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Coin tossing and Laplace inversion

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Abstract. An analysis of exchangeable sequences of coin tossings leads to inversion formulae for Laplace transforms of probability measures.

Keywords. Laplace inversion; moment problem; exchangeable probabilities.

1. Introduction

There is an intimate relationship between the Laplace transform

$$\phi(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} \mathrm{d}\nu(t), \quad \lambda \geq 0 \tag{1.1}$$

of a probability measure ν on $[0, \infty)$ and the moment sequence

$$c(n) = \int_0^1 x^n \mathrm{d}\mu(x), \quad n = 0, 1, 2, \ldots \tag{1.2}$$

of a probability measure μ on $(0, 1]$ via the obvious change of variables $\mathrm{e}^{-t} = x$. An inversion formula for μ in terms of its moments yields an inversion formula for ν in terms of the values of its Laplace transform at $n = 0, 1, 2, \ldots$ and vice versa. In our discussion we allow μ (respectively ν) to have positive mass at 0 (respectively ∞).

Let $X_1, X_2, \ldots$ be $0, 1$-valued random variables; one can identify 1 with 'heads' and 0 with 'tails'. These variables are said to be exchangeable if their joint distribution is invariant under finite permutations. Such variables can be generated in the following manner: first choose p at random according to a probability law μ on $[0, 1]$ and then let $X_1, X_2, \ldots$ be results of i.i.d tosses of a coin having probability p for 'heads'. The resulting measure

$$P(\cdot) = \int_0^1 P_p(\cdot) \mathrm{d}\mu(p) \tag{1.3}$$

on $\{0, 1\}^N$ is a mixture of i.i.d probabilities P_p.

Then under P the process of coordinate functions is exchangeable. By a theorem of De Finetti, any exchangeable sequence of $0, 1$-valued random variables arises in this manner for a suitable μ. The strong law of large numbers takes the form

$$Y_n := \frac{X_1 + X_2 + \cdots + X_n}{n} \longrightarrow Y_\infty \quad \text{a.s. } [P]. \tag{1.4}$$

Here the limit Y_∞ is a random variable. Further,

$$Y_\infty \sim \mu \quad \text{and} \quad \mathcal{L}(Y_n) \Longrightarrow \mu, \tag{1.5}$$

where $\mathcal{L}(Y_n)$ stands for the probability law of $Y_n, n \geq 1$ and '$\Longrightarrow$' denotes weak convergence. If $\{\tau_n\}_{n\geq 1}$ is a sequence of stopping times such that

$$\tau_n \to \infty \text{ a.s } [P], \tag{1.6}$$

then

$$Y_{\tau_n} \to Y_\infty \text{ a.s. } [P] \quad \text{and} \quad F_n \Longrightarrow F, \tag{1.7}$$

where $F, F_1, F_2, \ldots$ are the p.d.f.'s of $\mu, Y_{\tau_1}, Y_{\tau_2}, \ldots$ respectively. In the coin tossing situation many choices of $\{\tau_n\}_{n\geq 1}$ exist for which (1.6) holds and F_n can be explicitly written down in terms of $c(k)$, $k = 1, 2, \ldots$. Thus we get a host of inversion formulae for μ in terms of its moments.

The classical inversion formulae, e.g., those due to Hausdorff, Widder, and Feller can be obtained in the above manner. The methods of these authors were analytical although Feller was motivated by problems arising in stochastic theory of telephone traffic as he mentions in the introduction of his paper [2]. It is therefore satisfying to see that some of the results of Widder and Feller are consequences of the strong law of large numbers, conditioning and stopping times, ideas which are central to probability theory.

This paper is organized as follows. Section 1 deals with the 1-dimensional case while § 2 briefly deals with the 2-dimensional case; the generalization to higher dimensions is straightforward.

