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Inequalities for a polynomial and its derivative

V K JAIN

Mathematics Department, Indian Institute of Technology, Kharagpur 721 302, India

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Abstract. For an arbitrary entire function f and any $r > 0$, let $M(f, r) := \max_{|z|=r} |f(z)|$. It is known that if p is a polynomial of degree n having no zeros in the open unit disc, and $m := \min_{|z|=1} |p(z)|$, then

$$M(p', 1) \leq \frac{n}{2}\{M(p, 1) - m\},$$
$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right) M(p, 1) - m\left(\frac{R^n - 1}{2}\right), \quad R > 1.$$

It is also known that if p has all its zeros in the closed unit disc, then

$$M(p', 1) \geq \frac{n}{2}\{M(p, 1) + m\}.$$

The present paper contains certain generalizations of these inequalities.

Keywords. Inequalities; zeros; polynomial.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n. Concerning the estimate of $|p'(z)|$ on the disc $|z| \leq 1$, we have the following famous result known as Bernstein's inequality [11].

Theorem A. *If $p(z)$ is a polynomial of degree n, then*

$$M(p', 1) \leq nM(p, 1), \tag{1.1}$$

with equality only for $p(z) = \alpha z^n$.

For polynomials having no zeros in $|z| < 1$, Erdös conjectured and Lax [5] proved

Theorem B. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then*

$$M(p', 1) \leq \frac{n}{2} M(p, 1), \tag{1.2}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

For polynomials having all their zeros in $|z| \leq 1$, Turan [12] proved

Theorem C. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then*

$$M(p', 1) \geq \frac{n}{2} M(p, 1), \tag{1.3}$$

with equality for those polynomials, which have all their zeros on $|z| = 1$.

On the other hand, concerning the estimate of $|p(z)|$ on the disc $|z| \le R, R > 1$, we have, as a simple consequence of maximum modulus principle [7].

Theorem D. *If $p(z)$ is a polynomial of degree n, then*

$$M(p,R) \le R^n M(p,1), \quad R > 1, \tag{1.4}$$

with equality for $p(z) = \alpha z^n$.

For polynomials not vanishing in $|z| < 1$, Ankeny and Rivlin [1] proved

Theorem E. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then*

$$M(p,R) \le \frac{R^n + 1}{2} M(p,1), \quad R > 1, \tag{1.5}$$

with equality for $p(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

In [3], we had used a parameter β and obtained the following generalizations of inequalities (1.2), (1.5) and (1.3).

Theorem F. *Let $p(z)$ be a polynomial of degree n, having no zeros in $|z| < 1$. If $M(p,1) = 1$, then for $|\beta| \le 1$*

$$\left| zp'(z) + \frac{n\beta}{2} p(z) \right| \le \frac{n}{2} \left\{ \frac{|\beta|}{2} + \left| 1 + \frac{\beta}{2} \right| \right\}, \quad |z| = 1, \tag{1.6}$$

$$\left| p(Rz) + \beta \left(\frac{R+1}{2} \right)^n p(z) \right| \le \frac{1}{2} \left\{ \left| +\beta \left(\frac{R+1}{2} \right)^n \right| + \left| R^n + \beta \left(\frac{R+1}{2} \right)^n \right| \right\}, \\ R \ge 1, \quad |z| = 1. \tag{1.7}$$

The result is best possible and equality holds in (1.6) and (1.7) for $p(z) = \alpha + \gamma z^n$, with $|\alpha| = |\gamma|$.

