

Sampling and Π -sampling expansions

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Abstract. Using the hyperfinite representation of functions and generalized functions this paper develops a rigorous version of the so-called ‘delta method’ approach to sampling theory. This yields a slightly more general version of the classical WKS sampling theorem for band-limited functions.

Keywords. Sampling expansions; WKS sampling theorem; non-standard analysis; hyperfinite sums.

1. Preliminaries

The classical sampling expansion for band-limited functions can be derived rigorously by several distinct arguments, but the use of so-called ‘delta-methods’ offers an approach which is intuitively most satisfying. A rigorous form of a delta-method derivation of the sampling expansion has been presented, using standard analysis, by Nashed and Walter [1]. In this paper we consider instead a non-standard approach to sampling theory. The hyperfinite representation of functions and generalized functions has been studied in an earlier paper [2], and the same notation and conventions will be used here. In particular, $\kappa \in {}^*\mathbb{N}_\infty$ denotes a given even infinite hypernatural number, $\varepsilon = \kappa^{-1} \approx 0$ and

$$\begin{aligned}\Pi \equiv \Pi_\kappa &= \left\{ -\frac{\kappa}{2}, -\frac{\kappa}{2} + \varepsilon, \dots, 0, \dots, \frac{\kappa}{2} - \varepsilon \right\} \\ &= \left\{ \left(-\frac{\kappa^2}{2} + j - 1 \right) \varepsilon : j = 1, 2, \dots, \kappa^2 \right\} \subset {}^*\mathbb{R}\end{aligned}$$

is the (unbounded) *hyperfinite line*. Given a standard point $r \in \mathbb{R}$, define the Π -*monad* of r by

$$\text{mon}_\Pi(r) = \text{st}_\Pi^{-1}(r) = \text{mon}(r) \cap \Pi$$

where ‘mon’ denotes the usual monad of a standard number in ${}^*\mathbb{R}$. Then the set $\Pi_b = \bigcup_{r \in \mathbb{R}} \text{mon}_\Pi(r) = \text{st}_\Pi^{-1}(\mathbb{R}) \subset \Pi$ is the *nearstandard hyperfinite line* and $\Pi_\infty = \Pi \setminus \Pi_b$ is the set of *remote points* of the hyperfinite line. For every subset $A \subset \mathbb{R}$ define ${}^*A_\Pi = {}^*A \cap \Pi$ and $\text{ns}_\Pi({}^*A) = {}^*A \cap \Pi_b = \bigcup_{a \in A} \text{mon}_\Pi(a)$.

By \mathbb{F}_Π we denote the algebra of all internal functions $F : \Pi \rightarrow {}^*\mathbb{C}$ which are periodically extended to the infinite grid $\varepsilon \cdot {}^*\mathbb{Z}$. The two difference operators $\mathbf{D}_+, \mathbf{D}_- : \mathbb{F}_\Pi \rightarrow \mathbb{F}_\Pi$ defined, for every function F and $x \in \Pi$, by

$$\mathbf{D}_+ F(x) = \varepsilon^{-1} [F(x + \varepsilon) - F(x)] \quad \text{and} \quad \mathbf{D}_- F(x) = \varepsilon^{-1} [F(x) - F(x - \varepsilon)]$$

are called, respectively, the forward and the backward Π -difference operators (of first order). Iterating \mathbf{D}_+ (or \mathbf{D}_-) we obtain higher order Π -difference operators: for every

(finite or infinite) $n \in {}^*\mathbb{N}_0$

$$\mathbf{D}_+^n F(x) = \mathbf{D}_+(\mathbf{D}_+^{n-1} F(x)), \quad x \in \Pi$$

and similarly for \mathbf{D}_- . It is easily seen that for any two functions $F, G \in \mathbb{F}_\Pi$ we have, (both for \mathbf{D}_+ and \mathbf{D}_-),

$$\mathbf{D}(F + G) = \mathbf{D}F + \mathbf{D}G \quad \text{and} \quad \mathbf{D}(F \cdot G) = (\mathbf{D}F)G + F(\mathbf{D}G) \pm \varepsilon(\mathbf{D}F)(\mathbf{D}G),$$

where we take $\pm \varepsilon$ according to the use of \mathbf{D}_+ or \mathbf{D}_- , respectively.

For every $\alpha, x \in \Pi$ define the Π -intervals (containing only points in Π) $J_\alpha^+(x)$ and $J_\alpha^-(x)$ as follows:

$$J_\alpha^+(x) = \begin{cases} (x, \alpha]_\Pi & \text{if } x < \alpha \\ [x, \alpha)_\Pi & \text{if } x > \alpha \end{cases}$$

$$J_\alpha^-(x) = \begin{cases} [\alpha, x)_\Pi & \text{if } x < \alpha \\ (\alpha, x]_\Pi & \text{if } x > \alpha \end{cases}$$

while for $x = \alpha$ we have $J_\alpha^+(x) = \emptyset = J_\alpha^-(x)$. For any $F \in \mathbb{F}_\Pi$ define the functions \mathbf{S}_+F and \mathbf{S}_-F to be the forward and backward Π -sums of F which are zero at the origin and which, for every $x \in \Pi \setminus \{0\}$ are defined by

$$\mathbf{S}_+F(x) = \sum_{t \in J_0^+(x)} \varepsilon F(t) \quad \text{and} \quad \mathbf{S}_-F(x) = \sum_{t \in J_0^-(x)} \varepsilon F(t).$$

The Π -sum operators \mathbf{S}_+ and \mathbf{S}_- both transform \mathbb{F}_Π into \mathbb{F}_Π . Moreover, for every $F \in \mathbb{F}_\Pi$, we have

$$\mathbf{D}_+\mathbf{S}_+F = F \quad \text{and} \quad \mathbf{D}_-\mathbf{S}_-F = F$$

that is, \mathbf{S}_+ and \mathbf{S}_- are left inverses for \mathbf{D}_+ and \mathbf{D}_- , respectively.

