Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 4, November 2000, pp. 363–377.

Hyperfinite representation of distributions

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MS received 28 February 2000; revised 30 May 2000

Abstract. Hyperfinite representation of distributions is studied following the method introduced by Kinoshita [2, 3], although we use a different approach much in the vein of [4]. Products and Fourier transforms of representatives of distributions are also analysed.

Keywords. Generalized functions; Schwartz distributions; nonstandard analysis; hyperfinite representations.

1. Notation and preliminary results

A nonstandard treatment of the theory of distributions in terms of a hyperfinite representation has been presented in papers [2, 3] by Kinoshita. A further exploitation of this treatment in an N-dimensional context has been given by Grenier [1]. In the present paper we offer a different approach to the hyperfinite representation, based on the nonstandard theory of distributions developed in [4]. Some basic acquaintance with nonstandard analysis (NSA) is assumed. For the most part little more is needed than what is contained in the description in [4] of an elementary ultrapower model of the hyperreals. For a more detailed study of the fundamentals of NSA see, for example, Luxemburg [6] or Lindstrøm [5].

Let κ be any given infinite hypernatural number which, without any loss of generality, will be supposed to be even; then define $\varepsilon = \kappa^{-1} \approx 0$. Hence,

$$\begin{aligned}\Pi \equiv \Pi_\kappa &= \left\{-\frac{\kappa}{2}, -\frac{\kappa}{2}+\varepsilon, \ldots, 0, \ldots, \frac{\kappa}{2}-\varepsilon\right\} \\ &= \left\{\left(-\frac{\kappa}{2}+j-1\right)\varepsilon : j = 1, 2, \ldots, \kappa^2\right\} \subset {}^*\mathbb{R}\end{aligned}$$

is an (internal) hyperfinite set of hyperreal numbers with internal cardinality κ^2. Π will be referred to as the (unbounded) *hyperfinite line*. Given a standard point $r \in \mathbb{R}$, define the Π-monad of r by

$$\mathrm{mon}_\Pi(r) = \mathrm{st}_\Pi^{-1}(r) = \mathrm{mon}(r) \cap \Pi,$$

where mon denotes the usual monad of a standard number in ${}^*\mathbb{R}$. Then the set $\Pi_b = \cup_{r\in\mathbb{R}} \mathrm{mon}_\Pi(r) = \mathrm{st}_\Pi^{-1}(\mathbb{R}) \subset \Pi$ is the *nearstandard hyperfinite line* and $\Pi_\infty = \Pi \backslash \Pi_b$ is the set of *remote points* of the hyperfinite line. For every subset $A \subset \mathbb{R}$ define ${}^*A_\Pi = {}^*A \cap \Pi$ and $\mathrm{ns}_\Pi({}^*A) = {}^*A \cap \Pi_b = \cup_{a\in A} \mathrm{mon}_\Pi(a)$. The notation throughout will be the usual in the field.

Now consider the basic set of internal functions

$$^{\Pi}\mathbb{F} = \{F : \Pi \to {}^{*}\mathbb{C} : F \text{ is internal}\}$$

and suppose, if necessary, that each $F \in {}^{\Pi}\mathbb{F}$ is periodically extended to the infinite grid $\varepsilon \cdot {}^{*}\mathbb{Z}$. Defining addition and scalar multiplication componentwise, ${}^{\Pi}\mathbb{F}$ is a ${}^{*}\mathbb{C}$-linear space of hyperfinite dimension κ^2. Moreover, defining also componentwise the product of two functions, ${}^{\Pi}\mathbb{F}$ is in fact an algebra. The operators $\mathbf{D}_+, \mathbf{D}_- : {}^{\Pi}\mathbb{F} \to {}^{\Pi}\mathbb{F}$ defined, for every function F and $x \in \Pi$, by

$$\mathbf{D}_+F(x) = \varepsilon^{-1}[F(x+\varepsilon) - F(x)] \text{ and } \mathbf{D}_-F(x) = \varepsilon^{-1}[F(x) - F(x-\varepsilon)]$$

are called, respectively, the forward and the backward Π-difference operators (of first order). Iterating $\mathbf{D}_+$ (or $\mathbf{D}_-$) we obtain higher order Π-difference operators: for every (finite or infinite) $n \in {}^{*}\mathbb{N}_0$

$$\mathbf{D}_+^n F(x) = \mathbf{D}_+(\mathbf{D}_+^{n-1}F(x)), \quad x \in \Pi$$

and similarly for $\mathbf{D}_-^n$. It is easily seen that for any two functions $F, G \in {}^{\Pi}\mathbb{F}$ we have (both for $\mathbf{D}_+$ and $\mathbf{D}_-$),

$$\mathbf{D}(F+G) = \mathbf{D}F + \mathbf{D}G \text{ and } \mathbf{D}(F \cdot G) = (\mathbf{D}F)G + F(\mathbf{D}G) \pm \varepsilon(\mathbf{D}F)(\mathbf{D}G),$$

where we take $+\varepsilon$ or $-\varepsilon$ according as we use $\mathbf{D}_+$ or $\mathbf{D}_-$, respectively.

For every $\alpha, x \in \Pi$ define the Π-intervals (containing only points in Π) $J_\alpha^+(x)$ and $J_\alpha^-(x)$ as follows:

$$J_\alpha^+(x) = \begin{cases} (x,\alpha]_\Pi & \text{if } x < \alpha \\ [x,\alpha)_\Pi & \text{if } x > \alpha \end{cases}$$

$$J_\alpha^-(x) = \begin{cases} [\alpha,x)_\Pi & \text{if } x < \alpha \\ (\alpha,x]_\Pi & \text{if } x > \alpha \end{cases}$$

while for $x = \alpha$ we have $J_\alpha^+(x) = \emptyset = J_\alpha^-(x)$. For any $F \in {}^{\Pi}\mathbb{F}$ define the functions $\mathbf{S}_+F$ and $\mathbf{S}_-F$ to be the forward and backward Π-sums of F which are zero at the origin and which, for every $x \in \Pi\backslash\{0\}$ are defined by

$$\mathbf{S}_+F(x) = \sum_{t \in J_0^+(x)} \varepsilon F(t) \text{ and } \mathbf{S}_-F(x) = \sum_{t \in J_0^-(x)} \varepsilon F(t).$$

The Π-*sum* operators $\mathbf{S}_+$ and $\mathbf{S}_-$ both transform ${}^{\Pi}\mathbb{F}$ into ${}^{\Pi}\mathbb{F}$. Moreover, for every $F \in {}^{\Pi}\mathbb{F}$, we have

$$\mathbf{D}_+\mathbf{S}_+F = F \quad \text{and} \quad \mathbf{D}_-\mathbf{S}_-F = F$$

that is, $\mathbf{S}_+$ and $\mathbf{S}_-$ are left inverses for $\mathbf{D}_+$ and $\mathbf{D}_-$, respectively.

1.1 $\mathbf{S}\Pi$-*continuous functions*

Given a (standard) function $f : A \to \mathbb{C}$ defined on a subset A of $\mathbb{R}$ we always consider its extension to the whole of $\mathbb{R}$, denoted again by f, by setting $f(x) = 0$ on $A^c \equiv \mathbb{R}\backslash A$. For any such function consider the nonstandard extension ${}^{\star}f$ and then define ${}^{\star}f_\Pi$ to be the restriction of ${}^{\star}f$ to Π (periodically extended to $\varepsilon \cdot {}^{*}\mathbb{Z}$). Hence, for every standard function f, we clearly have ${}^{\star}f_\Pi \in {}^{\Pi}\mathbb{F}$.