2. Coin tossing and inversion formulae

Exchangeable probabilities on the coin-tossing space

Let $E_1 = \{0, 1\}$ and $\Omega = E_1^N$ where $N = \{1, 2, 3, \ldots\}$; the space Ω is sometimes called the coin-tossing space. Let $\omega = (\omega_1, \omega_2, \ldots)$ be a generic point of Ω and $X_1, X_2, \ldots$ be the coordinate variables, i.e., $X_n(\omega) := \omega_n$, $n = 1, 2, \ldots$. Let $\mathcal{F}_n = \sigma\langle X_1, X_2, \ldots, X_n\rangle$ and $\mathcal{F} = \sigma\langle X_n : n \geq 1\rangle$ be the σ-fields of subsets of Ω. Let Σ be the group of all permutations of natural numbers which shift only finitely many of them and let $\Sigma_n = \{\sigma \in \Sigma : \sigma(i) = i$ for $i > n\}$. For $\sigma \in \Sigma$, let $T_\sigma : \Omega \to \Omega$ be defined by $T_\sigma(\omega_1, \omega_2, \ldots) := (\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots)$. Let $\mathcal{S}_n = \{A \in \mathcal{F} : T_\sigma^{-1}A = A$ for all $\sigma \in \Sigma_n\}$ and $\mathcal{S} = \{A \in \mathcal{F} : T_\sigma^{-1}A = A$ for all $\sigma \in \Sigma\}$. Clearly $\mathcal{S}_n \downarrow \mathcal{S}$. A probability P on $(\Omega, \mathcal{F})$ is said to be exchangeable if $PT_\sigma^{-1} = P$ for all $\sigma \in \Sigma$. For each p, $0 \leq p \leq 1$, let P_p be the product probability on Ω under which $P_p(X_i = 1) = 1 - P_p(X_i = 0) = p$, $i = 1, 2, \ldots$. By a theorem of De Finetti, see, e.g., Meyer [7], under an exchangeable P on $(\Omega, \mathcal{F})$, $X_1, X_2, \ldots$ are conditionally independent given the symmetric σ-field $\mathcal{S}$. As a consequence, corresponding to an exchangeable P there exists a probability μ on $[0, 1]$ such that

$$P(F) = \int_0^1 P_p(F) \mathrm{d}\mu(p), \quad F \in \mathcal{F}. \tag{2.1}$$

The probability measure μ is called the *mixing probability* corresponding to P.

Inversion formulae

We fix a probability measure μ on $[0, 1]$ and the associated exchangeable probability P on $(\Omega, \mathcal{F})$. Let

$$\Delta c(k) = c(k+1) - c(k), \tag{2.2}$$

where $c(k)$, $k = 0, 1, 2, \ldots$ are given by (1.2). Then

$$(-1)^{n-k}\Delta^{n-k}c(k) = \int_0^1 (1-x)^{n-k}x^k \mathrm{d}\mu(x), \;\; k = 0, 1, 2, \ldots, n \geq k. \tag{2.3}$$

The atoms of $\mathcal{S}_n$ are

$$\Delta_{n,k} = \{\omega \in \Omega; \#(i : 1 \leq i \leq n, \omega_i = 1) = k\} \tag{2.4}$$

and

$$P(\Delta_{n,k}) = (-1)^{n-k}\binom{n}{k}\Delta^{n-k}c(k). \tag{2.5}$$

It is easily seen that the mixing probability μ is uniquely determined by $P(\Delta_{n,k})$, $n = 1, 2, \ldots,\ k = 1, 2, \ldots, n$.