Define the translation operator $\tau_\alpha : \mathbb{F}_\Pi \rightarrow \mathbb{F}_\Pi$ (with $\alpha \in \Pi$) by setting $\tau_\alpha F(x) = F(x - \alpha)$ for every function F and $x \in \Pi$.

2. Π -periodic functions and Π -Fourier sums

2.1 Π -Fourier sums

For any internal function $F \in \mathbb{F}_\Pi$ define the Π -periodic transform of F with period 1 (or, simply, the Π -periodic transform¹ of F) to be the internal function $\mathbf{T}_\Pi[F]$ in \mathbb{F}_Π which is such that

$$\mathbf{T}_\Pi[F](x) = \sum_{n \in {}^*\mathbf{Z}_\Pi} F(x - n), \quad x \in \Pi$$

where ${}^*\mathbf{Z}_\Pi \equiv {}^*\mathbb{Z} \cap \Pi$. (As usual we suppose that the function $\mathbf{T}_\Pi[F]$ is periodically extended to the whole of the discrete line ${}^*\mathbb{Z}$.) In particular for the function Δ_0 defined by

$$\Delta_0(x) = \begin{cases} \kappa & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

¹Unless explicitly stated, Π -periodic transforms will always be understood here to have period 1.

we obtain

$$\mathbf{T}_{\Pi}[\Delta_0](x) = \sum_{n \in {}^*\mathbf{Z}_{\Pi}} \Delta_0(x - n)$$

which is the internal function in \mathbb{F}_{Π} defined by

$$\mathbf{T}_{\Pi}[\Delta_0](x) = \begin{cases} \kappa & \text{if } x \in {}^*\mathbf{Z}_{\Pi} \\ 0 & \text{otherwise in } \Pi \end{cases}.$$

Considering the Π -convolution of an arbitrary function $F \in \mathbb{F}_{\Pi}$ with the function $\mathbf{T}_{\Pi}[\Delta_0]$, we get

$$\begin{aligned} F * \mathbf{T}_{\Pi}[\Delta_0](x) &= \sum_{y \in \Pi} \varepsilon F(x - y) \mathbf{T}_{\Pi}[\Delta_0](y) \\ &= \sum_{y \in \Pi} \varepsilon F(x - y) \left\{ \sum_{n \in {}^*\mathbf{Z}_{\Pi}} \Delta_0(y - n) \right\} \\ &= \sum_{n \in {}^*\mathbf{Z}_{\Pi}} \left\{ \sum_{y \in \Pi} \varepsilon F(x - y) \Delta_0(y - n) \right\} = \sum_{n \in {}^*\mathbf{Z}_{\Pi}} F(x - n) \end{aligned}$$

and, therefore, we may write

$$\mathbf{T}_{\Pi}[F] = F * \mathbf{T}_{\Pi}[\Delta_0]. \quad (1)$$

Taking the Π -Fourier transform of the internal function $\mathbf{T}_{\Pi}[\Delta_0]$, we obtain

$$\begin{aligned} \mathbf{F}_{\Pi}[\mathbf{T}_{\Pi}[\Delta_0]](y) &= \sum_{x \in \Pi} \varepsilon \mathbf{T}_{\Pi}[\Delta_0](x) e^{2\pi i x y} = \sum_{x \in \Pi} \varepsilon \left\{ \sum_{n \in {}^*\mathbf{Z}_{\Pi}} \Delta_0(x - n) \right\} e^{2\pi i x y} \\ &= \sum_{n=-\kappa/2}^{\kappa/2-1} \left\{ \sum_{x \in \Pi} \varepsilon \Delta_0(x - n) e^{2\pi i x y} \right\} \\ &= \sum_{n=-\kappa/2}^{\kappa/2-1} e^{2\pi i n y} = \sum_{n=0}^{\kappa-1} e^{2\pi i (n-\kappa/2)y} \\ &= e^{-i\kappa\pi y} \sum_{n=0}^{\kappa-1} e^{2\pi i n y} = \frac{1 - e^{2i\pi\kappa y}}{1 - e^{2i\pi y}} * \exp_{\Pi}(-i\pi\kappa y). \end{aligned}$$

Then, since $y = -\frac{\kappa}{2} + j\varepsilon$ where $j = 0, 1, \dots, \kappa^2 - 1$, this sum vanishes for all y such that $j \neq r\kappa$ and has the value κ for all y such that $j = r\kappa$, where $r = 0, 1, \dots, \kappa - 1$. Thus, we get

$$\mathbf{T}_{\Pi}[\Delta_0](y) = \mathbf{F}_{\Pi}[\mathbf{T}_{\Pi}[\Delta_0]](y) = \sum_{n \in {}^*\mathbf{Z}_{\Pi}} e^{2\pi i n y}, \quad y \in \Pi \quad (2)$$

which will be referred to as the Π -Fourier sum² for the internal function $\mathbf{T}_{\Pi}[\Delta_0]$.

²Similarly, we would obtain $\mathbf{T}_{\Pi}[\Delta_0](x) = \bar{\mathbf{F}}_{\Pi}[\mathbf{T}_{\Pi}[\Delta_0]](x) = \sum_{n \in {}^*\mathbf{Z}_{\Pi}} e^{-2\pi i n x}$, $x \in \Pi$.