DEFINITION 1.1

An internal function $F \in {}^{\Pi}\mathbb{F}$ is said to be $\mathbf{S}\Pi$-continuous on a nonempty subset Ω of Π if and only if

$$\forall_{x,y}[x, y \in \Omega \quad \text{and} \quad x \approx y \Rightarrow F(x) \approx F(y)].$$

From the nonstandard characterization of (standard) continuity and uniform continuity there follows

Theorem 1.2. *If $f : \mathbb{R} \to \mathbb{C}$ is a (standard) function which is continuous at a point $r \in \mathbb{R}$ then ${}^{\star}f_{\Pi} : \Pi \to {}^{*}\mathbb{C}$ is $\mathbf{S}\Pi$-continuous on $\mathrm{mon}_{\Pi}(r)$. If f is continuous on the set $A \subset \mathbb{R}$ then ${}^{\star}f_{\Pi}$ is $\mathbf{S}\Pi$-continuous on $\mathrm{ns}_{\Pi}({}^{\star}A)$. Moreover, if f is uniformly continuous on A, then ${}^{\star}f_{\Pi}$ is $\mathbf{S}\Pi$-continuous on ${}^{\star}A_{\Pi}$.*

The converse does not necessarily hold. The internal function ${}^{\star}f_{\Pi}$ may have infinitesimal variation over the Π-monad of a (standard) point, but this fact does not ensure that the variation is kept at an infinitesimal level over the entire monad of the same point. Consider, for example, the (standard) Dirichlet function

$$\mathbf{d}(x) = \begin{cases} 1 & \text{if } x \in [0,1]\backslash\mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Since Π contains only hyperrational points, ${}^{\star}\mathbf{d}_{\Pi}$ is zero for all $x \in \Pi$ and it follows that ${}^{\star}\mathbf{d}_{\Pi}$ is $\mathbf{S}\Pi$-continuous on Π_b while ${}^{\star}\mathbf{d}$ is not $\mathbf{S}$-continuous anywhere on ${}^{*}[0,1] \subset {}^{*}\mathbb{R}$.

Consider an internal function $F \in {}^{\Pi}\mathbb{F}$ such that

(a) $F(x)$ is finite on Π_b, and
(b) F is $\mathbf{S}\Pi$-continuous on Π_b.

Then it make sense to define the (standard) function $\mathrm{st}\,F : \mathbb{R} \to \mathbb{C}$ by setting for every $t \in \mathbb{R}$

$$\mathrm{st}\,F(t) = [\mathrm{st} \circ F](x), \quad \text{for any } x \in \mathrm{mon}_{\Pi}(t).$$

If ι is any *choice function* picking up one and only one point from each set to which it is applied then we may write $\mathrm{st}\,F = \mathrm{st} \circ F \circ \iota \circ \mathrm{st}_{\Pi}^{-1}$.

Denote by $\mathbf{SC}_{\Pi} \equiv \mathbf{SC}_{\Pi}(\mathbb{R})$ the set of all functions in ${}^{\Pi}\mathbb{F}$ which are finite and $\mathbf{S}\Pi$-continuous on Π_b.

As the above example concerning the Dirichlet function shows we cannot expect in the general case to recover the original function $f : \mathbb{R} \to \mathbb{C}$ from its Π-extension. However, it is not difficult to see that

Theorem 1.3. *If f is a continuous function on a subset A of $\mathbb{R}$ then we have $\mathrm{st}({}^{\star}f_{\Pi}) = f$ on A.*

and, more generally,

Theorem 1.4. *If f is a function which is k times continuously differentiable on a subset A of $\mathbb{R}$ then $\mathrm{st}\,\mathbf{D}_{+}^{j}({}^{\star}f_{\Pi}) = f^{(j)}$, $j = 0, 1, 2, \ldots, k$, hold on A. (The same holds if we consider $\mathbf{D}_{-}^{j}$ instead of $\mathbf{D}_{+}^{j}$.)*

2. SΠ-distributions

Given any $F \in \mathbf{SC}_\Pi$, the function $\mathbf{S}_+F$ (or $\mathbf{S}_-F$) is again in $\mathbf{SC}_\Pi$. In fact, for any $x \in \Pi_b$, considering $\mathbf{S}_+F$ for example, we have

$$|\mathbf{S}_+F(x)| \leq \sum_{t \in J_0^+(x)} \varepsilon |F(t)| \leq \left\{ \max_{t \in J_0^+(x)} |F(t)| \right\} |x|$$

and thus $\mathbf{S}_+F$ is finite on Π_b. Also, for any $x, y \in \Pi$,

$$|\mathbf{S}_+F(y) - \mathbf{S}_+F(x)| \leq \sum_{t \in J^+_{\{x,y\}}} \varepsilon\, |F(t)| \leq |y - x| \cdot \max_{t \in J^+_{\{x,y\}}} |F(t)|,$$

where $J^+_{\{x,y\}} \equiv J^+_{\min\{x,y\}}(\max\{x,y\})$. Hence, if $x \approx y$ then $\mathbf{S}_+F(x) \approx \mathbf{S}_+F(y)$ and therefore $\mathbf{S}_+F \in \mathbf{SC}_\Pi$, as asserted.

The same result is not generally true for Π-differences. If $F \in \mathbf{SC}_\Pi$ then the most we can say about the function $\mathbf{D}_+F$ (or $\mathbf{D}_-F$), in principle, is that it belongs to ${}^\Pi\mathbb{F}$.

2.1 *The* ${}^*\mathbb{C}_b$*-module* ${}^\Pi\mathbb{D}_\infty$

For any F in $\mathbf{SC}_\Pi$, $\operatorname{st} F$ is a (standard) continuous function on $\mathbb{R}$ which therefore defines a (regular) distribution in $\mathcal{D}'$. Denoting by ν_F either the function $\operatorname{st} F$ or the distribution it generates as the context demands, we have

$$\langle \nu_F, \varphi \rangle = \int_{\mathbb{K}} \nu_F(t)\varphi(t)\mathrm{d}t = \int_{\operatorname{st}_\Pi^{-1}(\mathbb{K})} (\operatorname{st} \circ F)\varphi_\Pi \mathrm{d}\Lambda_L, \tag{1}$$

where $\operatorname{st} \circ F$ and $\varphi_\Pi = \operatorname{st} \circ {}^\star\varphi_\Pi$ are (external) functions defined on Π, $\mathbb{K}$ is a compact of $\mathbb{R}$ containing the support of φ and Λ_L denotes the counting Loeb measure on Π. Since $F \cdot {}^\star\varphi_\Pi$ is an $\mathbf{S}\Pi$-lifting for the external function $(\operatorname{st} \circ F)\varphi_\Pi$ we may replace the Loeb integral in (1) by a proper Π-sum to obtain

$$\langle \nu_F, \varphi \rangle = \operatorname{st}\left(\sum_{x \in {}^\star\mathbb{K}_\Pi} \varepsilon\, F(x)\, {}^\star\varphi_\Pi(x) \right).$$

It is easy to see that $\varphi \rightsquigarrow {}^\star\varphi_\Pi$ is a linear and continuous map and therefore every internal function $F \in \mathbf{SC}_\Pi$ generates in this way a regular distribution. Since the map $f \rightsquigarrow {}^\star f_\Pi$ embeds $\mathcal{C} \equiv \mathcal{C}(\mathbb{R})$ into $\mathbf{SC}_\Pi$ and the distribution generated by f coincides with $\nu_{{}^\star f_\Pi}$, the map

$$\operatorname{st}_{\mathcal{D}} : \mathbf{SC}_\Pi \to \mathcal{D}'$$

defined by $\operatorname{st}_{\mathcal{D}}(F) = \nu_F$, establishes an onto correspondence between $\mathbf{SC}_\Pi$ and the subspace of $\mathcal{D}'$ comprising all regular distributions generated by continuous functions on $\mathbb{R}$.