Theorem G. *If $p(z)$ is a polynomial of degree n, having all its zeros in the closed unit disc, then for $|\beta| \le 1$*

$$\max_{|z|=1} \left| zp'(z) + \frac{n\beta}{2} p(z) \right| \ge \frac{n}{2} \{1 + \mathrm{Re}(\beta)\} M(p,1). \tag{1.8}$$

Aziz and Dawood [2] used

$$m = \min_{|z|=1} |p(z)| \tag{1.9}$$

to obtain certain refinements of inequalities (1.2), (1.5) and (1.3) and proved

Theorem H. *If $p(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then*

$$M(p',1) \le \frac{n}{2} [M(p,1) - m], \tag{1.10}$$

$$M(p,R) \le \left(\frac{R^n + 1}{2} \right) M(p,1) - \left(\frac{R^n - 1}{2} \right) m, \quad R > 1. \tag{1.11}$$

The result is best possible and equality holds in (1.10) and (1.11) for $p(z) = \alpha z^n + \gamma$, with $|\alpha| \le |\gamma|$.

Theorem I. *If $p(z)$ is a polynomial of degree n which has all its zeros in $|z| \le 1$, then*

$$M(p', 1) \ge \frac{n}{2}\{M(p, 1) + m\}. \tag{1.12}$$

The result is best possible and equality in (1.12) *holds for $p(z) = \alpha z^n + \gamma$, $|\gamma| \le |\alpha|$.*

In this paper, we have used a parameter β, to obtain generalizations of inequalities (1.10), (1.11) and (1.12), similar to the generalizations – namely Theorems F and G, of inequalities (1.2), (1.5) and (1.3), obtained earlier by us. More precisely, we prove

Theorem 1. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for β with $|\beta| \le 1$*

$$\max_{|z|=1}\left|zp'(z)+\frac{n\beta}{2}p(z)\right| \le \frac{n}{2}\left\{\left(\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right)M(p,1) - m\left(\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right)\right\}, \tag{1.13}$$

$$\begin{aligned}\max_{|z|=1}\left|p(Rz)+\beta\left(\frac{R+1}{2}\right)^n p(z)\right| &\le \frac{1}{2}\left\{\left|R^n+\beta\left(\frac{R+1}{2}\right)^n\right|\right.\\ &\quad \left.+\left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\}M(p,1)\\ &\quad -\frac{m}{2}\left\{\left|R^n+\beta\left(\frac{R+1}{2}\right)^n\right|-\left|1+\beta\left(\frac{R+1}{2}\right)^n\right|\right\}, \quad R>1.\end{aligned} \tag{1.14}$$

Equality holds in (1.13) *and* (1.14) *for $p(z) = \lambda + \mu z^n$ with $|\lambda| \ge |\mu|$.*

Theorem 2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \le 1$, then for β with $|\beta| \le 1$*

$$\max_{|z|=1}\left|zp'(z)+\frac{n\beta}{2}p(z)\right| \ge \frac{n}{2}[\{1+\mathrm{Re}(\beta)\}M(p,1) + m|\{1+\mathrm{Re}(\beta)\} - |\beta||]. \tag{1.15}$$

Equality holds in (1.15) *for $p(z) = Ce^{i\alpha}z^n, C > 0$ and $\beta \ge 0$.*

Remark 1. Theorem 1 is a refinement of Theorem F, it can be easily seen by observing that

$$\left|1+\frac{\beta}{2}\right| \ge \left|\frac{\beta}{2}\right|, \quad |\beta| \le 1,$$

and

$$\left|R^n+\beta\left(\frac{R+1}{2}\right)^n\right| \ge \left|1+\beta\left(\frac{R+1}{2}\right)^n\right|, \quad |\beta| \le 1 \quad \text{and} \quad R > 1.$$

Remark 2. Theorem 2 is a refinement of Theorem G.

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 1. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then*

$$|p'(z)| \geq \frac{n}{2}|p(z)|, \qquad |z| = 1.$$

This lemma is due to Malik and Vong [6]. It suffices to observe that if $p(z) = c\Pi_{\nu=1}^{n}(z - z_\nu)$, then for $|z| = 1$, we have

$$R\left(\frac{zp'(z)}{p(z)}\right) = \sum_{\nu=1}^{n} R\left(\frac{z}{z - z_\nu}\right) \geq \frac{n}{2}.$$

Lemma 2. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then*

$$|p(\mathrm{Re}^{i\theta})| \geq \left(\frac{R+1}{2}\right)^n |p(\mathrm{e}^{i\theta})|, \qquad R > 1 \quad \text{and} \quad 0 \leq 0 < 2\pi.$$

