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On (*N*, *p*, *q*) summability factors of infinite series

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Abstract. In this paper a necessary and sufficient condition has been obtained for $\Sigma\, a_n \epsilon_n$ to be summable $|\bar{N},\ q|$ whenever $\Sigma\, a_n$ is bounded $(N,\, p,\, q)$.

Keywords. Summability factors; conservative matrix.

1. Introduction

Let $\Sigma\, a_n$ be a given infinite series with s_n for its nth partial sum. Let $\{t_n\}$ denote the sequence of (N, p, q) mean of the sequence $\{s_n\}$. The (N, p, q) transform of $s_n = \Sigma_{\nu=0}^{n} a_\nu$ is defined as follows:

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^{n} p_{n-\nu}\, q_\nu\, s_\nu, \tag{1}$$

where

$$\begin{aligned} r_n &= p_0 q_n + \cdots + p_n q_0 \ (\neq 0) \\ p_{-1} &= q_{-1} = r_{-1} = 0. \end{aligned}$$

Necessary and sufficient conditions for the (N, p, q) method to be regular, that is for $s_n \to s$ to imply $s_n \to s(N, p, q)$ are

(i) $p_{n-\nu} q_\nu / r_n \to 0$ for each integer $\nu \geq 0$ as $n \to \infty$ and
(ii) $\sum_{\nu=0}^{n} |p_{n-\nu} q_\nu| < H|r_n|$, where H is a positive number independent of n.

Let $\{T_n\}$ denote the sequence of $(\bar{N}, q)$ mean of the sequence $\{s_n\}$ defined by

$$T_n = \frac{1}{Q_n} \sum_{\nu=0}^{n} q_\nu s_\nu;\ \ (Q_n \neq 0) \tag{2}$$

where $Q_n = \sum_{\nu=0}^{n} q_\nu \to \infty$, as $n \to \infty (Q_{-i} = q_{-i} = 0, i \geq 1)$.

We define the sequence of constants $\{c_n\}$ formally by means of the identity

$$\left(\sum_{n=0}^{\infty} p_n x^n \right)^{-1} = \sum_{n=0}^{\infty} c_n x^n,\ \ c_{-i} = 0, \quad i \geq 1. \tag{3}$$

We also write $c_n^{(1)} = c_0 + c_1 + \cdots + c_n$.

We denote by $\mathcal{M}$, the class of sequences $\{p_n\}$ for which the following holds:

$$p_n > 0, \quad \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1\ (n = 0, 1, \ldots). \tag{4}$$

Let $\{p_n\}$ and $\{q_n\}$ be positive sequences. A series $\Sigma\, a_n$ is said to be bounded $(N,\, p,\, q)$ or $\Sigma\, a_n = O(1)(N,\, p,\, q)$ if

$$\sum_{\nu=1}^{n} p_{n-\nu} q_\nu s_\nu = O(r_n), \quad \text{as } n \to \infty. \tag{5}$$

If X and Y are any two methods of summability, we say (ϵ_n) belongs to the class $[X, Y]$, if $\Sigma\, a_n \epsilon_n$ is summable – Y whenever $\Sigma\, a_n$ is summable X.

Recently Mishra [7], Sarigol and Bor [8] and Sulaiman [9] have obtained summability factor theorems of the type $[|\bar{N},\, p_n|_k, |\bar{N},\, q_n|_k]$, $[|\bar{N},\, p_n|,\, |\bar{N},\, q_n|_k]$, $[|\bar{N},\, p_n|_k, |\bar{N},\, q_n|]$.

In 1966 Das [2] has proved the following theorem:

Theorem A. *Let $\{p_n\} \in \mathcal{M},\; q_n \geq 0$. Then if $\Sigma\, a_n$ is summable $|N,\, p,\, q|$, it is summable $|\bar{N},\, q|$.*

It is therefore, natural to find a summability factor ϵ_n so that $\Sigma\, a_n \epsilon_n$ is summable $|\bar{N},\; q|$ whenever $\Sigma\, a_n$ is bounded $(N,\, p,\, q)$.