For an arbitrary internal function $F \in \mathbb{F}_\Pi$ and any $x \in \Pi$ we have

$$\begin{aligned} \mathbf{T}_\Pi[F](x) &= F * \mathbf{T}_\Pi[\Delta_0](x) = \sum_{y \in \Pi} \varepsilon F(x-y) \left\{ \sum_{n \in {}^*\mathbf{Z}_\Pi} e^{2\pi i n y} \right\} \\ &= \sum_{n \in {}^*\mathbf{Z}_\Pi} \left\{ \sum_{y \in \Pi} \varepsilon F(x-y) e^{2\pi i n y} \right\} \\ &= \sum_{n \in {}^*\mathbf{Z}_\Pi} e^{2\pi i n x} \left\{ \sum_{y \in \Pi} \varepsilon F(y) e^{2\pi i n y} \right\} = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{F}(n) e^{2\pi i n x}, \end{aligned}$$

where

$$\hat{F}(n) = \sum_{y \in \Pi} \varepsilon F(y) e^{-2\pi i n y}, \quad n \in {}^*\mathbf{Z}_\Pi.$$

Thus

$$\mathbf{T}_\Pi[F](x) = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{F}(n) e^{2\pi i n x}, \quad x \in \Pi \quad (3)$$

is the Π -Fourier sum for the internal function $\mathbf{T}_\Pi[F]$. If, in particular, we take $F = \Delta_0$, then $\hat{\Delta}_0(n) = 1$ for all $n \in {}^*\mathbf{Z}_\Pi$ and we recover (2).

Writing (3) in the form

$$\sum_{n \in {}^*\mathbf{Z}_\Pi} F(x+n) = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{F}(n) e^{2\pi i n x}, \quad x \in \Pi$$

and setting $x = 0$ we obtain

$$\sum_{n \in {}^*\mathbf{Z}_\Pi} F(n) = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{F}(n) \quad (4)$$

which is the Π -Poisson formula for the internal function $F(x), x \in \Pi$.

2.2 Π -periodic functions

An internal function $P \in \mathbb{F}_\Pi$ is said to be Π -periodic of period 1 (or, simply, Π -periodic)³ if and only if

$$P(x+1) = P(x)$$

for all $x \in \Pi$. We denote by \mathbb{P}_Π the subset of all Π -periodic internal functions in \mathbb{F}_Π .

Let F be any internal function in \mathbb{F}_Π . Since for any $x \in \Pi$ we have that

$$\begin{aligned} \mathbf{T}_\Pi[F](x+1) &= \sum_{n \in {}^*\mathbf{Z}_\Pi} F((x+1)-n) \\ &= \sum_{n \in {}^*\mathbf{Z}_\Pi} F(x-(n-1)) = \sum_{n \in {}^*\mathbf{Z}_\Pi} F(x-n) = \mathbf{T}_\Pi[F](x) \end{aligned}$$

then the Π -periodic transform of any internal function $F \in \mathbb{F}_\Pi$ belongs to \mathbb{P}_Π . Conversely, we have

³By Π -periodic functions we will always understand Π -periodic functions of period 1 defined on Π and periodically extended to the whole $\varepsilon^*\mathbb{Z}$.

Theorem 2.1. Every internal function P in \mathbb{P}_Π is the Π -periodic transform of an internal function Φ_P supported in $^*[-1/2, +1/2)_\Pi$.⁴

Proof. Let $\mathbf{H}(x)$, $x \in \mathbb{R}$, denote the (usual) Heaviside unit step function defined by

$$\mathbf{H}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then we have that

$$^*\mathbf{H}_\Pi(x) = \sum_{t=-\frac{\kappa}{2}}^x \varepsilon \Delta_0(t)$$

and, moreover,

$$^*\mathbf{H}_\Pi(x + 1/2) - ^*\mathbf{H}_\Pi(x - 1/2), \quad x \in \Pi$$

is an internal function supported in the Π -interval $^*[-1/2, 1/2)_\Pi$. Hence,

$$\Phi_P(x) = [^*\mathbf{H}_\Pi(x + 1/2) - ^*\mathbf{H}_\Pi(x - 1/2)]P(x), \quad x \in \Pi$$

defines an internal function in \mathbb{F}_Π supported in $^*[-1/2, +1/2)_\Pi$, and

$$\begin{aligned} \mathbf{T}_\Pi[\Phi_P](x) &= \sum_{n \in ^*\mathbf{Z}_\Pi} \Phi_P(x - n) \\ &= \sum_{n \in ^*\mathbf{Z}_\Pi} [^*\mathbf{H}_\Pi(x + 1/2 - n) - ^*\mathbf{H}_\Pi(x - 1/2 - n)]P(x - n) \\ &= P(x) \sum_{n \in ^*\mathbf{Z}_\Pi} [^*\mathbf{H}_\Pi(x + 1/2 - n) - ^*\mathbf{H}_\Pi(x - 1/2 - n)] = P(x) \end{aligned}$$

as stated. \square

From this proposition it follows that

$$\begin{aligned} P(x) &= \mathbf{T}_\Pi[\Phi_P](x) = \Phi_P * \mathbf{T}_\Pi[\Delta_0](x) \\ &= \sum_{y \in \Pi} \varepsilon \Phi_P(y) \mathbf{T}_\Pi[\Delta_0](x - y) = \sum_{n \in ^*\mathbf{Z}_\Pi} e^{2\pi i n x} \left\{ \sum_{y \in \Pi} \varepsilon \Phi_P(y) e^{-2\pi i n y} \right\} \\ &= \sum_{n \in ^*\mathbf{Z}_\Pi} e^{2\pi i n x} \left\{ \sum_{-1/2 \leq y < 1/2} \varepsilon \Phi_P(y) e^{-2\pi i n y} \right\} = \sum_{n \in ^*\mathbf{Z}_\Pi} c_{P,n} e^{2\pi i n x}, \end{aligned}$$

where, for every $n \in ^*\mathbf{Z}_\Pi$,

$$c_{P,n} = \sum_{-1/2 \leq y < 1/2} \varepsilon \Phi_P(y) e^{-2\pi i n y} = \hat{\Phi}_P(n)$$

is the n th Π -Fourier coefficient of the Π -periodic function $P \in \mathbb{P}_\Pi$. Hence

$$P(x) = \sum_{n \in ^*\mathbf{Z}_\Pi} \hat{\Phi}_P(n) e^{2\pi i n x}, \quad x \in \Pi$$

