

## Weighted approximation of continuous functions by sequences of linear positive operators

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MS received 24 November 1999; revised 7 July 2000

**Abstract.** In this work we obtain, under suitable conditions, theorems of Korovkin type for spaces with different weight, composed of continuous functions defined on unbounded regions. These results can be seen as an extension of theorems by Gadjiev in [4] and [5].

**Keywords.** Korovkin theorem; positive linear operators; weighted spaces; weight functions.

Let  $C(a, b)$  denote the space of all continuous functions on  $[a, b]$  and let  $B(a, b)$  be the space of all bounded functions on the same interval. If the sequence of positive linear operators  $A_n : C(a, b) \rightarrow B(a, b)$  satisfy the three conditions

$$\begin{aligned}\lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_{C(a, b)} &= 0, \\ \lim_{n \rightarrow \infty} \|A_n(t, x) - x\|_{C(a, b)} &= 0, \\ \lim_{n \rightarrow \infty} \|A_n(t^2, x) - x^2\|_{C(a, b)} &= 0,\end{aligned}$$

then

$$\lim_{n \rightarrow \infty} \|A_n(f, x) - f(x)\|_{C(a, b)} = 0$$

for all function  $f \in C(a, b)$  for which  $|f(x)| \leq M_f(1 + x^2)$  hold on  $\mathbb{R}$ . This theorem is known as Korovkin theorem ([6, 1]) and it is important in approximation theory. The theorem shows that convergence on three functions may be extended to all functions which are continuous on  $[a, b]$  and bounded on  $\mathbb{R}$ . Baskakov [2] generalized this result to unbounded functions on  $\mathbb{R}$ .

In refs [4] and [5] Gadjiev defined the weight spaces  $C_\rho$  and  $B_\rho$  of real functions defined on the real line and showed that Korovkin's theorem in general does not hold on these spaces. Here  $B_\rho := \{f : |f| \leq M_f \cdot \rho, \rho \geq 1 \text{ and } \rho \text{ unbounded}\}$  and  $C_\rho := \{f : f \in B_\rho \text{ and } f \text{ continuous}\}$  are spaces of functions which are defined on unbounded sets. However in [4] and [5] it has been shown that this theorem holds on a common subspace of the spaces  $B_\rho$  and  $C_\rho$ .

In ref. [3] it is proved that a theorem of Korovkin type does not hold on the spaces  $C_{\rho_1}$  and  $B_{\rho_2}$  with different weights  $\rho_1$  and  $\rho_2$ , respectively. But in this study we show that if we put some appropriate conditions on the weight functions it holds.

We firstly give the following lemma which will be needed for proving the other theorems.

Let

$$\psi_n(s) := \sup_{\|f\|_{\rho_1}=1} \sup_{|x| \leq s} \frac{|A_n(f, x)|}{\rho_1(x)},$$

for all  $f \in C_{\rho_1}$  and  $s \in \mathbb{R}$ .

**Lemma 1.** Suppose that for positive linear operators  $A_n : C_{\rho_1} \longrightarrow B_{\rho_2}$  the sequence  $\|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_2}}$  of operator norms is uniformly bounded, and

$$\lim_{x \rightarrow \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0 \quad (1)$$

and  $\lim_{n \rightarrow \infty} \psi_n(s) = 0$  for any  $s$ . Then

$$\lim_{n \rightarrow \infty} \|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_2}} = 0.$$

*Proof.* Since  $\lim_{x \rightarrow \infty} \frac{\rho_1(x)}{\rho_2(x)} = 0$ , there exists a number  $s_0$  for  $\varepsilon > 0$  such that  $\frac{\rho_1(x)}{\rho_2(x)} \leq \varepsilon$  for all  $|x| > s_0$ . Since  $\rho_1$  and  $\rho_2$  are continuous and strictly positive, the function  $\frac{\rho_1}{\rho_2}$  is also continuous and bounded for  $|x| \leq s_0$ . Then there exists a number  $c_1 > 0$  such that  $\frac{\rho_1(x)}{\rho_2(x)} \leq c_1$  for all  $|x| \leq s_0$ . On the other side there exists, from the hypothesis, a number  $c_2 > 0$  such that  $\|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_2}} \leq c_2$  for all  $n \in \mathbb{N}$ . Hence we have the inequality