Now, if $F \in \mathbf{SC}_\Pi$ and $\varphi \in \mathcal{D}$ is a function with support in the compact $\mathbb{K} \sqsubset \mathbb{R}$ then, taking Theorem 1.4 into account, we get

$$\begin{aligned}
&\sum_{x \in \Pi} \varepsilon\, \mathbf{D}_+F(x)^\star\varphi_\Pi(x) = \sum_{x \in \Pi} [F(x+\varepsilon) - F(x)]^\star\varphi_\Pi(x) \\
&= \sum_{x \in \Pi} \varepsilon F(x)(-\mathbf{D}_-{}^\star\varphi_\Pi(x)) \approx \int_{\operatorname{st}_\Pi^{-1}(\mathbb{K})} (\operatorname{st} \circ F)(-\varphi')_\Pi \mathrm{d}\Lambda_L \\
&= \langle \nu_F, -\varphi' \rangle = \langle \mathbf{D}\nu_F, \varphi \rangle,
\end{aligned}$$

where $\mathbf{D}\nu_F$ is the (standard) distributional derivative of ν_F. Let $\mathbf{D}_+(\mathbf{SC}_\Pi)$ be the set of first order Π-differences of all functions in $\mathbf{SC}_\Pi$. Since for every $F \in \mathbf{SC}_\Pi$ we have that $F = \mathbf{D}_+(\mathbf{S}_+F)$ where $\mathbf{S}_+F \in \mathbf{SC}_\Pi$ then $\mathbf{SC}_\Pi \subset \mathbf{D}_+(\mathbf{SC}_\Pi)$. Then the $\mathrm{st}_\mathcal{D}$-mapping may be extended onto $\mathbf{D}_+(\mathbf{SC}_\Pi)$, by setting

$$\mathrm{st}_\mathcal{D}(\mathbf{D}_+F) = \mathbf{D}\nu_F = \mathbf{D}(\mathrm{st}_\mathcal{D}(F)).$$

The same idea may be generalized to Π-differences of any *finite* order of a function in $\mathbf{SC}_\Pi$. Hence, if $F \in \mathbf{SC}_\Pi$ and $\varphi \in \mathcal{D}$, we obtain, for every $j \in \mathbb{N}_0$,

$$\begin{aligned}\sum_{x\in\Pi} \varepsilon\, \mathbf{D}_+^j F(x)^\star\varphi_\Pi(x) &= \sum_{x\in\Pi} \varepsilon\, F(x)[(-1)^j \mathbf{D}_-^{j\,\star}\varphi_\Pi(x)] \\ &\approx \int_{\mathrm{st}_\Pi^{-1}(\mathbb{K})} (\mathrm{st}\circ F)(-\varphi^{(j)})_\Pi \,\mathrm{d}\Lambda_L \\ &= \langle \nu_F, (-1)^j\varphi^{(j)}\rangle = \langle \mathbf{D}^j(\nu_F), \varphi\rangle,\end{aligned}$$

that is, $\mathrm{st}_\mathcal{D}(\mathbf{D}_+^j F) = \mathbf{D}^j(\mathrm{st}_\mathcal{D}(F))$.

Denoting by $\mathbf{D}_+^j(\mathbf{SC}_\Pi)$, for every $j \in \mathbb{N}_0$, the set of $\mathbf{D}_+^j$-differences of all functions in $\mathbf{SC}_\Pi$, then we have the inclusion $\mathbf{D}_+^j(\mathbf{SC}_\Pi) \subset \mathbf{D}_+^{j+1}(\mathbf{SC}_\Pi)$, and therefore

$${}^\Pi\mathbb{D}_\infty \equiv {}^\Pi\mathbb{D}_\infty(\mathbb{R}) = \bigcup_{j=0}^{\infty} \mathbf{D}_+^j(\mathbf{SC}_\Pi)$$

is the (external) set of all finite-order Π-differences of all functions in $\mathbf{SC}_\Pi$. Since for every $F \in \mathbf{SC}_\Pi$ the translate $\tau_\varepsilon F$ is also in $\mathbf{SC}_\Pi$ and, moreover, $\mathbf{D}_- = \mathbf{D}_+ \circ \tau_\varepsilon$ then ${}^\Pi\mathbb{D}_\infty$ may be obtained using indifferently either $\mathbf{D}_+$ or $\mathbf{D}_-$. Hence we may also write, more generally,

$${}^\Pi\mathbb{D}_\infty \equiv {}^\Pi\mathbb{D}_\infty(\mathbb{R}) = \bigcup_{j=0}^{\infty}\bigcup_{k=0}^{\infty} \mathbf{D}_+^j\mathbf{D}_-^k(\mathbf{SC}_\Pi).$$

We may now extend the map $\mathrm{st}_\mathcal{D}$ to the whole of ${}^\Pi\mathbb{D}_\infty$ as follows: for every $\Phi \in {}^\Pi\mathbb{D}_\infty$ there exist $F \in \mathbf{SC}_\Pi$ and $j \in \mathbb{N}_0$ so that $\Phi = \mathbf{D}_+^j F$. Hence, $\mathrm{st}_\mathcal{D}(\Phi) = \mathbf{D}^j\nu_F \in \mathcal{D}'$. Note that $\mathrm{st}_\mathcal{D}(\Phi)$ does not depend upon the representation of Φ as a finite order Π-difference of a function in $\mathbf{SC}_\Pi$. In fact, suppose we also have $\Phi = \mathbf{D}_+^m G$ with $G \in \mathbf{SC}_\Pi$ and $m \in \mathbb{N}_0$ (where, without any loss of generality we may assume $m \geq j$). Then from the equation $\mathbf{D}_+^j F = \mathbf{D}_+^m G$ it follows that $\mathbf{S}_+^{m-j}F + P_m = G$, where P_m is a polynomial of degree $< m$ (and coefficients in ${}^*\mathbb{C}$). Thus, for any $\varphi \in \mathcal{D}$, we get

$$\begin{aligned}\langle \mathbf{D}^m\nu_G, \varphi\rangle &= \langle \nu_G, (-1)^m\varphi^{(m)}\rangle \\ &\approx \sum_{x\in\Pi} \varepsilon\, G(x)[(-1)^m \mathbf{D}_-^{m\star}\varphi_\Pi(x)] \\ &= \sum_{x\in\Pi} \varepsilon[\mathbf{S}_+^{m-j}F(x) + P_m(x)][(-1)^m\mathbf{D}_-^{m\star}\varphi_\Pi(x)] \\ &= \sum_{x\in\Pi} \varepsilon\, F(x)(-1)^j\mathbf{D}_-^{j\,\star}\varphi_\Pi(x) + \sum_{x\in\Pi}\varepsilon\,\mathbf{D}_+^m P_m(x)^\star\varphi_\Pi(x) \\ &\approx \langle \nu_F, (-1)^j\varphi^{(j)}\rangle = \langle \mathbf{D}^j\nu_F, \varphi\rangle\end{aligned}$$

and therefore $\mathbf{D}^m\nu_G = \mathbf{D}^j\nu_F$ which proves the assertion made.

The *$\mathcal{D}$-standard part map* $\mathrm{st}_\mathcal{D} : {}^\Pi\mathbb{D}_\infty \to \mathcal{D}'$ is clearly linear; its kernel, $\mathcal{K}_\infty \equiv \mathcal{K}_\infty(\mathrm{st}_\mathcal{D})$, comprises all internal functions in ${}^\Pi\mathbb{D}_\infty$ which generate the null distribution. These are all the functions which are finite order derivatives of infinitesimal functions in $\mathbf{SC}_\Pi$. The factor space ${}^\Pi\mathcal{C}_\infty \equiv {}^\Pi\mathbb{D}_\infty/\mathcal{K}_\infty$ is a $\mathbb{C}$-vector space which may be shown to be isomorphic to $\mathcal{C}_\infty$, the space of all finite order Schwartz distributions.