⁴That is to say, a function which is zero outside $^*[-1/2, 1/2)_\Pi$.

is the Π -Fourier sum of the internal function $P \in \mathbb{P}_\Pi$. Moreover, restricting suitably the variable x , we get

$$\Phi_P(x) = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{\Phi}_P(n) e^{2\pi i n x}, \quad -1/2 \leq x < 1/2.$$

2.2.1 The Π -Fourier sum as an extension of the classical Fourier series for continuous periodic functions: Let f be a periodic function (with period 1), defined on the real line. Then there exists a function $g : [-1/2, 1/2) \rightarrow \mathbb{C}$ such that

$$f = \sum_{n=-\infty}^{+\infty} \tau_n \circ g.$$

The Π -nonstandard extension of g

$$\Phi(x) = {}^*g_\Pi(x)$$

is a finite $\mathbf{S}\Pi$ -continuous function on ${}^*[-1/2, 1/2)_\Pi$ and, moreover, $\mathbf{T}_\Pi[\Phi]$ is an $\mathbf{S}\Pi$ -continuous Π -periodic internal function. Then

$$\mathbf{T}_\Pi[\Phi](x) = \sum_{n \in {}^*\mathbf{Z}_\Pi} \hat{G}(n) e^{2\pi i n x}, \quad (5)$$

where

$$\hat{G}(n) = \sum_{-1/2 \leq x < 1/2} \varepsilon {}^*g_\Pi(x) e^{-2\pi i n x}.$$

Suppose in addition that g is twice differentiable in $[-1/2, 1/2)$. Thus $\mathbf{D}_\Pi^2 \Phi$ is always finite and, therefore, the sum

$$\Gamma(|\mathbf{D}_+^2 \Phi|) \equiv \sum_{-1/2 \leq x < 1/2} \varepsilon |\mathbf{D}_\Pi^2 \Phi(x)|$$

is also finite. Hence we have that

$$\mathbf{F}_\Pi[\mathbf{D}_+^2 \Phi](x) = \lambda(x)^2 \hat{G}(x)$$

and so, for $n \neq 0$, we get

$$\hat{G}(n) = \frac{1}{|\lambda(n)|^2} |\mathbf{F}_\Pi[\mathbf{D}_\Pi^2 \Phi](n)|.$$

Since $\lambda(n) = (2\pi i n) e^{i\pi \varepsilon n} {}^*\text{sinc}_\Pi(\varepsilon n)$ then, for every $n \in {}^*\mathbf{Z}_\Pi$, we have that $|\lambda(n)|^2 \geq 16|n|^2$. On the other hand, for any $n \in {}^*\mathbf{Z}_\Pi$, we have

$$|\bar{\mathbf{F}}_\Pi[\mathbf{D}_+^2 \Phi](n)| = \left| \sum_{-1/2 \leq x < 1/2} \varepsilon \mathbf{D}_+^2 \Phi(x) e^{-2\pi i n x} \right| \leq \Gamma(|\mathbf{D}_+^2 \Phi|),$$

where $\Gamma(|\mathbf{D}_+^2 \Phi|)$ is a finite number. Hence, for large values of $|n|$, $n \in {}^*\mathbf{Z}_\Pi$, it follows that

$$|\hat{G}(n)| \leq \frac{\Gamma(|\mathbf{D}_+^2 \Phi|)}{16} \cdot \frac{1}{|n|^2} \equiv C \cdot \frac{1}{|n|^2},$$

where C is a finite constant.

Let ν_1 and ν_2 be two infinite hypernatural numbers such that $\nu_1 \leq \nu_2 < \kappa/2$. Then, for any $x \in \Pi$, we have

$$\left| \sum_{|n|=\nu_1}^{\nu_2} \hat{G}(n) e^{2\pi i n x} \right| \leq \sum_{|n|=\nu_1}^{\nu_2} |\hat{G}(n)| \leq 2C \sum_{n=\nu_1}^{\nu_2} \frac{1}{n^2}.$$

Since we have that

$$\sum_{n=\nu_1}^{\nu_2} \frac{1}{n^2} \approx 0$$

for all infinite $\nu_1, \nu_2 \in {}^*\mathbb{N}_\infty$, ($\nu_1 \leq \nu_2$), it follows that

$$\left| \sum_{|n|=\nu_1}^{\nu_2} \hat{G}(n) e^{2\pi i n x} \right| \approx 0$$

for all $x \in \Pi_b$ and all $\nu_1, \nu_2 \in {}^*\mathbb{N}_\infty$, ($\nu_1 \leq \nu_2$).

Then we have

Theorem 2.2. *If t denotes any (standard) point in $[-1/2, 1/2) \subset \mathbb{R}$, then the Π -Fourier sum (5) reads as follows*

$$f(t) = \text{st} \circ \mathbf{T}_\Pi[\Phi](x) = \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} \hat{G}(n) e^{2\pi i n x} \right) = \sum_{n=-\infty}^{+\infty} \hat{g}(n) e^{2\pi i n t}$$

for all $x \in \text{mon}_\Pi(t)$.