$$\begin{aligned} \|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_2}} &= \sup_{\|f\|_{\rho_1}=1} \left\{ \sup_{x \in \mathbb{R}} \frac{|A_n(f, x)|}{\rho_2(x)} \right\} \\ &\leq \sup_{\|f\|_{\rho_1}=1} \left\{ \sup_{|x| > s_0} \frac{|A_n(f, x)|}{\rho_1(x)} \frac{\rho_1(x)}{\rho_2(x)} \right\} + \sup_{\|f\|_{\rho_1}=1} \left\{ \sup_{|x| \leq s_0} \frac{|A_n(f, x)|}{\rho_1(x)} \frac{\rho_1(x)}{\rho_2(x)} \right\} \\ &\leq \varepsilon \|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_1}} + c_1 \cdot \psi_n(s_0) \\ &\leq \varepsilon c_2 + c_1 \cdot \psi_n(s_0). \end{aligned}$$

From this and the hypothesis we obtain

$$\lim_{n \rightarrow \infty} \|A_n\|_{C_{\rho_1} \rightarrow B_{\rho_2}} = 0$$

and the proof is complete.  $\square$

**Theorem 1.** Let the weight functions  $\rho_1$  and  $\rho_2$  be as in Lemma 1 and let the sequence  $\|L_n\|_{C_{\rho_1} \rightarrow B_{\rho_1}}$  of operator norms be uniformly bounded. Here  $L_n : C_{\rho_1} \longrightarrow B_{\rho_2}$  are positive linear operators. If the equality

$$\lim_{n \rightarrow \infty} |L_n(f, x) - f(x)| = 0$$

holds for all  $s_0$  with  $|x| \leq s_0$ , then

$$\lim_{n \rightarrow \infty} \|L_n(f, x) - f(x)\|_{\rho_2} = 0$$

for all  $f \in C_{\rho_1}$ .

*Proof.* Let  $E$  be the identity operator on  $C_{\rho_1}$ , and replace the operators  $A_n$  in Lemma 1 by  $L_n - E$ . Since  $\rho_1(x) \geq 1$  for all  $x$ , we obtain the inequality

$$\psi_n(s_0) \leq \sup_{\|f\|_{\rho_1}=1} \left\{ \sup_{|x| \leq s_0} |L_n(f, x) - f(x)| \right\}.$$

By hypotheses it follows that

$$\lim_{n \rightarrow \infty} \psi_n(s_0) = 0.$$

Hence from Lemma 1 it follows  $\lim_{n \rightarrow \infty} \|L_n - E\|_{C_{\rho_1} \rightarrow B_{\rho_2}} = 0$ , and so

$$\|L_n(f, x) - f(x)\|_{\rho_2} \leq \|L_n - E\|_{C_{\rho_1} \rightarrow B_{\rho_2}} \|f\|_{\rho_1}.$$

From this we obtain the required result.  $\square$

*Remark.* Let  $(A_n)$ ,  $A_n : C_{\rho_1} \rightarrow B_{\rho_2}$ , be a sequence of positive linear operators for all  $n \in \mathbb{N}$ . Suppose that there exists  $M > 0$  such that for all  $x \in \mathbb{R}$  we have  $\rho_1(x) \leq M\rho_2(x)$ . If

$$\lim_{n \rightarrow \infty} \|A_n(\rho_1, x) - \rho_1(x)\|_{\rho_2} = 0,$$

then the sequence  $(A_n)_{n \in \mathbb{N}}$  is uniformly bounded.

Let  $\varphi_1$  and  $\varphi_2$  be two continuous functions, monotonically increasing on the real axis, such that  $\lim_{x \rightarrow \mp\infty} \varphi_1(x) = \lim_{x \rightarrow \mp\infty} \varphi_2(x) = \mp\infty$  and  $\rho_k(x) = 1 + \varphi_k^2(x)$ ,  $k = 1, 2$ .