This lemma is due to Jain [4]. It was observed by Rivlin [10] that if f is a polynomial of degree at most n such that $f(z) \neq 0$ in $|z| < 1$, then

$$|f(\rho\mathrm{e}^{i\theta})| \geq \left(\frac{1+\rho}{2}\right)^n |f(\mathrm{e}^{i\theta})|, \quad (0 \leq \rho < 1, 0 \leq \theta < 2\pi).$$

Applying this result to the polynomial $f(z) := z^n\overline{p(1/\bar{z})}$ with $\rho := 1/R$ we obtain the desired estimate.

Lemma 3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$, then for β with $|\beta| \leq 1$*

$$\min_{|z|=1}\left|z\rho'(z) + \frac{n\beta}{z}p(z)\right| \geq mn\left|1 + \frac{\beta}{2}\right|, \tag{2.1}$$

$$\min_{|z|=1}\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n\right| \geq m\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|, \qquad R > 1. \tag{2.2}$$

Equality holds in (2.1) *and* (2.2) *for $p(z) = m\mathrm{e}^{i\gamma} z^n, m > 0$.*

Proof of Lemma 3. If $p(z)$ has a zero on $|z| = 1$, then inequalities (2.1) and (2.2) are trivial. Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. Then $m > 0$ and for α with $|\alpha| < 1$, we have

$$|\alpha m z^n| < m \leq |p(z)|, \qquad |z| = 1, \quad (\text{by}(1.9)),$$

thereby implying by Rouché's theorem that the polynomial

$$p_1(z) = p(z) - \alpha m z^n$$

has all its zeros in $|z| < 1$. On applying Lemma 1, we get

$$|z\{p'(z) - \alpha m n z^{n-1}\}| \geq \frac{n}{2}|p(z) - \alpha m z^n|, \quad |z| = 1 \quad \text{and} \quad |\alpha| < 1.$$

Therefore for $|\beta| < 1$ and $|\alpha| < 1$, the polynomial

$$z\{p'(z) - \alpha m n z^{n-1}\} + \beta\frac{n}{2}\{p(z) - \alpha m z^n\}$$

i.e.

$$\left\{zp'(z) + \frac{n\beta}{2}p(z)\right\} - \alpha nmz^n\left\{1 + \frac{\beta}{2}\right\}$$

will have no zeros on $|z| = 1$. As $|\alpha| < 1$, we have for β with $|\beta| < 1$

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \geq \left|nmz^n\left(1 + \frac{\beta}{2}\right)\right|, \quad |z| = 1,$$

i.e.

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \geq mn\left|1 + \frac{\beta}{2}\right|, \quad |z| = 1. \tag{2.3}$$

For β with $|\beta| = 1$, (2.3) follows by continuity. And now, the inequality (2.1) follows.

On applying Lemma 2 to the polynomial $p_1(z)$, we get for $R > 1$ and $|\alpha| < 1$

$$|p(Rz) - \alpha mR^n z^n| \geq \left(\frac{R+1}{2}\right)^n |p(z) - \alpha mz^n|, \quad |z| = 1.$$

Therefore for $|\beta| < 1$ and $|\alpha| < 1$, the polynomial

$$p(Rz) - \alpha mR^n z^n + \beta\left(\frac{R+1}{2}\right)^n \{p(z) - \alpha mz^n\},$$

i.e.

$$\left\{p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right\} - \alpha mz^n\left\{R^n + \beta\left(\frac{R+1}{2}\right)^n\right\}$$

will have no zeros on $|z| = 1$. As $|\alpha| < 1$, we have for β with $|\beta| < 1$

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| \geq \left|mz^n\left\{R^n + \beta\left(\frac{R+1}{2}\right)^n\right\}\right|, \quad |z| = 1,$$

i.e.

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| \geq m\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|, \quad |z| = 1,$$

and the inequality (2.2) follows. This completes the proof of Lemma 3.