Proof. For each $x \in \text{mon}_\Pi(t)$ and for every real $r > 0$, the set

$$\left\{ \nu \in {}^*\mathbf{Z}_\Pi \cap {}^*\mathbb{N} : \left| \sum_{|n|=\nu}^{\frac{\kappa}{2}-1} \hat{G}(n) e^{2\pi i n x} \right| < r \right\}$$

is internal and contains all infinite positive numbers in ${}^*\mathbf{Z}_\Pi$. Then, by underflow it contains a finite number $n_r \in \mathbb{N}$. Hence

$$\left| \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} \hat{G}(n) e^{2\pi i n x} - \sum_{n=-n_r}^{n_r} \hat{G}(n) e^{2\pi i n x} \right| < r$$

and thus, taking standard parts, we obtain

$$\left| \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} \hat{G}(n) e^{2\pi i n x} \right) - \sum_{n=-n_r}^{n_r} \text{st} \hat{G}(n) \cdot \text{st} e^{2\pi i n x} \right| < r.$$

Taking into account that for $x \in \text{mon}_\Pi(t)$ and $|n| \in \mathbb{N}_0$ we have $e^{2\pi i n x} \approx e^{2\pi i n t}$ and $\text{st } \hat{G}(n) = \hat{g}(n)$ then, since the real number $r > 0$ is arbitrary, it follows that

$$\begin{aligned} f(t) &= \text{st} \circ \mathbf{T}_\Pi[\Phi](x) \\ &= \text{st} \left(\sum_{n=-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}-1} \hat{G}(n) e^{2\pi i n x} \right) = \sum_{n=-\infty}^{+\infty} \hat{g}(n) e^{2\pi i n t} \end{aligned}$$

for any $x \in \text{mon}_\Pi(t)$. □

3. Π -Bandlimited functions

3.1 Basic definitions and results

Denote by \mathbf{I} the unitary Π -interval $\mathbf{I} = \{-\frac{1}{2} \cdots \frac{1}{2} - \varepsilon\}$ and let the function \hat{F} in \mathbb{F}_Π be supported on \mathbf{I} (that is, $\hat{F}(y) = 0$ for all $y \notin \mathbf{I}$). Then, the internal function $F = \bar{\mathbf{F}}_\Pi[\hat{F}] \in \mathbb{F}_\Pi$ is said to be Π -bandlimited to \mathbf{I} . Denote by $\mathbb{B}_\Pi \equiv \mathbb{B}_\Pi(\mathbf{I})$ the subspace of \mathbb{F}_Π comprising all functions which are Π -bandlimited⁵.

For any function $F \in \mathbb{B}_\Pi$ we have

$$F(x) = \sum_{y \in \Pi} \varepsilon \hat{F}(y) e^{2\pi i x y} = \sum_{-1/2 \leq y < 1/2} \varepsilon \hat{F}(y) e^{2\pi i x y}$$

and therefore the inequality

$$|F(x)| \leq \Gamma(|\hat{F}|)$$

holds for all $x \in \Pi$. The function F may extend to the hyperfinite plane $\Pi + i\Pi$ by setting

$$\begin{aligned} F(\xi + i\eta) &= \sum_{-1/2 \leq y < 1/2} \varepsilon \hat{F}(y) e^{2\pi i (\xi + i\eta)y} \\ &= \sum_{-1/2 \leq y < 1/2} \varepsilon \{e^{-2\pi \eta y} \hat{F}(y)\} e^{2\pi i \xi y}. \end{aligned}$$

Hence we get

$$|F(\xi + i\eta)| \leq \Gamma(|\hat{F}|) \cdot \exp(\pi|\eta|)$$

for all $\xi + i\eta \in \Pi + i\Pi$.

Moreover, for any $j \in {}^*\mathbb{N}_0$, we obtain

$$\mathbf{D}_+^j F(x) = \sum_{-1/2 \leq y < 1/2} \varepsilon \lambda^j(y) \hat{F}(y) e^{2\pi i x y}$$

and therefore

$$|\mathbf{D}_+^j F(x)| \leq C_j \cdot \Gamma(|\hat{F}|),$$

where

$$C_j = \max_{-1/2 \leq y < 1/2} |\lambda^j(y)| \leq 2\pi \max_{-1/2 \leq y < 1/2} |t|^j = 2^{1-j}\pi.$$

For finite j the constant C_j is finite; for infinite j the constant C_j is infinitesimal.

⁵We will consider here only functions Π -bandlimited to \mathbf{I} and these will be referred to simply as Π -bandlimited functions. The generalization to other (finite or hyperfinite) intervals is immediate.

Hence, if $\Gamma(|\hat{F}|)$ is finite the function $F \equiv \bar{\mathbf{F}}_\Pi[\hat{F}]$ is finite and so also are its Π -derivatives over the whole hyperfinite line; moreover, F extends to $\Pi + i\Pi$ as a finite function of exponential type $\leq \pi$. In general F is not $\mathbf{S}\Pi$ -continuous on Π_b and therefore $\text{st} \circ F$ may not exist; however, the equation

$$\begin{aligned} \langle F, {}^*\hat{\varphi}_\Pi \rangle &= \sum_{x \in \Pi} \varepsilon F(x) {}^*\hat{\varphi}_\Pi(x) \\ &\approx \sum_{y \in \mathbf{I}} \varepsilon \hat{F}(y) {}^*\varphi_\Pi(y) = \langle \hat{F}, {}^*\varphi_\Pi \rangle \end{aligned}$$

holds for any function φ such that $\varphi \circ \text{st} = \text{st} \circ {}^*\varphi_\Pi$ and $\hat{\varphi} \circ \text{st} = \text{st} \circ {}^*\hat{\varphi}_\Pi$. This is certainly true for any function φ which is continuous and supported on the interval $[-1/2, 1/2]$; therefore F defines a continuous linear functional over the linear space which is the Fourier transform of the space $\mathcal{C}[-1/2, 1/2]$ (with the topology generated by the uniform norm).