**Theorem 2.** *If the positive linear operators sequence*

$$A_n : C_{\rho_1} \rightarrow B_{\rho_2}$$

*satisfies the following three conditions*

$$\lim_{n \rightarrow \infty} \|A_n(\varphi_1^\nu, x) - \varphi_1^\nu(x)\|_{\rho_2} = 0, \quad \nu = 0, 1, 2, \quad (2)$$

*and the condition expressed in equation (1). Then*

$$\lim_{n \rightarrow \infty} \|A_n(f, x) - f(x)\|_{\rho_2} = 0$$

*for all  $f \in C_{\rho_1}$ .*

*Proof.* To prove the theorem, it is sufficient to show that the conditions of Theorem 1 should be satisfied, i.e., the sequence of operator norms of  $A_n$  is uniformly bounded and  $\lim_{n \rightarrow \infty} |A_n(f, x) - f(x)| = 0$  for  $|x| \leq s_0$ . Let us show that the sequence of operator norms of  $A_n$  is uniformly bounded. From the hypotheses

$$\lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_{\rho_2} = 0$$

and

$$\lim_{n \rightarrow \infty} \|A_n(\varphi_1^2(x), x) - \varphi_1^2(x)\|_{\rho_2} = 0.$$

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|A_n(\rho_1, x) - \rho_1\|_{\rho_2} &\leq \lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_{\rho_2} \\ &\quad + \lim_{n \rightarrow \infty} \|A_n(\varphi_1^2(x), x) - \varphi_1^2(x)\|_{\rho_2} = 0. \end{aligned}$$

That means the sequence of operator norms are uniformly bounded from the Remark above.

Now let us examine the difference  $|A_n(f, x) - f(x)|$ :

$$\begin{aligned} |A_n(f, x) - f(x)| &\leq A_n(|f(t) - f(x)|, x) + |f(x)| |A_n(1, x) - 1| \\ &= I'_n(x) + I''_n(x). \end{aligned}$$

Firstly, we investigate the limit of  $I''_n(x)$  for  $n \rightarrow \infty$ .

Since  $f(x)$  is a continuous function, it is bounded on the interval  $|x| \leq s_0$  for any  $s_0$ . Now set  $M_1 := \max_{|x| \leq s_0} |f(x)|$ . Note that from the hypothesis we have

$$\lim_{n \rightarrow \infty} \|A_n(1, x) - 1\|_{\rho_2} = \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{|A_n(1, x) - 1|}{\rho_2(x)} = 0.$$

Hence, for a zero sequence,  $\varepsilon_n$

$$\sup_{x \in \mathbb{R}} \frac{|A_n(1, x) - 1|}{\rho_2(x)} = \varepsilon_n.$$

From this we have

$$|A_n(1, x) - 1| \leq \varepsilon_n \rho_2(x)$$

for all  $x \in \mathbb{R}$ , and so

$$\lim_{n \rightarrow \infty} |A_n(1, x) - 1| = 0.$$

Therefore this implies

$$\lim_{n \rightarrow \infty} I''_n(x) = \lim_{n \rightarrow \infty} |f(x)| |A_n(1, x) - 1| \leq \lim_{n \rightarrow \infty} M_1 |A_n(1, x) - 1| = 0$$

for all  $|x| \leq s_0$ .

Let us obtain some inequalities that can be used to find the limit of  $I'_n$  as  $n$  tends to infinity. It is easy to see that the inequality

$$\begin{aligned} |f(t) - f(x)| &\leq 2M_f \rho_1(t) \rho_1(x) \\ &\leq 4M_f \rho_1(x) [1 + (\varphi_1(t) - \varphi_1(x))^2 + \varphi_1^2(x)] \end{aligned} \quad (*)$$

holds. Setting

$$\Delta_\rho(\varphi_1, x) := \min\{\varphi_1(x + \delta) - \varphi_1(x); \varphi_1(x) - \varphi_1(x - \delta)\}, \quad (3)$$

we obtain

$$|\varphi_1(t) - \varphi_1(x)| > \min\{\varphi_1(x + \delta) - \varphi_1(x); \varphi_1(x) - \varphi_1(x - \delta)\},$$

and therefore

$$\frac{1}{|\varphi_1(t) - \varphi_1(x)|} < \frac{1}{\Delta_\rho(\varphi_1, x)}.$$