DEFINITION 2.1

The internal functions in ${}^\Pi\mathbb{D}_\infty \subset {}^\Pi\mathbb{F}$ will be called finite order Π-predistributions and the classes $[\Phi] \in {}^\Pi\mathcal{C}_\infty$, with $\Phi \in {}^\Pi\mathbb{D}_\infty$, will be called *finite order SΠ-distributions.*

The Π-predistributions are internal functions in ${}^\Pi\mathbb{F}$ which do not grow too fast on Π_b according to the following result:

Theorem 2.2. *For every internal function $\Phi \in {}^\Pi\mathbb{D}_\infty$ there exists a finite nonnegative integer $m \equiv m_\Phi$ such that for every compact $\mathbb{K}$ of $\mathbb{R}$*

$$|\Phi(x)| \leq \mathbf{C}_{\mathbb{K},\Phi} \cdot \kappa^m, \quad \text{on } {}^*\mathbb{K}_\Pi \tag{2}$$

where $\mathbf{C}_{\mathbb{K},\Phi}$ is a finite positive constant (depending on $\mathbb{K}$ and Φ).

Proof. The inequality (2) clearly holds for every $\Phi \in \mathbf{SC}_\Pi$ with $m = 0$. Now, if we have $\Phi = \mathbf{D}_+ F$ with $F \in \mathbf{SC}_\Pi$, then

$$\Phi(x) = \mathbf{D}_+ F(x) = \kappa\,[F(x+\varepsilon) - F(x)]$$

and therefore, for every compact $\mathbb{K} \sqsubset \mathbb{R}$, we obtain

$$\max_{x \in {}^*\mathbb{K}_\Pi} |\Phi(x)| \leq 2\left\{ \max_{x \in {}^*\mathbb{K}_\Pi} |F(x)| \right\} \cdot \kappa.$$

Hence the inequality holds with $C_{\mathbb{K},\Phi} = 2\max_{x \in {}^*\mathbb{K}_\Pi} |F(x)| \in {}^*\mathbb{R}_b$ and $m = 1$.

Suppose now that the inequality holds for all internal functions of the form $\mathbf{D}_+^j F$ with $F \in \mathbf{SC}_\Pi$. If $\Phi = \mathbf{D}_+^{j+1} F$ with $F \in \mathbf{SC}_\Pi$ then we obtain,

$$\max_{x \in {}^*\mathbb{K}_\Pi} |\Phi(x)| \leq 2\left\{ \max_{x \in {}^*\mathbb{K}_\Pi} |\mathbf{D}_+^j F(x)| \right\} \cdot \kappa \leq C_{\mathbb{K},\Phi}^{[j+1]} \cdot \kappa^{j+1},$$

where $C_{\mathbb{K},\Phi}^{[j+1]}$ is, for every fixed $j \in \mathbb{N}_0$, a positive bounded constant. Therefore the result follows by finite induction. Note that there are functions $F \in \mathbf{SC}_\Pi$ such that $\mathbf{D}_+ F \in \mathbf{SC}_\Pi$; then, for a general function of the form $\Phi = \mathbf{D}_+^j F$ with $F \in \mathbf{SC}_\Pi$, equation (2) may be satisfied with $m \leq j$. $\square$

Now, define ${}^\Pi\mathbb{G}_\infty$ to be the subset of ${}^\Pi\mathbb{F}$ comprising all internal functions Φ satisfying (2) for some number $m \in \mathbb{N}_0$ and every compact $\mathbb{K}$ of $\mathbb{R}$ with $\mathbf{C}_{\mathbb{K},\Phi}$ a bounded positive constant. ${}^\Pi\mathbb{G}_\infty$ is a Π-difference algebra which contains ${}^\Pi\mathbb{D}_\infty$. Within ${}^\Pi\mathbb{G}_\infty$ the ordinary product of Π-predistributions make sense although the product of two Π-predistributions is not generally a Π-predistribution. By imposing appropriate restrictions on the factors, however, the product of two elements in ${}^\Pi\mathbb{D}_\infty$ may still be a Π-predistribution. In particular, we have

Theorem 2.3. *Let $\Theta, \Phi \in {}^\Pi\mathbb{D}_\infty$ be such that $\mathbf{D}_+^m\Theta \in \mathbf{SC}_\Pi$ and $\Phi \in \mathbf{D}_+^m(\mathbf{SC}_\Pi)$ for some given $m \in \mathbb{N}_0$. Then, $\Phi = \mathbf{D}_+^m F$ with $F \in \mathbf{SC}_\Pi$, and*

$$\mathrm{st}_{\mathcal{D}}\left(\Theta\Phi - \mathbf{D}_+^m\left(\sum_{j=0}^{m} \binom{m}{j} (-1)^j \mathbf{S}_+^{m-j}[(\mathbf{D}_+^{m-j}\Theta)F] \right) \right) = 0,$$

where $G \equiv \sum_{j=0}^{m} \binom{m}{j} (-1)^j \mathbf{S}_+^{m-j}[(\mathbf{D}_+^{m-j}\Theta)F]$ *is a function in* $\mathbf{SC}_\Pi$.

Proof. For $m = 1$ we have

$$\Theta \mathbf{D}_+ F = \mathbf{D}_+(\Theta F) - (\mathbf{D}_+\Theta)F - \varepsilon(\mathbf{D}_+\Theta)(\mathbf{D}_+F)$$

and therefore

$$\mathrm{st}_{\mathcal{D}}(\Theta\Phi - \mathbf{D}_+[\Theta F - \mathbf{S}_+(\mathbf{D}_+\Theta)F]) = \mathrm{st}_{\mathcal{D}}(-\varepsilon(\mathbf{D}_+\Theta)(\mathbf{D}_+F)).$$

For any $\varphi \in \mathcal{D}$, with support within the compact $\mathbb{K} \sqsubset \mathbb{R}$, we have that

$$\langle \mathrm{st}_{\mathcal{D}}(-\varepsilon(\mathbf{D}_+\Theta)(\mathbf{D}_+F)), \varphi\rangle = \mathrm{st}\left(\sum_{x \in {}^*\mathbb{K}_\Pi} \varepsilon\, \mathbf{D}_+\Theta(x)\, [F(x+\varepsilon) - F(x)]\right)$$

and the result follows from the fact that $F(x+\varepsilon) \approx F(x)$ for all $x \in {}^*\mathbb{K}_\Pi$. The proof now proceeds by induction on $m \in \mathbb{N}$. $\square$

This result allow us to introduce the notion of Schwartz product in ${}^\Pi\mathcal{C}_\infty$ by setting

$$\Theta \cdot [\Phi] = [\Theta\Phi],$$

where Θ and Φ are as above.

2.2 *The* ${}^*\mathbb{C}_b$*-Module* ${}^\Pi\mathbb{D}$

For any subset A of $\mathbb{R}$ let $\kappa(A)$ be the family of all compact subsets of A. Denote by ${}^\Pi\mathbb{D}$ the subset of all functions $\Phi \in {}^\Pi\mathbb{F}$ such that for each $\mathbb{K} \in \kappa(\mathbb{R})$ there exist $\Phi_K \in {}^\Pi\mathbb{D}_\infty$ so that $\Phi = \Phi_K$ on ${}^*\mathbb{K}_\Pi$. Every function in ${}^\Pi\mathbb{D}$ determines a family

$$\{\Phi_\mathbb{K}\}_{\mathbb{K} \in \kappa(\mathbb{R})}$$

which is such that

$$\text{if } \mathbb{K}, \mathbb{L} \in \kappa(\mathbb{R}) \text{ and } \mathbb{K} \subset \mathbb{L} \text{ then } \Phi_\mathbb{K} = \Phi_\mathbb{L} \text{ on } {}^*\mathbb{K}_\Pi.$$

Such a family of ${}^\Pi\mathbb{D}_\infty$-functions is said to be *compatible*. Moreover the converse also holds, that is, if $\{\Phi_\mathbb{K}\}_{\mathbb{K} \in \kappa(\mathbb{R})}$ is a compatible family of internal functions in ${}^\Pi\mathbb{D}_\infty$ then we can define $\Phi \in {}^\Pi\mathbb{D}$ by setting

$$\Phi_{|_\mathbb{K}} = \Phi_\mathbb{K} \text{ on } {}^*\mathbb{K}_\Pi$$

for all $\mathbb{K} \in \kappa(\mathbb{R})$. Hence $\Phi \in {}^\Pi\mathbb{D}$.