As $\mathcal{S}_n \downarrow \mathcal{S}$, by the reverse martingale convergence theorem, we have

$$Y_n := E(X_1 \| \mathcal{S}_n) = \frac{X_1 + X_2 + \cdots + X_n}{n} \longrightarrow Y_\infty := E(X_1 \| \mathcal{S}) \text{ a.s. } [P]. \tag{2.6}$$

Further, for $k = 1, 2, \ldots,$

$$\begin{aligned} Y_\infty^k &= E(X_1\|\mathcal{S}) \cdot E(X_2\|\mathcal{S}) \ldots E(X_k\|\mathcal{S}) \text{ a.s. } [P] \\ &= E(X_1X_2\ldots X_k\|\mathcal{S}) \text{ a.s. } [P] \text{ by De Finetti's theorem} \\ &= P(X_1 = X_2 = \cdots = X_k = 1\|\mathcal{S}) \text{ a.s. } [P] \end{aligned}$$

and consequently,

$$E(Y_\infty^k) = \int_0^1 p^k \mathrm{d}\mu(p) = c(k).$$

Thus

$$Y_\infty \sim \mu. \tag{2.7}$$

Now let $\{\tau_n\}_{n\geq 1}$ be a sequence of stopping times with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ such that

$$\tau_n \to \infty \text{ a.s. } [P]. \tag{2.8}$$

Then

$$Y_{\tau_n} \to Y_\infty \text{ a.s. } [P]. \tag{2.9}$$

By the compactness of $[0, 1]$, it follows by Prokhorov's theorem, see, e.g., [1], that the sequence of probability laws of Y_{τ_n} converges in the weak $*$-topology to μ. Thus

$$F_n \Longrightarrow F, \tag{2.10}$$

where $F, F_1, F_2, \ldots$ are the p.d.f.'s of $\mu, Y_{\tau_1}, Y_{\tau_2}, \ldots$ respectively.

For each choice of $\{\tau_n\}_{n\geq 1}$ for which (2.8) holds, we get an inversion formula for μ. We give some examples.

Example 1. Take $\tau_n \equiv n$. Then (2.8) holds and

$$P\left(Y_n = \frac{k}{n}\right) = P(\Delta_{n,k}) = (-1)^{n-k}\binom{n}{k}\Delta^{n-k}c(k) \text{ by (2.5).}$$

Thus

$$F_n(t) = \sum_{k:k\le[nt]} (-1)^{n-k}\binom{n}{k}\Delta^{n-k}c(k) \to F(t)$$

at the points of continuity of F. This is the inversion formula of Hausdorff [3].

Example 2. Let τ_n be the waiting time for the appearance of the nth tail, i.e.,

$$\tau_n(\omega) = \inf\{m : X_1(\omega) + X_2(\omega) + \cdots + X_m(\omega) = m - n\};$$

we adopt the convention that the infimum of the empty set is ∞. Here $\tau_n(\omega) \ge n$ for all ω and (2.8) holds. Further,

$$P_p(\tau_n = n+k) = \binom{n+k-1}{k}p^k(1-p)^n, \quad k = 0,1,2,\ldots,\ 0<p<1,$$
$$P_0(\tau_n = n) = 1 \quad \text{and} \quad P_1(\tau_n = \infty) = 1.$$

Also,

$$P_p\left(Y_{\tau_n} = \frac{k}{n+k}\right) = \binom{n+k-1}{k}p^k(1-p)^n, \quad k = 0,1,2,\ldots,\ 0<p<1,$$
$$P_0(Y_{\tau_n} = 0) = 1 \quad \text{and} \quad P_1(Y_{\tau_n} = 1) = P_1(Y_\infty = 1) = 1$$

by the strong law of large numbers. Thus, by (2.1), the distribution function F_n places mass $(-1)^n\binom{n+k-1}{k}\Delta^n c(k)$ at $\frac{k}{k+n}$, $k = 0,1,2,\ldots$ and the remaining mass $\mu(\{1\}) = c(\infty)$ at 1.

This is essentially the inversion formula derived by Widder – see Theorem 42 and the footnote on p. 193 of [8]. The nth approximant of Widder places mass $(-1)^{n+1}\binom{n+k}{k}$ $\Delta^{n+1}c(k)$ at $\frac{k}{k+n}$, $k = 0,1,2,\ldots$ and mass $c(\infty)$ at 1 which agrees with our F_{n+1} except that $\frac{k}{k+n}$ is replaced by $\frac{k}{k+n+1}$, which hardly matters. It may be observed that F_n contains infinitely many jumps, they cluster at 1 and their amount uses differences of a fixed order.