Mazur and Orlicz [5] stated that, if a conservative (i.e. convergence preserving) matrix sums a bounded nonconvergent sequence, then it must sum an unbounded sequence. Zeller [10] obtained a proof of this theorem as a consequence of his study of the summability of slowly oscillating sequences whereas the proof of Mazur and Orlicz [6] was functional analytic, based on rather deep topological properties of FK-spaces. A simple direct proof of this theorem was also given by Fridy [3] which used only the well known Silverman–Toeplitz conditions for regularity.

In view of this remark we state and prove the following summability factor theorem.

Theorem 1. *Let $\{p_n\} \in \mathcal{M},\, q_0 > 0,\; q_n \geq 0$ and let $\{q_n\}$ be monotonic non-increasing sequence for $n \geq 0$. The necessary and sufficient condition that $\Sigma\, a_n \epsilon_n$ should be summable $|\bar{N},\, q|$, whenever*

$$\sum a_n = O(1)(N, p, q), \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\epsilon_n| < \infty, \tag{7}$$

$$\sum_{n=0}^{\infty} |\Delta \epsilon_n| < \infty, \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \epsilon_n| < \infty, \tag{9}$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n|\, |\epsilon_n| < \infty. \tag{10}$$

Our theorem generalizes and unifies several known results of Mazhar [4], Daniel [1] and others.

2. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. *Let* $\{q_n\}$ *be positive and monotonic non-increasing sequence. If* $\{\epsilon_n\}$ *is such that*

(i) $\Delta\epsilon_n = o(1)$, *as* $n \to \infty$, *and*

(ii) $\sum \frac{Q_n}{q_{n+1}} |\Delta^2 \epsilon_n| < \infty$,

then

$$\sum \frac{Q_n \Delta q_n}{q_n q_{n+1}} |\Delta \epsilon_n| < \infty.$$

Remark. This lemma holds only if $\{q_n\}$ is monotonic non-increasing.

If $\{q_n\}$ is not monotonic non-increasing then conclusion of the lemma may not be true.

Lemma 2 [2]. *Let* $\{p_n\} \in \mathcal{M}$. *Then*

(iii) $c_0 > 0,\ c_n \leq 0\ (n = 1, 2, \ldots)$,

(iv) $\sum_{n=0}^{\infty} c_n x^n$ *is absolutely convergent for* $|x| \leq 1$,

(v) $\sum_{n=0}^{\infty} c_n > 0$,

except when $\sum_{n=0}^{\infty} p_n = \infty$, *in which case*

(vi) $\sum_{n=0}^{\infty} c_n = 0$.

Lemma 3 [2]. *If*

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^{n} p_{n-\nu} q_\nu s_\nu.$$

then

$$s_n = \frac{1}{q_n} \sum_{\nu=0}^{n} c_{n-\nu} r_\nu t_\nu.$$

Lemma 4 [2].

$$\sum_{\mu=0}^{n} c_{n-\mu}^{(1)} r_\mu = Q_n,$$

where c_n, r_n *and* Q_n *are defined as above.*

3. Proof of theorem 1

Let

$$t_n = \frac{1}{r_n} \sum_{\nu=0}^{n} p_{n-\nu} q_\nu s_\nu,$$

and

$$T_n = \frac{1}{Q_n}\sum_{\nu=0}^{n} q_\nu \sum_{r=0}^{\nu} a_r\epsilon_r$$
$$= \frac{1}{Q_n}\sum_{\nu=0}^{n}(Q_n - Q_{\nu-1})a_\nu\epsilon_\nu.$$

Then for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n} Q_{\nu-1}a_\nu\epsilon_\nu.$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{q_n}{Q_nQ_{n-1}}\left[\sum_{\nu=0}^{n-1} s_\nu\Delta(Q_{\nu-1}\epsilon_\nu) + s_n\epsilon_nQ_{n-1}\right]$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} q_\nu s_\nu\epsilon_\nu + \frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} Q_\nu s_\nu\Delta\epsilon_\nu + \frac{q_n}{Q_n}s_n\epsilon_n.$$

Let

$$T_n - T_{n-1} = \sum_1 + \sum_2 + \sum_3, \quad \text{say.} \tag{11}$$

The theorem is proved if we show that $\Sigma\,|\Sigma_1|$ and $\Sigma\,|\Sigma_2|$ are convergent.