If $\Phi \in {}^\Pi\mathbb{D}_\infty$ then the '*constant*' family $\{\Phi\}_{\mathbb{K} \in \kappa(\mathbb{R})}$ is certainly a compatible family and therefore defines an element in ${}^\Pi\mathbb{D}$; hence ${}^\Pi\mathbb{D}_\infty \subset {}^\Pi\mathbb{D}$. Every function in ${}^\Pi\mathbb{D}$ will be called a *global* Π-*predistribution*. Finite order Π-predistributions are global Π-predistributions, but the converse is not true, as the example that follows shows.

Example 2.4. Given the internal function

$$\Delta_0(x) = \begin{cases} \kappa & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

it is easy to see that for any $m \in {}^*\mathbb{N}_0$,

$$\mathbf{D}_+^m \Delta_0(x) = \begin{cases} (-1)^j \binom{m}{j} \kappa^{m+1} & \text{if } x = -(m-j)\varepsilon,\ j = 0, 1, \ldots, m \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for any $\varphi \in \mathcal{D}$, we get

$$^{\circ}\left(\sum_{x \in \Pi} \varepsilon \mathbf{D}_+^m \Delta_0(x)^{\star}\varphi_{\Pi}(x) \right) = (-1)^m \varphi^{(m)}(0) = \langle \mathbf{D}^m \delta, \varphi \rangle$$

and hence Δ_0 is an hyperfinite representation of the (standard) delta distribution. $\mathbf{D}_+^m \Delta_0$ is, for every $m \in \mathbb{N}_0$, a function in ${}^{\Pi}\mathbb{D}_{\infty}$ and so is any finite linear combination (over ${}^{*}\mathbb{C}_b$) of these (finite order) Π-differences of Δ_0. However, the internal function

$$\Phi(x) = \sum_{n=-\kappa/2}^{\kappa/2-1} \mathbf{D}_+^{|n|} \Delta_0(x-n) \equiv \sum_{n=-\kappa/2}^{\kappa/2-1} \mathbf{D}_+^{|n|} \Delta_n(x)$$

is not in ${}^{\Pi}\mathbb{D}_{\infty}$ although, as it will be seen shortly, it belongs to ${}^{\Pi}\mathbb{D}$. To see this, note that for finite n the function $\mathbf{D}_+^{|n|}\Delta_0(x-n)$ is zero outside the Π-monad of n and for infinite n it is zero outside the Π-interval $[n-1/2, n+1/2]_{\Pi}$ which is completely contained in Π_{∞}. Thus for every compact $\mathbb{K} \in \kappa(\mathbb{R})$ the intersection of ${}^{*}\mathbb{K}_{\Pi}$ with the support of $\mathbf{D}_+^{|n|}\Delta_0(x-n)$ is empty, provided that $|n| \in {}^{*}\mathbb{N}_{\infty}$. Hence, for every $\mathbb{K} \in \kappa(\mathbb{R})$ there is only a finite number of finite-order Π-differences of finite-translates of Δ_0. Consequently, the restriction of Φ to ${}^{*}\mathbb{K}_{\Pi}$ is equal to a finite-order Π-difference of a function in $\mathbf{SC}_{\Pi}$.

The mapping $\mathrm{st}_{\mathcal{D}}$, defined on ${}^{\Pi}\mathbb{D}_{\infty}$, may now be extended to the whole of ${}^{\Pi}\mathbb{D}$ by setting

$$\mathrm{st}_{\mathcal{D}}(\Phi) = \{\mathrm{st}_{\mathcal{D}_{\mathbb{K}}}(\Phi_{\mathbb{K}})\}_{\mathbb{K} \sqsubset \mathbb{R}},$$

where $\mathrm{st}_{\mathcal{D}_{\mathbb{K}}}(\Phi_{\mathbb{K}})$ denotes the restriction of $\mathrm{st}_{\mathcal{D}}(\Phi_{\mathbb{K}})$ to $\mathcal{D}_{\mathbb{K}}$, for every $\mathbb{K} \in \kappa(\mathbb{R})$. That is to say, if $\varphi \in \mathcal{D}_{\mathbb{K}}$

$$\langle \mathrm{st}_{\mathcal{D}}(\Phi), \varphi \rangle = \mathrm{st}\left(\sum_{x \in {}^{*}\mathbb{K}_{\Pi}} \varepsilon \Phi_{\mathbb{K}}(x)^{\star}\varphi_{\Pi}(x) \right).$$

$\mathrm{st}_{\mathcal{D}}$ is a linear map whose kernel, $\mathcal{K} \equiv \mathcal{K}(\mathrm{st}_{\mathcal{D}})$, comprises all internal functions in ${}^{\Pi}\mathbb{D}$ whose $\mathcal{D}$-standard part is the null distribution. Hence ${}^{\Pi}\mathbb{D}/\mathcal{K}$ is a linear space whose elements will be called *global* SΠ-*distributions*.

Note that for each $\mathbb{K} \in \kappa(\mathbb{R})$ there exist $m_{\mathbb{K}} \in \mathbb{N}_0$ and $F_{\mathbb{K}} \in \mathbf{SC}_{\Pi}$ such that $\Phi_{\mathbb{K}} = \mathbf{D}_+^{m_{\mathbb{K}}} F_{\mathbb{K}}$ on ${}^{*}\mathbb{K}_{\Pi}$. Thus, from Theorem 2 it follows that if $\Phi \in {}^{\Pi}\mathbb{D}$ then for every compact $\mathbb{K} \in \kappa(\mathbb{R})$ there exist a bounded positive constant $C_{\Phi,K}$ and an integer $m_K \in \mathbb{N}_0$, such that

$$\max_{x \in {}^{*}\mathbb{K}_{\Pi}} |\Phi(x)| \leq C_{\Phi,\mathbb{K}} \cdot \kappa^{m_{\mathbb{K}}}. \tag{3}$$

Define ${}^{\Pi}\mathbb{G}$ to be the set of all functions $\Phi \in {}^{\Pi}\mathbb{F}$ which satisfy the following property: for every compact $\mathbb{K} \in \kappa(\mathbb{R})$ there exist an integer $m_K \in \mathbb{N}_0$ and a finite number $C_{\Phi,\mathbb{K}}$ so that (3) holds. ${}^{\Pi}\mathbb{G}$ is a Π-difference algebra which contains ${}^{\Pi}\mathbb{D}$ as a linear submodule and ${}^{\Pi}\mathbb{G}_{\infty}$ as a subalgebra. Global Π-predistributions may therefore be multiplied within ${}^{\Pi}\mathbb{G}$. The product of two global Π-predistributions in general will not be a global

Π-predistribution. However, if $\Theta \in {}^{\Pi}\mathbb{D}$ is an internal function such that $\mathbf{D}_{+}^{j}\Theta \in \mathbf{SC}_{\Pi}$ for all (finite) $j \in \mathbb{N}_0$, and if $\Phi \in {}^{\Pi}\mathbb{D}$ then $\Theta\Phi$ is a global Π-predistribution in the sense that

$$\mathrm{st}_{\mathcal{D}_{\mathbb{K}}}\left(\Theta\Phi - \mathbf{D}_{+}^{m_K}\left(\sum_{j=0}^{m_K}\binom{m_K}{j}(-1)^j\mathbf{S}_{+}^{m_K-j}[(\mathbf{D}_{+}^{m_K-j}\Theta)F]\right)\right) = 0$$

for all compact $\mathbb{K} \sqsubset \mathbb{R}$. Hence, we define the product $\Theta[\Phi]$ to be the global Π-distribution $[\Theta\Phi]$.