Example 3. Let σ_n be the waiting time for the appearance of the nth head, i.e.,

$$\sigma_n(\omega) = \inf\{m : X_1(\omega) + X_2(\omega) + \cdots + X_m(\omega) = n\},$$

the infimum of the empty set being ∞. Here $\sigma_n(\omega) \ge n$ for all ω and (2.8) holds. Further,

$$P_p(\sigma_n = n+k) = \binom{n+k-1}{k}p^n(1-p)^k, \quad k = 0,1,2,\ldots,0<p<1,$$
$$P_1(\sigma_n = n) = 1 \quad \text{and} \quad P_0(\sigma_n = \infty) = 1.$$

Also,

$$P_p\left(Y_{\sigma_n} = \frac{n}{n+k}\right) = \binom{n+k-1}{k}p^n(1-p)^k, \quad k = 0,1,2,\ldots,0<p<1,$$
$$P_1(Y_{\sigma_n} = 1) = 1 \quad \text{and} \quad P_0(Y_{\sigma_n} = 0) = 1.$$

Thus, by (2.1), F_n places mass $(-1)^k\binom{n+k-1}{k}\Delta^k c(n)$ at $\frac{n}{n+k}$, $k = 0,1,2,\ldots$ and the remaining mass $\mu(\{0\}) = 1 - \Sigma_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}\Delta^k c(n)$ at 0. It may be observed that F_n contains infinitely many jumps, they cluster at 0 and their amount uses differences belonging to a fixed point.

The above formula was derived by very different methods by Feller – see theorem 5 and remark on pages 673–74 of [2]; in his case μ is supported on $(0,1]$ so that $\mu(\{0\}) = 0$.

Example 4. Let $\rho_n = \sigma_n \wedge \tau_n$ where τ_n and σ_n are as in examples 2 and 3 respectively. Then $\rho_n(\omega) \geq n$ for all ω and (2.8) holds. Further, $P_p(\rho_n = n+k) = \binom{n+k-1}{k}\{p^n q^k + p^k q^n\}$, $k = 0,1,2,\ldots,n-1$ and it is easily seen that F_n places mass $(-1)^k \binom{n+k-1}{k}\Delta^k c(n)$ at $\frac{n}{n+k}$ and mass $(-1)^n\binom{n+k-1}{k}\Delta^n c(k)$ at $\frac{k}{n+k}$, $k = 0,1,2,\ldots,n-1$.

The reader is invited to choose his favourite $\{\tau_n\}_{n\geq 1}$ satisfying (2.8) and write down the corresponding sequence of approximants of the d.f. F of μ.

3. Inversion formulae in higher dimensions

We restrict ourselves to two dimensions; the generalization to higher dimensions is straightforward. The problem of inversion of the Laplace transform

$$\phi(\lambda_1,\lambda_2) = \int_0^\infty \int_0^\infty \mathrm{e}^{-(\lambda_1 t_1 + \lambda_2 t_2)} \mathrm{d}\nu(t_1,t_2), \quad \lambda_i \geq 0, i = 1,2 \tag{3.1}$$

of a probability measure ν on $[0,\infty)\times[0,\infty)$ in terms of $\phi(k,\ell), k,\ell = 0,1,2,\ldots$ is same as that of finding an inversion formula for a probability measure μ on $(0,1]\times(0,1]$ in terms of its moments

$$c(k,\ell) = \int_0^1 \int_0^1 x_1^k x_2^\ell \mathrm{d}\mu(x_1,x_2). \tag{3.2}$$

In our discussion we consider probability measures μ on $I^2 := [0,1]\times[0,1]$.

