom{n+k}{k}\sum\sum a(\boldsymbol{\alpha}) a(\boldsymbol{\beta}) \sigma(\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ by (3.17)} \end{aligned}$$

(ii) $$\begin{aligned} \frac{1}{s_n^3}\sum_{\gamma \in \mathcal{L}_{n,k}} E|X_n(\boldsymbol{\gamma})|^3 &= \frac{1}{s_n^3}\sum_{\gamma \in \mathcal{L}_{n,k}} |b_n(\boldsymbol{\gamma})|^3 E|Z(\boldsymbol{\gamma}) - 1|^3 \\ &\to 0, \quad \text{as } n \to \infty, \end{aligned}$$

since $|b_n(\boldsymbol{\gamma})| \leq \sum_{\beta \in \tilde{\mathcal{L}}_{m,k}} |a(\boldsymbol{\beta})|$ by (3.19), $|\mathcal{L}_{n,k}| = \binom{n+k}{k}$ and $s_n^3 = O\left(\binom{n+k}{k}^{3/2}\right)$ by (i).

Hence, by Loéve [7], p. 275, $U_n/s_n \Longrightarrow N(0,1)$, or equivalently,

$$\sum_{\beta\in\tilde{\mathcal{L}}_{m,k}} a(\boldsymbol{\beta})T_n(\boldsymbol{\beta}) = \binom{n+k}{k}^{-1/2} U_n \Longrightarrow N\Big[0, \sum\sum a(\boldsymbol{\alpha})a(\boldsymbol{\beta})\sigma(\boldsymbol{\alpha},\boldsymbol{\beta})\Big].$$

This completes the proof. □

4. Remarks

The moment spaces $\mathbf{M_n}$ of $[0,1]$ – in our notation $\mathbf{M_n} = \mathbf{M_{n,1}}$ – have been comprehensively treated in the literature. Karlin and McGregor [5] study a random walk $\{X_n\}_{n\geq 0}$ on the nonnegative integers Z_+ such that

$$P(X_{n+1} = j | X_n = i) = \begin{cases} p_i & \text{if } j = i-1 \\ 1-p_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

and $0 < p_i < 1$ for $i \geq 1$, while $p_0 = 0$; they show that there exists a unique probability measure λ of infinite support on $[0,1]$ such that

$$P_{00}^{(2n)} = P(X_{2n} = 0 | X_0 = 0) = \int_0^1 x^n \,\mathrm{d}\lambda \text{ for all } n \geq 0. \tag{4.2}$$

This essentially establishes a one-to-one correspondence between the moment sequences on one hand and the parameters $\{p_i\}_{i\geq 0}$ of the associated random walks on the other; taking $0 < p_i < 1$ amounts to confining oneself to the interior of $\mathbf{M_n}$. The parameters $p_i, i \geq 1$, associated with λ turn out to be canonical moments of λ introduced by Skibinsky [8]. Chang, Kemperman and Studden [1] employ $\{p_i\}$ as the canonical coordinates of the space $\mathbf{M_n}$ and prove a normal limit theorem for moment sequences.

All our efforts to find analogues of the random walk of Karlin and McGregor [5] or of canonical moments associated with a probability measure on S_k were unsuccessful. It would be desirable to do an asymptotic analysis of the moment spaces $\mathbf{M_{n,k}}$ for $k \geq 2$. That, in the present state of our knowledge, appears to be a formidable task.

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References

[1] Chang F C, Kemperman J H B and Studden W J, A normal limit theorem for moment sequences. *Ann. Probab.* **21** (1993) 1295–1309
[2] Gupta J C, The moment problem for the standard k-dimensional simplex. *Sankhyā, Series A* **61** (1999) 286–291
[3] Gupta J C, Partial Hausdorff sequences and symmetric probabilities on finite products of $\{0,1\}$. *Sankhyā, Series A* **61** (1999) 347–357
[4] Hausdorff F, Momentprobleme für ein endliches Intervall, *Math. Z.* **16** (1923) 220–248

[5] Karlin S and McGregor J, Random walks, *Illinois J. Math.* **3** (1959) 66–81
[6] Kemperman J H B and Skibinsky M, Covariance spaces for measures on polyhedral sets. In *Stochastic Inequalities* (eds) M Shaked, and Y L Tong, *IMS Lecture Notes* **22** (1993) 182–195
[7] Loéve M, *Probability Theory*, 3rd edition, (New York: Van Nostrand) (1963)
[8] Skibinsky M, The range of $(r+1)$th moment for distributions on $[0,1]$, *J. Appl. Probab.* **4** (1967) 543–552