3. The Π-Fourier transform

If F is a function in ${}^{\Pi}\mathbb{F}$ then, for each $y \in \Pi$, the sum

$$\hat{F}(y) = \sum_{x\in\Pi} \varepsilon^{*} \exp_{\Pi}(-2\pi ixy)F(x) \tag{4}$$

is a well-defined hypercomplex number. Thus, the right-hand side of (4) defines, for every $F \in {}^{\Pi}\mathbb{F}$, the internal function $\hat{F} : \Pi \to {}^{*}\mathbb{C}$ which is also in ${}^{\Pi}\mathbb{F}$. Conversely, after some easy manipulations, we obtain

$$F(x) = \sum_{y\in\Pi} \varepsilon^{*} \exp_{\Pi}(2\pi ixy)\hat{F}(y) \tag{5}$$

which allows us to recover F from $\hat{F}$.

DEFINITION 3.1

Given $F \in {}^{\Pi}\mathbb{F}$, the function $\hat{F} \in {}^{\Pi}\mathbb{F}$, defined by (4), is called the Π-*Fourier transform* of F. Conversely F, as given by (5), is called the *inverse* Π-*Fourier transform* of $\hat{F}$.

Denoting the Π-Fourier transforms by $\mathcal{F}_{\Pi}$ and $\bar{\mathcal{F}}_{\Pi}$, respectively, then $\hat{F} = \mathcal{F}_{\Pi}[F]$ and $F = \bar{\mathcal{F}}_{\Pi}[\hat{F}]$. $\mathcal{F}_{\Pi}$ and $\bar{\mathcal{F}}_{\Pi}$ are linear transformations of ${}^{\Pi}\mathbb{F}$ onto ${}^{\Pi}\mathbb{F}$ and, moreover, $\mathcal{F}_{\Pi} \circ \bar{\mathcal{F}}_{\Pi} = \bar{\mathcal{F}}_{\Pi} \circ \mathcal{F}_{\Pi} = \mathrm{id}$.

Nonstandard hyperfinite versions for many of the properties of the (standard) Fourier transform and its inverse may be obtained. In particular, for any function $F \in {}^{\Pi}\mathbb{F}$, we obtain

$$\mathcal{F}_{\Pi}[\mathbf{D}_{+}F](y) = \sum_{x\in\Pi} \varepsilon^{\star} \exp_{\Pi}(-2\pi ixy)\mathbf{D}_{+}F(x) = [-\lambda(y)]\hat{F}(y)$$

and, more generally, for any $j \in {}^{\star}\mathbb{N}_0$,

$$\mathcal{F}_{\Pi}[\mathbf{D}_{+}^{j}F](y) = [-\lambda(y)]^{j}\hat{F}(y), \tag{6}$$

where $\lambda : \Pi \to {}^{\star}\mathbb{C}$ is the internal function defined by

$$\lambda(y) = \frac{1}{\varepsilon}[{}^{\star}\exp_{\Pi}(2\pi i\varepsilon y) - 1]$$

and which is such that $\lambda(y) \approx 2\pi i(\mathrm{st}\ y)$ for every $y \in \Pi_b$. Also, for any $j \in {}^{\star}\mathbb{N}_0$, we get

$$\mathbf{D}_{+}^{j}\hat{F}(y) = \mathcal{F}_{\Pi}[\bar{\lambda}^{j}F](y) \tag{7}$$

and, therefore, from (6) and (7), by inversion we obtain

$$\bar{\lambda}^j(x)F(x) = \mathcal{F}_\Pi[\mathbf{D}_+^j \hat{F}](x), \tag{8}$$

$$\mathbf{D}_+^j F(x) = \bar{\mathcal{F}}_\Pi[(-\lambda)^j F](x). \tag{9}$$

Let $F, G \in {}^\Pi\mathbb{F}$ be any two internal function. Then, by simple manipulation, we may obtain the equation

$$\sum_{x\in\Pi} \varepsilon F(x)\hat{G}(x) = \sum_{y\in\Pi} \varepsilon \hat{F}(y)G(y) \tag{10}$$

which is the Π-*Parseval formula* in ${}^\Pi\mathbb{F}$.

3.1 *The* Π-*Fourier transform as an extension of the classical Fourier transform*

The (standard) classical Fourier transform is defined on $\mathbf{L}^1$, the space of all Lebesgue integrable functions on $\mathbb{R}$, by the integral

$$\mathcal{F}[f](\omega) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-2\pi i\omega t}\mathrm{d}t, \quad \omega \in \mathbb{R}.$$

We denote by $\mathcal{F}_0$ the restriction of that transformation to $\mathcal{C}_0 \cap \mathbf{L}^1 \subset \mathbf{L}^1$, the subspace of all continuous and integrable functions on $\mathbb{R}$ which tend monotonically to zero at infinity. Now we want to show that $\mathcal{F}_\Pi$ is an extension of $\mathcal{F}_0$ in the following sense:

Theorem 3.2. *For every $f \in \mathcal{C}_0 \cap \mathbf{L}^1$ the equality*

$$\mathcal{F}_0[f](\mathrm{st}\ y) = \mathrm{st}\ \circ \mathcal{F}_\Pi[{}^\star f_\Pi](y)$$

holds for all $y \in \Pi_b$.

Proof. For any (fixed) $\omega \in \mathbb{R}$, let y be an arbitrarily given point in $\mathrm{st}_\Pi^{-1}(\omega)$. Defining for every $t \in \mathbb{R}$

$$f_y(t) = f(t)\exp[-2\pi i(\mathrm{st}\, y)t]$$

and extending this function to $\bar{\mathbb{R}}$ so that $f_y(\pm\infty) = 0$, consider the (external) function $f_y \circ \mathrm{st}_\infty(x)$, $x \in \Pi$ (where $\mathrm{st}_\infty x = \mathrm{st}\, x$ if $x \in \Pi_b$ and $\mathrm{st}_\infty x = \pm\infty$ if $x \in \Pi_\infty^\pm$, respectively). Then we have that

$$\mathcal{F}_0[f](\omega) = \int_{\mathbb{R}} f_y(t)\mathrm{d}t = \int_\Pi f_y \circ \mathrm{st}_\infty(x)\mathrm{d}\Lambda_L(x),$$

where the last integral is the Loeb integral with respect to the Loeb counting measure on the hyperfinite grid. The proof will be complete provided it is shown that the equality

$$\int_\Pi f_y \circ \mathrm{st}_\infty(x)\mathrm{d}\Lambda_L(x) = \mathrm{st}\left(\sum_{x\in\Pi} \varepsilon^\star f_\Pi(x)^\star \exp_\Pi(-2\pi i x y)\right) \tag{11}$$

holds for all $y \in \Pi_b$. For this purpose it is necessary to prove that the internal function

$${}^\star f_\Pi(x)^\star \exp_\Pi(-2\pi i x y)$$

is an $\mathbf{S}\Pi$-integrable lifting for the external function $f_y \circ \mathrm{st}_\infty(x)$, $x \in \Pi$.