To do an analysis similar to the 1-dimensional case, we introduce a special kind of exchangeable probability on the space of a sequence of tosses of a pair of coins. First we set up the notation. Let $E_2 = \{(00),(01),(10),(11)\}$, $\Omega = E_2^N$, $\omega = (\omega_1,\omega_2,\ldots)$ with $\omega_i = (\omega_{i1}\,\omega_{i2})$, $i = 1,2,\ldots,$ be a generic point of Ω and $X_n = (X_{n1},X_{n2})$ with $X_{n1}(\omega) = \omega_{n1}$, $X_{n2}(\omega) = \omega_{n2}, n = 1,2,\ldots,$ be the coordinate variables. Let $\mathcal{F}_n, \mathcal{F}, \Sigma, \Sigma_n, T_\sigma, \mathcal{S}_n$ and $\mathcal{S}$ be defined as before. For $p = (p_1,p_2)\in I^2$ let θ_p be the probability on E_2 defined by

$$\begin{aligned}&\theta_p(00) = (1-p_1)(1-p_2),\ \theta_p(01) = (1-p_1)p_2,\\ &\theta_p(10) = p_1(1-p_2) \text{ and } \theta_p(11) = p_1p_2\end{aligned} \tag{3.3}$$

and let $P_p = \theta_p \times \theta_p \times \ldots$ be the corresponding product probability on Ω. We fix a probability μ on I^2 and introduce the exchangeable probability P on $(\Omega,\mathcal{F})$ by

$$P(F) = \int_{I^2} P_p(F)\mathrm{d}\mu(p), \quad F\in\mathcal{F}. \tag{3.4}$$

Let

$$\begin{aligned}&\Delta_1 c(k,\ell) := c(k+1,\ell) - c(k,\ell) \quad \text{and}\\ &\Delta_2 c(k,\ell) := c(k,\ell+1) - c(k,\ell),\end{aligned} \tag{3.5}$$

where $c(k,\ell), k,\ell = 0,1,2,\dots$ are given by (3.2). We have

$$(-1)^{2n-k-\ell}\Delta_1^{n-k}\Delta_2^{n-\ell}c(k,\ell) = \int_{I^2} x_1^k x_2^\ell (1-x_1)^{n-k}(1-x_2)^{n-\ell}\mathrm{d}\mu(x_1,x_2). \tag{3.6}$$

The probability of the atoms of $\mathcal{S}_n$ can be written in terms of these differences and it is easily seen that the mixing probability μ is uniquely determined by the values of P on the atoms of $\mathcal{S}_n, n \geq 1$.

By De Finetti's theorem, under $P, X_n, n \geq 1$ are conditionally independent given $\mathcal{S}$; further, by our construction, the mixing probability μ is supported on the set of probabilities of type θ_p on E_2 as given in (3.3). Therefore

(a) for almost all $\omega[P], (E(X_{n1}\|\mathcal{S})(\omega), E(X_{n2}\|\mathcal{S})(\omega)), n = 1,2,\dots$ are like tosses of a pair of coins having probability of 'heads', say $p_1(\omega)$ and $p_2(\omega)$, *all* the tosses being independent and
(b) $(p_1, p_2) \sim \mu$.

By the reverse martingale convergence theorem,

$$\begin{aligned} Y_{n1} &:= E(X_{11}\|\mathcal{S}_n) = \frac{X_{11}+X_{21}+\cdots+X_{n1}}{n} \to Y_{\infty 1} := E(X_{11}\|\mathcal{S}) \text{ a.s. } [P] \\ Y_{n2} &:= E(X_{12}\|\mathcal{S}_n) = \frac{X_{12}+X_{22}+\cdots+X_{n2}}{n} \to Y_{\infty 2} := E(X_{12}\|\mathcal{S}) \text{ a.s. } [P]. \end{aligned} \tag{3.7}$$