Now

$$\sum_1 = -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} q_\nu\epsilon_\nu s_\nu$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} q_\nu\epsilon_\nu \frac{1}{q_\nu}\sum_{\mu=0}^{\nu} c_{\nu-\mu}r_\mu t_\mu$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} \epsilon_\nu \sum_{\mu=0}^{\nu} c_{\nu-\mu}r_\mu t_\mu$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}\epsilon_\nu$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \left[\sum_{\nu=\mu}^{n-1}\left(\sum_{k=0}^{\nu} c_{k-\mu}\right)\Delta\epsilon_\nu + \epsilon_n\sum_{k=0}^{n-1} c_{k-\mu} - \sum_{k=0}^{\mu-1} c_{k-\mu}\epsilon_\mu\right]$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \left[\sum_{\nu=\mu}^{n-1} c^{(1)}_{\nu-\mu}\Delta\epsilon_\nu + \epsilon_n c^{(1)}_{n-1-\mu}\right]$$
$$= -\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c^{(1)}_{\nu-\mu}\Delta\epsilon_\nu - \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} \epsilon_n c^{(1)}_{n-1-\mu}r_\mu t_\mu$$
$$= \sum_{11} + \sum_{12}, \quad \text{say.}$$

Then as $t_n = O(1)$, using lemmas 2, 3 and 4,

$$\sum_{n=1}^{\infty}\left|\sum_{11}\right| \leq \sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu|t_\mu|\sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}|\Delta\epsilon_\nu|, \text{ since } c_{\nu-\mu}^{(1)} > 0 \text{ by Lemma 2.}$$

$$\leq K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}|\Delta\epsilon_\nu|\sum_{\mu=0}^{\nu} c_{\nu-\mu}^{(1)} r_\mu$$

$$= K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}|\Delta\epsilon_\nu|Q_\nu$$

$$= K\sum_{\nu=0}^{\infty}|\Delta\epsilon_\nu|Q_\nu\sum_{n=\nu+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}$$

$$= K\sum_{\nu=0}^{\infty}|\Delta\epsilon_\nu| < \infty.$$

Similarly,

$$\sum_{n=1}^{\infty}\left|\sum_{12}\right| \leq K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}|\epsilon_n|\sum_{\mu=0}^{n-1} c_{n-1-\mu}^{(1)} r_\mu$$

$$= K\sum_{n=1}^{\infty}\frac{q_n}{Q_n}|\epsilon_n| < \infty.$$

Hence

$$\sum_{n=1}^{\infty}\left|\sum_{1}\right| < \infty.$$

Again,

$$\sum_{2} = \frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1} Q_\nu\Delta\epsilon_\nu\frac{1}{q_\nu}\sum_{\mu=0}^{\nu} c_{\nu-\mu}r_\mu t_\mu$$

$$= \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu\sum_{\nu=\mu}^{n-1}\frac{Q_\nu}{q_\nu}\Delta\epsilon_\nu c_{\nu-\mu} = \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu L_1, \quad \text{say,}$$

where

$$L_1 = \sum_{\nu=\mu}^{n-1}\frac{Q_\nu}{q_\nu}\Delta\epsilon_\nu c_{\nu-\mu}$$

$$= \sum_{\nu=\mu}^{n-1}\left(\sum_{k=0}^{\nu} c_{k-\mu}\right)\Delta\left(\frac{Q_\nu}{q_\nu}\Delta\epsilon_\nu\right) + \frac{Q_n}{q_n}\Delta\epsilon_n\sum_{k=0}^{n-1} c_{k-\mu}$$