Lemma 4. *Let $Q(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq 1$ and $S(z)$ be a polynomial of degree not exceeding that of $Q(z)$. If*

$$|S(z)| \leq |Q(z)| \tag{2.4}$$

for $|z| = 1$, then for any $|\beta| \leq 1$,

$$\left|\frac{zS'(z)}{n} + \beta\frac{S(z)}{2}\right| \leq \left|\frac{zQ'(z)}{n} + \beta\frac{Q(z)}{2}\right| \tag{2.5}$$

for $|z| = 1$.

This lemma is due to Malik and Vong [6]. However, this result is contained in ([9], Theorem 3.4) where it is shown that under the conditions of lemma 4,

$$|B_n S(z)| \leq |B_n Q(z)|, \quad (|z| = 1),$$

for every B_n-operator. It may be added that a linear operator T, which carries polynomials of degree at most n into polynomials of degree at most n, is called a B_n-operator provided that $T[f]$ has all its zeros in the open unit disc if f is of exact degree n and has all its zeros in the open unit disc.

Lemma 5. *If $p(z)$ is a polynomial of degree n, with $M(p,1)=1$, then for $|\beta|\le 1$ and $|z|=1$*

$$\left|zp'(z)+\frac{n\beta}{2}p(z)\right|+\left|zq'(z)+\frac{n\beta}{2}q(z)\right|\le n\left(\left|\frac{\beta}{2}\right|+\left|1+\frac{\beta}{2}\right|\right),$$

where

$$q(z)=z^n\overline{p(1/\bar{z})}. \tag{2.6}$$

This lemma is due to Rahman ([8], inequality (5.3)).

Lemma 6. *Let $Q(z)$ be a polynomial of degree n, having all its zeros in $|z|\le 1$. If $S(z)$ is a polynomial of degree at most n such that*

$$|S(z)|\le|Q(z)|,\quad \text{for}\quad |z|=1, \tag{2.7}$$

then for β with $|\beta|\le 1$ and $R\ge 1$, we have

$$\left|S(Rz)+\beta\left(\frac{R+1}{2}\right)^n S(z)\right|\le\left|Q(Rz)+\beta\left(\frac{R+1}{2}\right)^n Q(z)\right|,\quad |z|=1. \tag{2.8}$$

This lemma is due to Jain [4].

Lemma 7. *If $p(z)$ is a polynomial of degree at most n such that $M(p,1)=1$, then for β with $|\beta|\le 1, R\ge 1$ and $|z|=1$*

$$\begin{aligned}&\left|p(Rz)+\beta\left(\frac{R+1}{2}\right)^n p(z)\right|+\left|q(Rz)+\beta\left(\frac{R+1}{2}\right)^n q(z)\right|\\ &\quad\le\left|1+\beta\left(\frac{R+1}{2}\right)^n\right|+\left|R^n+\beta\left(\frac{R+1}{2}\right)^n\right|.\end{aligned}$$

where $q(z)$ is, as in lemma 5.
This lemma is due to Jain [4].

Lemma 8. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z|\le 1$, then for β with $|\beta|\le 1$ and $|z|=1$,*

$$\left|zp'(z)+\frac{n\beta}{2}p(z)\right|\ge\frac{n}{2}\{1+\mathrm{Re}(\beta)\}|p(z)|.$$

This lemma is due to Jain ([3], Remark 2).

3. Proofs of the theorems

Proof of Theorem 1. If $p(z)$ has a zero on $|z|=1$, then Theorem 1 reduces to ([3], Theorem 1). Therefore we assume that $p(z)$ has all its zeros in $|z|>1$ (i.e. $m>0$). Now for α with $|\alpha|<1$, we have

$$|\alpha m|<m\le|p(z)|,\quad |z|=1,\quad (\text{by } (1.9)),$$

thereby implying by Rouché's theorem that the polynomial

$$p_2(z) = p(z) - \alpha m$$

has no zeros in $|z| < 1$. Therefore the polynomial

$$\begin{aligned} q_2(z) &= z^n \overline{p_2(1/\bar{z})} \\ &= q(z) - \overline{\alpha} m z^n, \quad \text{(by (2.6))} \end{aligned}$$