First, we have that

$$\mathrm{st}\{{}^\star f_\Pi(x)^\star \exp_\Pi(-2\pi ixy)\} = \mathrm{st}({}^\star f_\Pi(x)) \cdot \mathrm{st}({}^\star \exp_\Pi(-2\pi ixy))$$

and therefore:

- if $x \in \Pi_b$ then, from the continuity of the functions f and 'exp', it follows that
$$\mathrm{st}\{{}^\star f_\Pi(x)^\star \exp_\Pi(-2\pi ixy)\} = f(\mathrm{st}\, x)\exp[-2\pi i(\mathrm{st}\, x)(\mathrm{st}\, y)] = f_y \circ \mathrm{st}_\infty(x).$$
- if $x \in \Pi_\infty$ then since $f \in \mathcal{C}_0 \cap \mathbf{L}^1$ we have ${}^\star f_\Pi(x) \approx 0$; moreover the function ${}^\star \exp_\Pi(-2\pi ixy)$ is finitely bounded and therefore
$$\mathrm{st}\{{}^\star f_\Pi(x)^\star \exp_\Pi(-2\pi ixy)\} = f(\mathrm{st}\, x)\exp[-2\pi i(\mathrm{st}\, x)(\mathrm{st}\, y)] = 0 = f_y \circ \mathrm{st}_\infty(x).$$

Now it remains to show that the internal function ${}^\star f_\Pi(x)^\star \exp_\Pi(-2\pi ixy)$ is, for every (fixed) $y \in \Pi_b$, an $\mathbf{S}\Pi$-integrable function, that is, satisfies the following requirements:

(a) $\displaystyle\sum_{x \in \Pi_0^+} \varepsilon|{}^\star f_\Pi(x)|$ is finite,

(b) if $A \subset \Pi$ is internal and $\Lambda(A) \approx 0$ then $\displaystyle\sum_{x \in A} \varepsilon|{}^\star f_\Pi(x)| \approx 0$,

(c) if $A \subset \Pi$ is internal and ${}^\star f_\Pi(x) \approx 0, \forall_{x \in A}$ then $\displaystyle\sum_{x \in A} \varepsilon|{}^\star f_\Pi(x)| \approx 0$.

Since ${}^\star f_\Pi(x)$ is finitely bounded, taking into account that

$$\left|\sum_{x \in A} \varepsilon^\star f_\Pi(x)\right| \le \sum_{x \in A} \varepsilon|{}^\star f_\Pi(x)| \le \left\{\max_{x \in A} |{}^\star f_\Pi(x)|\right\} \cdot \Lambda(A)$$

shows that (b) follows immediately. We proceed now by proving the following lemma:

Lemma 3.3. *The hyperfinite* Π*-sum*

$$\sum_{|\gamma_1| \le x \le |\gamma_2|} \varepsilon|{}^\star f_\Pi(x)|$$

is infinitesimal for every two remote points $\gamma_1, \gamma_2 \in \Pi_\infty^+$ *(or, alternatively,* $\gamma_1, \gamma_1 \in \Pi_\infty^-$*) with* $|\gamma_1| \le |\gamma_2| < \kappa/2$.

Proof of Lemma 3.3. Without any loss of generality we may take γ_1 and γ_2 to belong to $\Pi_\infty^+ \cap {}^\star\mathbb{N}_\infty$. Then we have

$$\sum_{\gamma_1 \le x < \gamma_2} \varepsilon|{}^\star f_\Pi(x)| = \sum_{j=\gamma_1\kappa}^{\gamma_2\kappa - 1} \varepsilon|{}^\star f_\Pi(j\varepsilon)| = \sum_{n=\gamma_1}^{\gamma_2}\left\{\sum_{m=0}^{\kappa-1} \varepsilon|{}^\star f_\Pi(x)|\right\}$$

and therefore, taking into account that $|{}^\star f_\Pi(x)|$ is monotonically decreasing, we obtain

$$\sum_{\gamma_1 \le x < \gamma_2} \varepsilon|{}^\star f_\Pi(x)| \le \sum_{n=\gamma_1}^{\gamma_2} |{}^\star f_\Pi(x)|\left\{\sum_{m=0}^{\kappa-1} \varepsilon\right\} = \sum_{n=\gamma_1}^{\gamma_2} |{}^\star f_\Pi(n)|.$$

From the integral test it follows that the (standard) series

$$\sum_{n=1}^{\infty} |f(n)|$$

and the (standard) integral

$$\int_{1}^{+\infty} |f| \mathrm{d}\lambda$$

both converge or both diverge. Since the integral, by the hypothesis, is convergent, then the series also converges and therefore, from the nonstandard Cauchy convergence criterion for series, it follows that the hyperfinite sum

$$\sum_{n=\gamma_1}^{\gamma_2} |{}^\star f_\Pi(n)|$$

is infinitesimal. ■

Now, for any arbitrarily fixed real number $e > 0$, define

$$\mathcal{N}_e = \left\{ n \in {}^\star\mathbb{N} : \sum_{|j|=n\kappa}^{\frac{\kappa^2}{2}-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| < e \right\}.$$

From Lemma 3.3, $\mathcal{N}_e$ contains arbitrarily small infinite numbers; since $\mathcal{N}_e$ is internal, then by underflow it contains a finite number, say $n_e \in \mathbb{N}$. That is,

$$\forall_n \left[n \in {}^\star\mathbb{N} \wedge n_e \leq n \leq \kappa/2 \Rightarrow \sum_{|j|=n\kappa}^{\frac{\kappa^2}{2}-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| < e \right].$$

Hence, since we have

$$\sum_{x\in\Pi} \varepsilon |{}^\star f_\Pi(x)| = \sum_{|j|=0}^{n_e\kappa-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| + \sum_{|j|=n_e\kappa}^{\frac{\kappa^2}{2}-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| < \sum_{|j|=0}^{n_e\kappa-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| + e$$

and, moreover,

$$\sum_{|j|=0}^{n_e\kappa-1} \varepsilon |{}^\star f_\Pi(j\varepsilon)| \leq n_e \left\{ \max_{-n_e \leq x \leq n_e} |{}^\star f_\Pi(x)| \right\} < +\infty,$$

then (a) follows.

To prove (c) we reason as follows: (i) if $\Lambda(A)$ is finite, then the result follows from the fact that

$$\sum_{x\in A} \varepsilon |{}^\star f_\Pi(x)| = \left\{ \max_{x\in A} |{}^\star f_\Pi(x)| \right\} \cdot \Lambda(A) \approx 0;$$

(ii) if $\Lambda(A)$ is not finite then A certainly contains an infinite point in $\Pi^{\pm}_{\infty}$. Again from lemma 3.3 it follows that for any real $e > 0$ there exists (standard) $n_e \in \mathbb{N}$ such that

$$\sum_{|x|\in A\cap[n_e,\kappa/2-\varepsilon]} \varepsilon |{}^\star f_\Pi(x)| < e,$$

while

$$\sum_{|x|\in A\cap[0,n_e-\varepsilon]} \varepsilon|^\star f_\Pi(x)| \approx 0.$$

Thus,

$$\sum_{x\in A} \varepsilon|^\star f_\Pi(x)| < e,$$

and, since $e > 0$ is arbitrary, the proof of (c) is complete.

Taking into account the definition of the (external) function $f \circ \mathrm{st}_\infty$, to prove the equality sign in (11) we need yet to show that the equality

$$\mathrm{st}\left(\sum_{x\in\Pi} \varepsilon^\star f_\Pi(x)\mathrm{e}^{-2\pi ixy}\right) = \mathrm{st}\left(\sum_{x\in\Pi} \varepsilon^\star f_\Pi(x)\mathrm{e}^{-2\pi ix(\mathrm{st}\, y)}\right)$$

holds. For this it is enough to show that the internal function

$$\hat{F}(y) = \sum_{x\in\Pi} \varepsilon^\star f_\Pi(x)\mathrm{e}^{-2\pi ixy}$$

is $\mathbf{S}\Pi$-continuous on Π_b. For $y, y' \in \Pi_b$ we have that

$$|\hat{F}(y) - \hat{F}(y')| \le \sum_{x\in\Pi} \varepsilon|^\star f_\Pi(x)|\,|1 - \mathrm{e}^{-2\pi ix(y-y')}|.$$