3.2 The Π -sampling expansion

Let F be a Π -bandlimited function in \mathbb{B}_Π . Then we have

$$F(x) = \sum_{y \in \Pi} \varepsilon \hat{F}(y) e^{2\pi ixy} = \sum_{-1/2 \leq y < 1/2} \varepsilon \hat{F}(y) e^{2\pi ixy} \quad (6)$$

for every $x \in \Pi$. Now, for $y \in \mathbf{I}$, we have

$$\hat{F}(y) = \mathbf{T}_\Pi[\hat{F}](y) = \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} c_n e^{-2\pi iny},$$

where, for every $n \in {}^*\mathbf{Z}_\Pi$, the coefficient c_n is given by

$$c_n = \sum_{-1/2 \leq y < 1/2} \varepsilon \hat{F}(y) e^{2\pi iny} = F(n).$$

Replacing $\hat{F}(y)$ for its Π -Fourier sum in (6), we obtain

$$\begin{aligned} F(x) &= \sum_{-1/2 \leq y < 1/2} \varepsilon \left\{ \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) e^{-2\pi iny} \right\} e^{2\pi ixy} \\ &= \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \left\{ \sum_{-1/2 \leq y < 1/2} \varepsilon e^{2\pi i(x-n)y} \right\} = \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n), \end{aligned}$$

where $\Gamma : \Pi \rightarrow {}^*\mathbb{C}$ is defined by

$$\begin{aligned} \Gamma(x) &= \sum_{-1/2 \leq y < 1/2} \varepsilon e^{2\pi ixy} = \sum_{m=0}^{\kappa-1} \varepsilon e^{2\pi ix(-\frac{1}{2}+m\varepsilon)} \\ &= \varepsilon e^{-i\pi x} \sum_{m=0}^{\kappa-1} e^{2\pi i\varepsilon xm} = \varepsilon \frac{1 - e^{2\pi ix}}{1 - e^{2\pi i\varepsilon x}} e^{-i\pi x} = \varepsilon \frac{{}^*\sin_\Pi(\pi x)}{{}^*\sin_\Pi(\pi \varepsilon x)} e^{-i\pi \varepsilon x}. \end{aligned}$$

Since

$$\lambda(x) = 2\pi i x e^{i\pi \varepsilon x} * \text{sinc}_{\Pi}(\varepsilon x)$$

then, finally, we get

$$\Gamma(x) = \frac{2\pi i x}{\lambda(x)} * \text{sinc}_{\Pi}(x), \quad x \in \Pi. \quad (7)$$

We thus obtain

$$F(x) = \sum_{n=-\kappa/2}^{\kappa/2-1} F(n) \frac{2\pi i(x-n)}{\lambda(x-n)} * \text{sinc}_{\Pi}(x-n), \quad x \in \Pi \quad (8)$$

which will be called the Π -sampling expansion for the internal function $F \in \mathbb{B}_{\Pi}$. The internal function F is expressed in terms of its values at the hyperinteger points $n \in {}^*\mathbf{Z}_{\Pi}$ and the Π -interpolating function Γ .

We consider now some properties of the Π -interpolating function Γ . First, extending Γ by continuity, we have

$$\Gamma(0) = \frac{2\pi i x}{\lambda(x)} * \text{sinc}_{\Pi}(x) \Big|_{x=0} = 1. \quad (9)$$

Since the zeros of the function $\sin(\pi \varepsilon x)$ are of the form $x = j\kappa, j \in {}^*\mathbf{Z}$, then the function $\lambda(x)$ has no zeros inside the hyperfinite line Π other than $x = 0$. Hence $\Gamma(x)$ is a well-defined function on Π . Moreover we have that $\Gamma(x) = 0$ for $x = -\frac{\kappa}{2}, \dots, -2, -1, 1, 2, \dots, \frac{\kappa}{2} - 1$.

For any $\alpha \in \Pi$ such that $|\alpha| \leq \frac{\pi}{2}$ we have⁶ that $\frac{2}{\pi}|\alpha| \leq |\sin \alpha| \leq |\alpha|$; thus, since $|x| \leq \kappa/2$, we obtain

$$\frac{2}{\pi}|\pi \varepsilon x| \leq |\sin(\pi \varepsilon x)| \leq |\pi \varepsilon x|$$

and therefore

$$4|x| \leq |\lambda(x)| \leq 2\pi|x|.$$

Hence $|(2\pi i x)/\lambda(x)| \leq \pi/2$ and so

$$|\Gamma(x)| \leq \pi/2, \quad \text{for all } x \in \Pi$$

that is, Γ is a finitely bounded internal function on Π . Moreover, since

$$\mathbf{D}_+^j \Gamma(x) = \sum_{-1/2 \leq y < 1/2} \varepsilon \lambda^j(y) e^{2\pi i x y}$$

and $\lambda^j(\cdot) e^{2\pi i x \cdot}$ is $\mathbf{S}\Pi$ -integrable on \mathbf{I} for every $j \in \mathbb{N}_0$, then the function Γ is easily seen to be such that

$$\mathbf{D}_+^j \Gamma \text{ is } \mathbf{S}\Pi\text{-continuous on } \Pi_b$$

⁶See Abramowitz and Stagen, *Handbook of Mathematical Functions* (Dover) (1972).