$\rho_1(x) \geq 1$  and the inequality (\*) implies

$$\begin{aligned} |f(t) - f(x)| &< 4M_f \rho_1(x) (\varphi_1(t) - \varphi_1(x))^2 \left[ \frac{1}{\Delta_\rho^2(\varphi_1, x)} + 1 + \frac{\varphi_1(x)^2}{\Delta_\rho^2(\varphi_1, x)} \right] \\ &\leq 4M_f \rho_1(x)^2 (\varphi_1(t) - \varphi_1(x))^2 \left[ \frac{1}{\Delta_\rho^2(\varphi_1, x)} + 1 \right]. \end{aligned}$$

On the other hand, from the continuity of the function  $f(x)$ , there exists a number  $\varepsilon > 0$  such that

$$|f(t) - f(x)| < \varepsilon$$

for all  $t$  and  $x$  for which  $|t - x| < \delta$ . If we set

$$K_{\rho_1}(x) := 4M_f \rho_1(x)^2 \left[ 1 + \frac{1}{\Delta_\rho^2(\varphi_1, x)} \right],$$

then we obtain

$$|f(t) - f(x)| < \varepsilon + K_{\rho_1}(x) (\varphi_1(t) - \varphi_1(x))^2 \quad (4)$$

for all  $t \in \mathbb{R}$  and  $x$  for which  $|x| \leq s_0$ . Since the function  $\varphi_1$  is monotonically increasing, then  $\Delta_\rho(\varphi_1, x) \neq 0$ , and therefore  $K_{\rho_1}(x)$  is a continuous function. Now suppose

$$M_2 := \max_{|x| \leq s_0} K_{\rho_1}(x).$$

The monotonicity of the operators  $A_n$  and eq. (4) yield the inequality

$$A_n(|f(t) - f(x)|, x) \leq \varepsilon [A_n(1, x) - 1] + \varepsilon + M_2 A_n((\varphi_1(t) - \varphi_1(x))^2, x).$$

We obtain

$$\begin{aligned} |A_n(1, x) - 1| &< \varepsilon_n \rho_2(x), \\ |A_n(\varphi, x) - \varphi(x)| &< \varepsilon_n \rho_2(x), \\ |A_n(\varphi^2, x) - \varphi^2(x)| &< \varepsilon_n \rho_2(x), \end{aligned}$$

from the hypothesis

$$\lim_{n \rightarrow \infty} \|A_n(\varphi^\nu(x), x) - \varphi^\nu(x)\|_{\rho_2} = 0, \quad \nu = 0, 1, 2.$$

We can write from these inequalities that

$$A_n((\varphi_1(t) - \varphi_1(x))^2, x) < \varepsilon_n \rho_2(x) (1 + \varphi_1(x))^2.$$

The boundness of  $\rho_k(x)$  ( $k = 1, 2$ ) for  $|x| \leq s_0$  yields

$$\lim_{n \rightarrow \infty} A_n((\varphi_1(t) - \varphi_1(x))^2, x) < \lim_{n \rightarrow \infty} \varepsilon_n \rho_1^2(x) \rho_2(x) = 0.$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} I'_n(x) &\leq \lim_{n \rightarrow \infty} [\varepsilon (A_n(1, x) - 1) + \varepsilon + M_2 A_n((\varphi_1(t) - \varphi_1(x))^2, x)] \\ &< \lim_{n \rightarrow \infty} [\varepsilon \cdot \varepsilon_n \cdot \rho_2(x) + M_2 A_n((\varphi_1(t) - \varphi_1(x))^2, x)] = 0 \end{aligned}$$

for all  $|x| \leq s_0$ . This proves the theorem.  $\square$

*Note.* Lemma 1, Theorems 1 and 2 have been obtained by Gadjiev, cf. [5], for  $\rho_1 = \rho_2$ . The results in ref. [5] cannot be deduced from our theorems from the condition (1).

### Acknowledgements

The author would like to thank Prof. A D Gadjiev for several helpful discussions and comments.

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