From the fact that $f \in \mathcal{C}_0 \cap \mathbf{L}^1$ we have that, given a real number $r > 0$, the subset

$$\{x \in \Pi_b : |^\star f_\Pi(x)| < r/3\}$$

contains arbitrarily small infinite points; since the set is internal then by underflow there exists $x_r \in \Pi_b^+$ such that

$$\forall_x[x \in \Pi \wedge |x| > x_r \Rightarrow |^\star f_\Pi(x)| < r/3].$$

Then

$$\begin{aligned}
|\hat{F}(y) - \hat{F}(y')| &\le \left\{\sum_{|x|\le x_r} + \sum_{|x|>x_r}\right\}\varepsilon|^\star f_\Pi(x)|\,|1 - \mathrm{e}^{-2\pi ix(y-y')}| \\
&\le \sum_{|x|\le x_r} \varepsilon|^\star f_\Pi(x)|\,|1 - \mathrm{e}^{-2\pi ix(y-y')}| + 2\sum_{|x|>x_r} \varepsilon|^\star f_\Pi(x)| \\
&< \frac{2r}{3} + \sum_{|x|\le x_r} \varepsilon|^\star f_\Pi(x)|\,|1 - \mathrm{e}^{-2\pi ix(y-y')}|.
\end{aligned}$$

Now, if $y \approx y'$ and x is finite then $2\pi ix(y - y') \approx 0$ and thus

$$\begin{aligned}
&\sum_{|x|\le x_r} \varepsilon|^\star f_\Pi(x)|\,|1 - \mathrm{e}^{-2\pi ix(y-y')}| \\
&\quad\le \left\{\max_{|x|\le x_r} |1 - \mathrm{e}^{-2\pi ix(y-y')}|\right\} \sum_{|x|\le x_r} \varepsilon|^\star f_\Pi(x)| \approx 0 < \frac{r}{3}.
\end{aligned}$$

Hence

$$|\hat{F}(y) - \hat{F}(y')| < r$$

and, since this is true for all real $r > 0$, it follows that

$$\forall_{y,y'}[y, y' \in \Pi_b \wedge y \approx y' \Rightarrow \hat{F}(y) \approx \hat{F}(y')]$$

that is, $\hat{F}$ is $\mathbf{S}\Pi$-continuous on Π_b and we have

$$\hat{f}(\operatorname{st} y) = \operatorname{st}(\hat{F}(y)), \quad y \in \Pi_b.$$

The proof is thereby complete. □

Given $f \in \mathcal{C}_0 \cap \mathbf{L}^1$ we may therefore obtain the Fourier transform of f by

$$\hat{f}(\operatorname{st} y) = \operatorname{st} \hat{F}(y), \quad y \in \Pi_b$$

where $\hat{F}$ is an $\mathbf{S}\Pi$-continuous function over Π. Hence $\hat{f}(\operatorname{st} y)$ is a continuous (and even uniformly continuous) function. Moreover, for every $y \in \Pi$,

$$|\hat{F}(y)| \leq \sum_{x \in \Pi} \varepsilon |{}^\star f_\Pi(x)|$$

which, since the right-hand side is finite, allow us to conclude that $\hat{F}(y)$, $y \in \Pi$ and $\hat{f}(\operatorname{st} y)$, $y \in \Pi_b$ are bounded functions.

The function $\hat{f}$, in general, does not belong to $\mathbf{L}^1$ and therefore, the inverse Fourier transform as defined by

$$\bar{\mathcal{F}}_\Pi[{}^\star \hat{f}_\Pi](x) = \sum_{y \in \Pi} \varepsilon {}^\star \hat{f}_\Pi(y) {}^\star \exp_\Pi(2\pi i x y)$$

in general, does not allow us to recover the original function ${}^\star f_\Pi$ (and therefore f). For this purpose we have to take the inverse Π-Fourier transform of the function $\hat{F} = \mathcal{F}_\Pi[{}^\star f_\Pi]$

$$\begin{aligned}\bar{\mathcal{F}}_\Pi[{}^\star \hat{f}_\Pi](x) &\neq \sum_{y \in \Pi} \varepsilon \hat{F}(y) {}^\star \exp_\Pi(2\pi i x y) \\ &= \hat{\mathcal{F}}_\Pi[\hat{F}](x) = {}^\star f_\Pi(x), \quad x \in \Pi.\end{aligned}$$

However, a nonstandard version of the Parseval's formula involving two functions $f, g \in \mathcal{C} \cap \mathbf{L}^1$, of the form

$$\sum_{y \in \Pi} \varepsilon {}^\star \hat{f}_\Pi(y) {}^\star g_\Pi(y) \approx \sum_{x \in \Pi} \varepsilon {}^\star f_\Pi(x) {}^\star \hat{g}_\Pi(x) \tag{12}$$

can be derived. Note that this is not the Π-Parseval's formula (10). To prove (12) it is enough to show that

$$\begin{aligned}\sum_{x \in \Pi} \varepsilon {}^\star f_\Pi(x) \hat{G}(x) &\approx \sum_{x \in \Pi} \varepsilon {}^\star f_\Pi(x) {}^\star \hat{g}_\Pi(x), \\ \sum_{y \in \Pi} \varepsilon \hat{F}(y) {}^\star g_\Pi(y) &\approx \sum_{y \in \Pi} \varepsilon {}^{\star\star} \hat{f}_\Pi(y) {}^\star g_\Pi(y).\end{aligned}$$

We will prove, for example, the second one since the other may be obtained similarly. Consider therefore

$$\sum_{x \in \Pi} \varepsilon \{\hat{F}(y) - {}^\star \hat{f}_\Pi(y)\} {}^\star g_\Pi(y).$$

For any $y \in \Pi_b$ we have that

$$ {}^\star\hat{f}_\Pi(y) \approx \hat{f}(\operatorname{st} y) \approx \hat{F}(y) $$

and therefore the set

$$ \{y \in \Pi : |y| > 0 \wedge |\hat{F}(y) - {}^\star\hat{f}_\Pi(y)| < 1/|y|\} $$

is internal and contains all finite $y \in \Pi_b$, $y \neq 0$. By overflow it contains $\eta \in \Pi_\infty^+$ such that

$$ \forall_y[y \in \Pi \wedge |y| \leq \eta \Rightarrow \hat{F}(y) \approx {}^\star\hat{f}_\Pi(y)]. $$

This fact, however, does not imply that the difference $\hat{F}(y) - {}^\star\hat{f}_\Pi(y)$ is kept at an infinitesimal level over the whole of the hyperfinite grid. Nevertheless, we have

$$ \begin{aligned} \left|\sum_{y\in\Pi} \varepsilon\{\hat{F}(y) - {}^\star\hat{f}_\Pi(y)\}{}^\star g_\Pi(y)\right| &\leq \left|\left\{\sum_{|y|\leq\eta} + \sum_{|y|>\eta}\right\}\varepsilon\{\hat{F}(y) - {}^\star\hat{f}_\Pi(y)\}{}^\star g_\Pi(y)\right| \\ &= \left\{\max_{|y|\leq\eta} |\hat{F}(y) - {}^\star\hat{f}_\Pi(y)|\right\} \sum_{|y|\leq\eta} \varepsilon|{}^\star g_\Pi(y)| \\ &\quad + \left\{\max_{|y|>\eta} |\hat{F}(y) - {}^\star\hat{f}_\Pi(y)|\right\} \sum_{|y|>\eta} \varepsilon|{}^\star g_\Pi(y)|. \end{aligned} \tag{13} $$

Now, because $g \in \mathcal{C}_0 \cap \mathbf{L}^1$, then

$$ \sum_{|y|\leq\eta} \varepsilon|{}^\star g_\Pi(y)| \leq \int_{\mathbb{R}} |g(t)|\mathrm{d}t < +\infty $$

and

$$ \sum_{|y|>\eta} \varepsilon|{}^\star g_\Pi(y)| \approx 0. $$

Moreover, $\max_{|y|\leq\eta}|\hat{F}(y) - {}^\star\hat{f}_\Pi(y)| \approx 0$ and $\max_{|y|>\eta}|\hat{F}(y) - {}^\star\hat{f}_\Pi(y)|$ is finite. Using all these facts in (13) we obtain finally

$$ \sum_{y\in\Pi} \varepsilon\{\hat{F}(y) - {}^\star\hat{f}_\Pi(y)\}{}^\star g_\Pi(y) \approx 0 $$

as asserted.

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