$$= \sum_{\nu=\mu}^{n-1}\left(\sum_{k=\mu}^{\nu} c_{k-\mu}\right)\left[\Delta\left(\frac{Q_\nu}{q_\nu}\right)\Delta\epsilon_\nu + \frac{Q_{\nu+1}}{q_{\nu+1}}\Delta^2\epsilon_\nu\right] + \frac{Q_n}{q_n}\Delta\epsilon_n\sum_{k=\mu}^{n-1} c_{k-\mu}$$

$$= \sum_{\nu=\mu}^{n-1}\left(\sum_{k=\mu}^{\nu} c_{k-\mu}\right)\left[-\frac{q_{\nu+1}}{q_\nu}\Delta\epsilon_\nu + Q_{\nu+1}\Delta\left(\frac{1}{q_\nu}\right)\Delta\epsilon_\nu + \frac{Q_{\nu+1}}{q_{\nu+1}}\Delta^2\epsilon_\nu\right] + \frac{Q_n}{q_n}\Delta\epsilon_n\sum_{k=\mu}^{n-1} c_{k-\mu}$$

$$= \sum_{\nu=\mu}^{n-1}\left[-c_{\nu-\mu}^{(1)}\frac{q_{\nu+1}}{q_\nu}\Delta\epsilon_\nu + c_{\nu-\mu}^{(1)}Q_{\nu+1}\Delta\left(\frac{1}{q_\nu}\right)\Delta\epsilon_\nu + c_{\nu-\mu}^{(1)} \times \frac{Q_{\nu+1}}{q_{\nu+1}}\Delta^2\epsilon_\nu\right] + \frac{Q_n}{q_n}\Delta\epsilon_n c_{n-1-\mu}^{(1)}.$$

So

$$\begin{aligned}
\sum_2 =& -\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}\frac{q_{\nu+1}}{q_\nu}\Delta\epsilon_\nu \\
&+ \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}Q_{\nu+1}\Delta\left(\frac{1}{q_\nu}\right)\Delta\epsilon_\nu \\
&+ \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}\frac{Q_{\nu+1}}{q_{\nu+1}}\Delta^2\epsilon_\nu + \frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu t_\mu c_{n-1-\mu}^{(1)}\frac{Q_n}{q_n}\Delta\epsilon_n \\
=& \sum_{21} + \sum_{22} + \sum_{23} + \sum_{24}, \quad \text{say.}
\end{aligned}$$

Therefore to show that

$$\sum_{n=1}^{\infty}\left|\sum_2\right| < \infty$$

it is enough to show that

$$\sum_{n=1}^{\infty}\left|\sum_{2i}\right| < \infty, \quad i = 1,2,3,4.$$

Now as $c_k^{(1)} > 0$ for $k > 0$ and as $t_n = O(1)$

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|\sum_{21}\right| &\le \sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu |t_\mu| \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}\frac{q_{\nu+1}}{q_\nu}|\Delta\epsilon_\nu| \\
&\le K\left[\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1} r_\mu \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}^{(1)}\frac{q_{\nu+1}}{q_\nu}|\Delta\epsilon_\nu|\right] \\
&= K\left[\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}\frac{q_{\nu+1}}{q_\nu}|\Delta\epsilon_\nu|\sum_{\mu=0}^{\nu} r_\mu c_{\nu-\mu}^{(1)}\right] \\
&= K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}\frac{q_{\nu+1}}{q_\nu}|\Delta\epsilon_\nu| Q_\nu \\
&= K\sum_{\nu=0}^{\infty}\frac{q_{\nu+1}}{q_\nu}Q_\nu|\Delta\epsilon_\nu|\sum_{n=\nu+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}} \\
&= K\sum_{\nu=0}^{\infty}\frac{q_{\nu+1}}{q_\nu}Q_\nu|\Delta\epsilon_\nu|\frac{1}{Q_\nu} = K\sum_{\nu=0}^{\infty}\frac{q_{\nu+1}}{q_\nu}|\Delta\epsilon_\nu| \\
&\le K\sum_{\nu=0}^{\infty}|\Delta\epsilon_\nu| < \infty, \quad \text{as } \{q_n\} \text{ is non-increasing.}
\end{aligned}$$

Note that K is a positive constant not necessarily same at each occurance.