Further, by (a) and (b) above, for $k,\ell = 0,1,2,\dots,$ we have

$$\begin{aligned} Y_{\infty 1}^k Y_{\infty 2}^\ell &= \{\Pi_{i=1}^k E(X_{i1}\|\mathcal{S})\}\{\Pi_{j=1}^\ell E(X_{j2}\|\mathcal{S})\} \text{ a.s. } [P] \\ &= E\{(\Pi_{i=1}^k X_{i1})(\Pi_{j=1}^\ell X_{j2})\|\mathcal{S}\} \text{ a.s. } [P] \\ &= P(X_{11} = X_{21} = \cdots = X_{k1} = X_{12} = X_{22} = \cdots = X_{\ell 2} = 1\|\mathcal{S}) \text{ a.s. } [P] \\ &= p_1^k p_2^\ell \end{aligned}$$

and

$$\begin{aligned} E(Y_{\infty 1}^k Y_{\infty 2}^\ell) &= \int_{I^2} p_1^k p_2^\ell \mathrm{d}\mu(p) \\ &= c(k,\ell). \end{aligned}$$

Thus

$$(Y_{\infty 1}, Y_{\infty 2}) \sim \mu. \tag{3.8}$$

Now let $\{\tau_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ be two sequences of stopping times with respect to $\{\mathcal{F}_n\}_{n\geq 1}$ such that

$$\tau_n \to \infty \text{ a.s. } [P], \quad \sigma_n \to \infty \text{ a.s. } [P]. \tag{3.9}$$

Then, by (3.7) and (3.9),

$$(Y_{\tau_n 1}, Y_{\sigma_n 2}) \to (Y_{\infty 1}, Y_{\infty 2}) \text{ a.s. } [P]. \tag{3.10}$$

By (3.8), (3.10) and the compactness of I^2 it follows that the sequence of probability laws of $(Y_{\tau_n 1}, Y_{\sigma_n 2})$ converges in the weak $*$-topology to μ. Thus

$$G_n \Longrightarrow G, \tag{3.11}$$

where $G, G_1, G_2, \ldots$ are the p.d.f.'s of $\mu, (Y_{\tau_1 1}, Y_{\sigma_1 2}), (Y_{\tau_2 1}, Y_{\sigma_2 2}), \ldots$ respectively.

For each choice of $\{\tau_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ for which (3.9) holds we get an inversion formula for μ. We give some examples.

Example 1. Let $\tau_n \equiv n$ and $\sigma_n \equiv n$. Then (3.9) holds and

$$P\left(Y_{n1} = \frac{k}{n}, Y_{n2} = \frac{\ell}{n}\right) = (-1)^{2n-k-\ell}\Delta_1^{n-k}\Delta_2^{n-\ell}c(k,\ell) \text{ by } (3.6),$$

$k, \ell = 0, 1, 2, \ldots, n$. Thus

$$G_n(s,t) = \sum_{\substack{k: k\leq [ns] \\ \ell: \ell \leq [nt]}} (-1)^{2n-k-\ell}\Delta_1^{n-k}\Delta_2^{n-\ell}c(k,\ell) \to G(s,t)$$

at the points of continuity of G. This inversion formula can be found in [6]; also see [4] and [5].

Example 2. Let τ_n (respectively σ_n) be the waiting time for the appearance of the nth tail for the first (respectively second) coin, i.e.,

$$\tau_n(\omega) = \inf\{m : X_{11}(\omega) + X_{21}(\omega) + \cdots + X_{m1}(\omega) = m - n\},$$
$$\sigma_n(\omega) = \inf\{m : X_{12}(\omega) + X_{22}(\omega) + \ldots + X_{m2}(\omega) = m - n\},$$

the infimum of an empty set being ∞. Then (3.9) holds and for $k, \ell = 0, 1, 2, \ldots,$