for all $j \in \mathbb{N}_0$. Therefore, we may define the infinitely differentiable (standard) function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$, by setting for every $t \in \mathbb{R}$,

$$\gamma(t) = \text{st } \Gamma(x),$$

for any $x \in \text{mon}_{\Pi}(t)$. For $x + i\eta \in {}^*\mathbb{C}$ we have that

$$\Gamma(x + i\eta) = \sum_{-1/2 \leq y < 1/2} e^{-2\pi\eta y} e^{2\pi i x y}$$

and so

$$|\Gamma(x + i\eta)| \leq C \exp(\pi|\eta|).$$

Hence γ also extends into the complex plane as an entire function of exponential type $\leq \pi$. Since

$$\Gamma(x) = \frac{e^{-i\pi\epsilon x}}{{}^*\text{sinc}_{\Pi}(\epsilon x)} {}^*\text{sinc}_{\Pi}(x)$$

and $e^{i\pi\epsilon x} / {}^*\text{sinc}_{\Pi}(\epsilon x) \approx 1$ for every $x \in \Pi_b$, we have

$$\Gamma(x) \approx {}^*\text{sinc}_{\Pi}(x)$$

for all $x \in \Pi_b$; on the other hand,

$$\Gamma(x) \approx 0$$

for all infinite $x \in \Pi$. In fact,

- if $\epsilon x = x/\kappa$ is infinitesimal, then

$$\Gamma(x) \approx {}^*\text{sinc}_{\Pi}(x) \approx 0,$$

- if $\epsilon x = x/\kappa$ is not infinitesimal then, taking into account the estimate for the function $|\lambda(x)|$, we obtain

$$|\Gamma(x)| \leq \frac{\pi}{2} \left| \frac{\sin(\pi x)}{\pi x} \right| \approx 0.$$

3.2.1 The Π -sampling expansion as an extension of the sampling expansion for a (standard) bandlimited function with continuous and differentiable spectrum: Let \hat{f} be a continuous function with compact support within the (standard) interval $[-1/2, +1/2] \subset \mathbb{R}$. Then, the internal function

$$\hat{F}(y) = {}^*\hat{f}_{\Pi}(y), \quad -1/2 \leq y < 1/2$$

is finite, $\mathbf{S}\Pi$ -continuous and such that $\text{st } \hat{F} = \hat{f} \circ \text{st}$; moreover, $\Gamma(|\hat{F}|)$ is certainly a finite number.

The inverse Π -Fourier transform of \hat{F} is, itself, an internal function in $\mathbf{S}\mathbf{C}_{\Pi}$. In fact, for $x, \xi \in \Pi_b$, we have

$$\begin{aligned} F(x) - F(\xi) &= \sum_{-1/2 \leq y < 1/2} \epsilon \hat{F}(y) e^{2\pi i \xi y} \{e^{2\pi i(x-\xi)y} - 1\} \\ &= 2\pi i(x - \xi) \sum_{-1/2 \leq y < 1/2} \epsilon y \hat{F}(y) e^{2\pi i \xi y} [1 + 2\pi i(x - \xi)y \mathbf{O}(1)] \end{aligned}$$

and therefore

$$\begin{aligned}
 |F(x) - F(\xi)| &\leq 2\pi|x - \xi| \cdot \sum_{y \in \mathbf{I}} \varepsilon |y \hat{F}(y)| |1 + 2\pi i(x - \xi)y \mathbf{O}(1)| \\
 &= 2\pi|x - \xi| \left\{ \max_{y \in \mathbf{I}} |1 + 2\pi i(x - \xi)y \mathbf{O}(1)| \right\} \sum_{y \in \mathbf{I}} \varepsilon |y \hat{F}(y)| \\
 &= \pi|x - \xi| \left\{ \max_{y \in \mathbf{I}} |1 + 2\pi i(x - \xi)y \mathbf{O}(1)| \right\} \Gamma(|\hat{F}|).
 \end{aligned}$$

Hence, if $x \approx \xi$ we have that $F(x) \approx F(\xi)$ on Π_b , as asserted. Thus the projection $f = \text{st } F$ is a well-defined continuous function on \mathbb{R} and is the inverse Fourier transform of the given function \hat{f} . Moreover, since

$$\mathbf{D}_+^j F(x) = \sum_{-1/2 \leq y < 1/2} \varepsilon \lambda^j(y) \hat{F}(y) e^{2\pi i x y}$$

then it is easily seen that $\mathbf{D}_+^j F$ belongs to \mathbf{SC}_Π for all $j \in \mathbb{N}_0$ (but not necessarily so for infinite $j \in {}^*\mathbb{N}_0$). Thus since

$$\text{st}(\mathbf{D}_+^j F) = f^{(j)}$$

we may assert that $f \equiv \text{st } F$ is a C^∞ -function. Further, we have

$$F(x + i\eta) = \sum_{y \in \mathbf{I}} \varepsilon \{e^{-2\pi \eta y} \hat{F}(y)\} e^{2\pi i x y}$$

and so F extends finitely to $\Pi_b + i\Pi_b$ such that

$$|F(x + i\eta)| \leq C \exp(\pi|\eta|).$$

The projection $f = \text{st } F$, therefore, extends over the whole finite complex plane as an entire function of exponential type $\leq \pi$.