Similarly

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left|\sum_{22}\right| &\leq \sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1}r_\mu|t_\mu|\sum_{\nu=\mu}^{n-1}c_{\nu-\mu}^{(1)}\frac{Q_{\nu+1}\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu| \\
&\leq K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1}r_\mu\sum_{\nu=\mu}^{n-1}c_{\nu-\mu}^{(1)}\frac{Q_{\nu+1}\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu| \\
&= K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}\frac{Q_{\nu+1}\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu|\sum_{\mu=0}^{\nu}r_\mu c_{\nu-\mu}^{(1)} \\
&= K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}\frac{Q_{\nu+1}Q_\nu\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu| \\
&= K\sum_{\nu=0}^{\infty}\frac{Q_{\nu+1}Q_\nu\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu|\sum_{n=\nu+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}} \\
&= K\sum_{\nu=0}^{\infty}\frac{Q_{\nu+1}\Delta q_\nu}{q_\nu q_{\nu+1}}|\Delta\epsilon_\nu| < \infty,\ \text{by Lemma 1,}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left|\sum_{23}\right| &\leq K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\mu=0}^{n-1}r_\mu\sum_{\nu=\mu}^{n-1}c_{\nu-\mu}^{(1)}\frac{Q_{\nu+1}}{q_{\nu+1}}|\Delta^2\epsilon_\nu| \\
&= K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\sum_{\nu=0}^{n-1}\frac{Q_{\nu+1}}{q_{\nu+1}}|\Delta^2\epsilon_\nu|\sum_{\mu=0}^{\nu}r_\mu c_{\nu-\mu}^{(1)} \\
&= K\sum_{\nu=0}^{\infty}\frac{Q_{\nu+1}Q_\nu}{q_{\nu+1}}|\Delta^2\epsilon_\nu|\times\frac{1}{Q_\nu} \\
&= K\sum_{\nu=0}^{\infty}\frac{Q_{\nu+1}}{q_{\nu+1}}|\Delta^2\epsilon_\nu| < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left|\sum_{24}\right| &\leq K\sum_{n=1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}|\Delta\epsilon_n|\frac{Q_n}{q_n}\sum_{\mu=0}^{n-1}r_\mu c_{n-1-\mu}^{(1)} \\
&= K\sum_{n=1}^{\infty}\frac{1}{Q_{n-1}}|\Delta\epsilon_n|Q_{n-1} \\
&= K\sum_{n=1}^{\infty}|\Delta\epsilon_n| < \infty.
\end{aligned}
$$

Hence

$$
\sum_{n=1}^{\infty}\left|\sum_{2}\right| < \infty
$$

and the proof of the theorem is completed.

Theorem 2. *Let* $\{p_n\}\in\mathcal{M}, q_0>0, q_n\geq 0$ *and let* $\{q_n\}$ *be monotonic non-increasing sequence for* $n\geq 0$. *The necessary and sufficient condition that* $\Sigma\, a_n\epsilon_n$ *should be* *summable $|\bar{N}, q|$ whenever*

$$\sum a_n = O(\mu_n)(N, p, q), \tag{12}$$

where $\{\mu_n\}$ is positive and monotonic non-decreasing and $\{\epsilon_n\}$ is such that

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n} |\epsilon_n| \mu_n < \infty, \tag{13}$$

$$\sum_{n=0}^{\infty} |\Delta \epsilon_n| \mu_n < \infty, \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{Q_{n+1}}{q_{n+1}} |\Delta^2 \epsilon_n| \mu_n < \infty, \tag{15}$$

is that

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n} |s_n| \, |\epsilon_n| < \infty. \tag{16}$$

The proof of theorem 2 is similar to that of theorem 1.

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