$$\begin{aligned}
P_{(p_1,p_2)}\left\{Y_{\tau_n 1} = \frac{k}{n+k}, Y_{\sigma_n 2} = \frac{\ell}{n+\ell}\right\} &= P_{(p_1,p_2)}\{\tau_n = n+k, \sigma_n = n+\ell\} \\
&= \binom{n+k-1}{k}\binom{n+\ell-1}{\ell} p_1^k p_2^\ell (1-p_1)^n (1-p_2)^n, 0 \leq p_1, p_2 < 1, \\
P_{(1,p_2)}\left\{Y_{\tau_n 1} = 1, Y_{\sigma_n 2} = \frac{\ell}{n+\ell}\right\} &= P_{(1,p_2)}\{\tau_n = \infty, \sigma_n = n+\ell\} \\
&= \binom{n+\ell-1}{\ell} p_2^\ell (1-p_2)^n, \quad 0 \leq p_2 < 1, \\
P_{(p_1,1)}\left\{Y_{\tau_n 1} = \frac{k}{n+k}, Y_{\sigma_n 2} = 1\right\} &= P_{(p_1,1)}\{\tau_n = n+k, \sigma_n = \infty\} \\
&= \binom{n+k-1}{k} p_1^k (1-p_1)^n, \quad 0 \leq p_1 < 1
\end{aligned}$$

and

$$P_{(1,1)}\{Y_{\tau_n 1} = 1, Y_{\sigma_n 2} = 1\} = P_{(1,1)}\{\tau_n = \infty, \sigma_n = \infty\} = 1.$$

The p.d.f. G_n of $(Y_{\tau_n 1}, Y_{\sigma_n 2})$ places mass $(-1)^{2n}\binom{n+k-1}{k}\binom{n+\ell-1}{\ell}\Delta_1^n\Delta_2^n c(k,\ell)$ at $\left(\frac{k}{n+k}, \frac{\ell}{n+\ell}\right)$, $(-1)^n\binom{n+\ell-1}{\ell}\Delta_2^n c(\infty,\ell)$ at $\left(1, \frac{\ell}{n+\ell}\right)$, $(-1)^n\binom{n+k-1}{k}\Delta^n c(k,\infty)$ at $\left(\frac{k}{n+k}, 1\right)$ and $c(\infty,\infty)$ at

(1,1). Here $c(\infty,\ell)$ stands for $\lim_{m\to\infty} c(m,\ell)$, $c(k,\infty)$ for $\lim_{m\to\infty} c(k,m)$ and $c(\infty,\infty) = \lim_{m\to\infty} c(m,m)$. This gives an inversion formula which is a 2-dimensional analogue of Widder's formula.

Example 3. Let τ_n (respectively σ_n) be the waiting time for the appearance of nth head or tail, whichever is earlier, for the first (respectively second) coin. Then (3.9) holds and it is easily seen that G_n places mass

$$(-1)^{k+\ell}\binom{n+k-1}{k}\binom{n+\ell-1}{\ell}\Delta_1^k\Delta_2^\ell c(n,n) \text{ at } \left(\frac{n}{n+k},\frac{n}{n+\ell}\right),$$
$$(-1)^{k+n}\binom{n+k-1}{k}\binom{n+\ell-1}{\ell}\Delta_1^k\Delta_2^n c(n,\ell) \text{ at } \left(\frac{n}{n+k},\frac{\ell}{n+\ell}\right),$$
$$(-1)^{n+\ell}\binom{n+k-1}{k}\binom{n+\ell-1}{\ell}\Delta_1^n\Delta_2^\ell c(k,n) \text{ at } \left(\frac{k}{n+k},\frac{n}{n+\ell}\right)$$

and

$$(-1)^{2n}\binom{n+k-1}{k}\binom{n+\ell-1}{\ell}\Delta_1^n\Delta_2^n c(k,\ell) \text{ at } \left(\frac{k}{n+k},\frac{\ell}{n+\ell}\right),$$
$$k,\ell = 0,1,2,\ldots,n-1.$$

The reader is invited to choose his favourite $\{\tau_n\}_{n\geq 1}$ and $\{\sigma_n\}_{n\geq 1}$ for which (3.9) holds and write down the corresponding inversion formula.

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