Additionally suppose now that \hat{f} is such that⁷

$$\Gamma(|\mathbf{D}_+ \hat{F}|) = \sum_{y \in \mathbf{I}} \varepsilon |\mathbf{D}_+ \hat{F}(y)|$$

is a finite number. For any infinite $n \in {}^*\mathbf{Z}_\Pi \cap {}^*\mathbb{Z}_\infty$ we have that

$$\bar{\lambda}(n) F(n) = \sum_{-1/2 \leq y < 1/2} \varepsilon \mathbf{D}_+ \hat{F}(y) e^{2\pi i n y}$$

and therefore

$$|\bar{\lambda}(n) F(n)| \leq \Gamma(|\mathbf{D}_+ \hat{F}|)$$

or, taking into account that $|\bar{\lambda}(n)| \geq 4|n|$, we obtain

$$|F(n)| \leq C \cdot \frac{1}{|n|},$$

⁷This condition certainly holds if the (standard) function \hat{f} is differentiable everywhere on $[-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$.

where $C = \Gamma(|\mathbf{D}_+ \hat{F}|)/4$. Now, for every $x \in \Pi_b$ and $n \in {}^*\mathbf{Z}_\Pi \cap {}^*\mathbf{Z}_\infty$, we have that

$$\begin{aligned} |\Gamma(x-n)| &\leq \frac{\pi}{2} \left| \frac{\sin(\pi(x-n))}{\pi(x-n)} \right| \leq \frac{1}{2} \frac{1}{|x-n|} \\ &\leq \frac{1}{2} \frac{1}{||x|-|n||} \leq \frac{1}{2|n|} \frac{1}{1-|x/n|} \leq \frac{1}{|n|} \end{aligned}$$

since $x/n \approx 0$ and thus $1 - |x/n| \geq 1/2$. Hence, for $\nu_1, \nu_2 \in {}^*\mathbb{N}_\infty$ such that $\nu_1 \leq \nu_2 < \kappa/2$, we have

$$\left| \sum_{|n|=\nu_1}^{\nu_2} F(n) \Gamma(x-n) \right| \leq 2C \cdot \sum_{n=\nu_1}^{\nu_2} \frac{1}{n^2}.$$

On the other hand,

$$\sum_{n=\nu_1}^{\nu_2} \frac{1}{n^2} \approx 0$$

for all infinite $\nu_1, \nu_2 \in {}^*\mathbb{N}_\infty$, ($\nu_1 \leq \nu_2$) and therefore

$$\sum_{|n|=\nu_1}^{\nu_2} F(n) \Gamma(x-n) \approx 0.$$

Then we have

Theorem 3.1. *Let f be any standard function whose Fourier transform \hat{f} is a continuous and differentiable function on $[-\frac{1}{2}, \frac{1}{2}]$. If t denotes any (standard) point in \mathbb{R} , the Π -sampling expansion (8) reads as follows:*

$$f(t) = \text{st } F(x) = \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \right) = \sum_{n=-\infty}^{+\infty} f(n) \text{sinc}(t-n),$$

where x is any point in $\text{mon}_\Pi(t)$. Moreover, the convergence of the (standard) series on the right-hand side is almost uniform on \mathbb{R} (that is to say is uniform on compacts).

Proof. If \hat{f} is differentiable everywhere on $[-\frac{1}{2}, \frac{1}{2}]$ then, although $\text{st} \circ \mathbf{D}_+ \hat{F}$ may not be equal to $\hat{f}' \circ \text{st}$, we certainly have that $\mathbf{D}_+ \hat{F}$ is finite everywhere on \mathbf{I} and therefore $\Gamma(|\mathbf{D}_+ \hat{F}|)$ is a finite number. Hence, from above, for each $x \in \text{mon}_\Pi(t)$ and every real $r > 0$, the set

$$\left\{ \nu \in {}^*\mathbf{Z}_\Pi \cap {}^*\mathbb{N} : \left| \sum_{|n|=\nu}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \right| < r/2 \right\}$$

is internal and contains all infinite positive numbers in ${}^*\mathbf{Z}_\Pi$. By underflow it contains a finite number, say $n_r \in \mathbb{N}$, for which we have

$$\left| \sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) - \sum_{n=-n_r}^{n_r} F(n) \Gamma(x-n) \right| < r/2.$$

Thus, taking standard parts, gives

$$\left| \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \right) - \sum_{n=-n_r}^{n_r} \text{st } F(n) \text{st } \Gamma(x-n) \right| < r.$$

Taking into account that for $x \in \text{mon}_{\Pi}(t)$ and $|n| \in \mathbb{N}_0$ we have $\Gamma(x-n) \approx \Gamma(t-n)$ and therefore

$$\text{st } \Gamma(x-n) = \text{st } \Gamma(t-n) = \text{sinc}(t-n).$$

Since the real number $r > 0$ is arbitrary, it follows that

$$\begin{aligned} f(t) &= \text{st } F(x) \\ &= \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \right) = \sum_{n=-\infty}^{+\infty} f(n) \text{sinc}(t-n) \end{aligned}$$

for any $x \in \text{mon}_{\Pi}(t)$, where the rightmost hand side is to be interpreted in the standard sense. Moreover, since for all $x, z \in \Pi_b$ such that $x \approx z$ and for all $\nu \in {}^*\mathbb{Z}_{\Pi} \cap {}^*\mathbb{N}_{\infty}$, we have that

$$\sum_{|n|=\nu}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \approx \sum_{|n|=\nu}^{\frac{\kappa}{2}-1} F(n) \Gamma(z-n)$$

and so the series converges almost uniformly on \mathbb{R} . \square

From the whole proof above it follows that this result also holds under the following slightly more general form:

Theorem 3.2. *Let f be any standard function whose Fourier transform \hat{f} is continuous on $[-\frac{1}{2}, \frac{1}{2}]$ and such that $\Gamma(|\mathbf{D}_+ \hat{f}|)$ is finite. If t denotes any (standard) point in \mathbb{R} , the Π -sampling expansion (8) reads as follows*

$$f(t) = \text{st } F(x) = \text{st} \left(\sum_{n=-\frac{\kappa}{2}}^{\frac{\kappa}{2}-1} F(n) \Gamma(x-n) \right) = \sum_{n=-\infty}^{+\infty} f(n) \text{sinc}(t-n),$$

where x is any point in $\text{mon}_{\Pi}(t)$. Moreover, the convergence of the (standard) series on the right-hand side is almost uniform on \mathbb{R} .

References

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