will have all its zeros in $|z| \le 1$. Also

$$|p_2(z)| = |q_2(z)|, \quad |z| = 1.$$

On applying Lemma 4, we get for $|z| = 1$,

$$\left| z p_2'(z) + \frac{n\beta}{2} p_2(z) \right| \le \left| z q_2'(z) + \frac{n\beta}{2} q_2(z) \right|,$$

i.e.

$$\left| \{ z p'(z) + \frac{n\beta}{2} p(z) \} - \frac{n\beta}{2} \alpha m \right| \le \left| \left\{ z q'(z) + \frac{n\beta}{2} q(z) \right\} - \overline{\alpha} m n z^n \left(1 + \frac{\beta}{2} \right) \right|, \quad |\alpha| < 1,$$

i.e.

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| - nm|\alpha| \frac{|\beta|}{2} \le \left| \left| z q'(z) + \frac{n\beta}{2} q(z) \right| - |\alpha| \, mn \left| 1 + \frac{\beta}{2} \right| \right|, \quad |\alpha| < 1. \tag{3.1}$$

The polynomial $q(z)$, given by (2.6) has all its zeros in $|z| \le 1$ and

$$\min_{|z|=1} |q(z)| = \min_{|z|=1} |p(z)| = m, \quad \text{(by (1.9))}.$$

And so, by Lemma 3 (inequality (2.1))

$$\min_{|z|=1} \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \ge mn \left| 1 + \frac{\beta}{2} \right|,$$

thereby allowing us to rewrite (3.1) as

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| - mn|\alpha| \frac{|\beta|}{2} \le \left| z q'(z) + \frac{n\beta}{2} q(z) \right| - |\alpha| mn \left| 1 + \frac{\beta}{2} \right|, \quad |z| = 1 \quad \text{and} \quad |\alpha| < 1.$$

As $|\alpha| \to 1$, we get for $|z| = 1$,

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| - \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \le -mn \left(\left| 1 + \frac{\beta}{2} \right| - \left| \frac{\beta}{2} \right| \right). \tag{3.2}$$

Now, by lemma 5, we have for $|z| = 1$,

$$\left| z p'(z) + \frac{n\beta}{2} p(z) \right| + \left| z q'(z) + \frac{n\beta}{2} q(z) \right| \le n \left(\left| 1 + \frac{\beta}{2} \right| + \left| \frac{\beta}{2} \right| \right) M(p, 1). \tag{3.3}$$

Addition of inequalities (3.2) and (3.3) easily leads to inequality (1.13)

On applying lemma 6 to the polynomials $p_2(z)$ and $q_2(z)$, we get for $R > 1$ and $|z| = 1$,

$$\left|p_2(Rz) + \beta\left(\frac{R+1}{2}\right)^n p_2(z)\right| \le \left|q_2(Rz) + \beta\left(\frac{R+1}{2}\right)^n q_2(z)\right|,$$

i.e.

$$\left|\left\{p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right\} - \alpha m\left\{1 + \beta\left(\frac{R+1}{2}\right)^n\right\}\right|$$
$$\le \left|\left\{q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right\} - \overline{\alpha} m z^n\left\{R^n + \beta\left(\frac{R+1}{2}\right)^n\right\}\right|, \quad |\alpha| < 1,$$

i.e.

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| - m|\alpha|\left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|$$
$$\le \left|\left|q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right| - |\alpha|\, m\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|\right|,$$
$$|\alpha| < 1. \qquad (3.4)$$

Further on applying lemma 3 (inequality (2.2)) to the polynomial $q(z)$, we get for $R > 1$

$$\min_{|z|=1}\left|q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right| \ge \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| \min_{|z|=1}|q(z)|,$$
$$= m\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|,$$

thereby allowing us to rewrite (3.4) as

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| - m|\alpha|\left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|$$
$$\le \left|q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right| - |\alpha| m\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right|,$$
$$|z| = 1, \quad R > 1 \quad \text{and} \quad |\alpha| < 1.$$

As $|\alpha| \to 1$, we get for $|z| = 1$ and $R > 1$,

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| - \left|q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right|$$
$$\le -m\left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| - \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\}. \qquad (3.5)$$

Now, by lemma 7, we have for $|z| = 1$ and $R > 1$,

$$\left|p(Rz) + \beta\left(\frac{R+1}{2}\right)^n p(z)\right| + \left|q(Rz) + \beta\left(\frac{R+1}{2}\right)^n q(z)\right|$$
$$\le \left\{\left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| + \left|1 + \beta\left(\frac{R+1}{2}\right)^n\right|\right\} M(p, 1). \qquad (3.6)$$

Addition of inequalities (3.5) and (3.6) easily leads to inequality (1.14). This also completes the proof of Theorem 1.

Proof of Theorem 2. If $p(z)$ has a zero on $|z| = 1$, then Theorem 2 reduces to ([3], Remark 2). Therefore we assume that $p(z)$ has all its zeros in $|z| < 1$. Now as in the proof of lemma 3, for α with $|\alpha| < 1$, the polynomial

$$p_1(z) = p(z) - \alpha m$$

will have all its zeros in $|z| < 1$. On applying lemma 8, we get for α with $|\alpha| < 1$ and $|z| = 1$,

$$\left|zp'(z) + \frac{n\beta}{2}\{p(z) - \alpha m\}\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}|p(z) - \alpha m|, \tag{3.7}$$

i.e.

$$\left|\left|zp'(z) + \frac{n\beta}{2}p(z)\right| - \left|\frac{n\beta}{2}\alpha m\right|\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}\{|p(z)| - |\alpha m|\}. \tag{3.8}$$

Further, by lemma 3 (inequality (2.1)), we have for $|z| = 1$.

$$\begin{aligned}\left|zp'(z) + \frac{n\beta}{2}p(z)\right| &\geq nm\left|1 + \frac{\beta}{2}\right|,\\ &\geq nm\frac{|\beta|}{2},\\ &\geq \left|\frac{n\beta}{2}\alpha m\right|, \quad \text{for} \quad |\alpha| < 1,\end{aligned}$$

thereby allowing us to rewrite (3.8) as

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| - \left|\frac{n\beta}{2}\alpha m\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}\{|p(z)| - |\alpha m|\},$$
$$|z| = 1 \quad \text{and} \quad |\alpha| < 1.$$

As $|\alpha| \to 1$, we get for $|z| = 1$,

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}|p(z)| + \frac{nm}{2}[|\beta| - \{1 + \mathrm{Re}(\beta)\}],$$

thereby implying

$$\max_{|z|=1}\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \geq \frac{n}{2}(\{1 + \mathrm{Re}(\beta)\}M(p, 1) + m[|\beta| - \{1 + \mathrm{Re}(\beta)\}]). \tag{3.9}$$

Again, by (3.7), we have for $|\alpha| < 1$ and $|z| = 1$,

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| + \left|\frac{n\beta}{2}\alpha m\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}\{|p(z)| + |\alpha m|\}.$$

As $|\alpha| \to 1$, we get for $|z| = 1$,

$$\left|zp'(z) + \frac{n\beta}{2}p(z)\right| \geq \frac{n}{2}\{1 + \mathrm{Re}(\beta)\}|p(z)| + \frac{nm}{2}[\{1 + \mathrm{Re}(\beta)\} - |\beta|],$$

thereby implying

$$\max_{|z|=1}\left|zp'(z)+\frac{n\beta}{2}p(z)\right| \geq \frac{n}{2}(\{1+\mathrm{Re}(\beta)\}M(p,1)+m[\{1+\mathrm{Re}(\beta)\}-|\beta|]). \tag{3.10}$$

From (3.9) and (3.10), Theorem 2 follows.

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