Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, No. 1, February 2000, pp. 35–53.


# Connections for small vertex models

R SRINIVASAN

Mehta Research Institute of Mathematics and Mathematical Physics,
Allahabad 211 019, India

MS received 19 May 1999

**Abstract.** This paper is a first attempt at classifying connections on small vertex models i.e., commuting squares of the form displayed in (1.2) below. More precisely, if we let $B(k,n)$ denote the collection of matrices $W$ for which (1.2) is a commuting square then, we: (i) obtain a simple model form for a representative from each equivalence class in $B(2,n)$, (ii) obtain necessary conditions for two such 'model connections' in $B(2,n)$ to be themselves equivalent, (iii) show that $B(2,n)$ contains a $(3n-6)$-parameter family of pairwise inequivalent connections, and (iv) show that the number $(3n-6)$ is sharp. Finally, we deduce that every graph that can arise as the principal graph of a finite depth subfactor of index 4 actually arises for one arising from a vertex model corresponding to $B(2,n)$ for some $n$.

**Keywords.** Subfactor; vertex model; biunitary.

## 1. Introduction

We first recall certain facts about commuting squares and biunitaries. These facts can be found in [USC] or [JS].

1.1 Consider the following commuting square:

$$\begin{array}{ccc} A_0^1 & \overset{L}{\subset} & A_1^1 \\ K\cup & & \cup H \\ A_0^0 & \overset{G}{\subset} & A_1^0 \end{array} . \tag{1.1}$$

Then the following are equivalent. (i) $G = L = [n]$ and $H = K = [k]$; (ii) the square (1.1) is isomorphic to a commuting square of the form

$$\begin{array}{ccc} W(1\otimes M_k(\mathbb{C}))W^* & \subset & M_n(\mathbb{C})\otimes M_k(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & M_n(\mathbb{C})\otimes 1 \end{array} , \tag{1.2}$$

where $W = ((W^{\alpha a}_{\beta b})) \in M_n(\mathbb{C})\otimes M_k(\mathbb{C})$ is unitary. (We use the convention that $1\le \alpha$, $\beta \le n$, $1 \le a$, $b \le k$.)

1.2 If $W = ((W^{\alpha a}_{\beta b})) \in M_n(\mathbb{C})\otimes M_k(\mathbb{C})$, then the square (1.2) is a commuting square iff $W$ is biunitary i.e., both $W$ and $\tilde{W}$ given by $\tilde{W}^{\alpha a}_{\beta b} = W^{\beta a}_{\alpha b}$ are unitary.

We shall use the symbol $B(k,n)$ to denote the set of such biunitary matrices.

1.3 Two biunitary matrices $W$ and $W'$ are said to be equivalent if the corresponding commuting squares are isomorphic. It is true that if $W, W' \in B(k,n)$, then $W$ and $W'$ are equivalent if and only if there exists unitary matrices $U, U' \in M_n$, $A, A' \in M_k$ such that $(U \otimes A)W = W'(U' \otimes A')$.

1.4 Given $W \in B(k,n)$, the basic construction yields a grid of commuting squares and consequently, a *horizontal* (respectively *vertical*) subfactor $A_\infty^0 \subset A_\infty^1$ (respectively $A_0^\infty \subset A_1^\infty$) with index $k^2$ (respectively $n^2$). This construction is canonical, and so, isomorphic commuting squares (i.e., equivalent biunitary matrices) yield isomorphic horizontal (respectively vertical) subfactors.

1.5 When $n = k = 2$, any $W \in B(2,2)$ is equivalent to a biunitary matrix of the form

$$W(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix},$$

where $\omega \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Further neither the vertical nor the horizontal subfactor is irreducible. It is in fact true, although not mentioned in [USC], that $W(\omega)$ is equivalent to $W(\omega')$ if and only if $\mathrm{Re}(\omega) = \mathrm{Re}(\omega')$.

## 2. A model form for a matrix in $B(2, n)$

In this section we prove that every biunitary matrix in $B(2,n)$ is equivalent to a biunitary matrix in a model form with $(3n-5)$ independent parameters.

PROPOSITION 2.1

*Any biunitary matrix $W \in B(2,n)$ is equivalent to a matrix of the form*

$$\begin{pmatrix} C & US \\ VS & -UVC \end{pmatrix},$$

*where $U, V$ are diagonal unitary matrices and $C, S$ are positive diagonal matrices such that $C^2 + S^2 = 1$.*

*Proof.* Let $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that $\tilde{W} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ where $a, b, c, d \in M_n$. Then $W$ is a biunitary matrix if and only if both $W$ and $\tilde{W}$ are unitary matrices. The unitarity of $W$ and $\tilde{W}$ (i.e. the relation $WW^* = 1 = \tilde{W}\tilde{W}^*$) implies the following equations:

$$\begin{aligned} aa^* + bb^* &= 1, \qquad & cc^* + dd^* &= 1 \\ a^*a + c^*c &= 1, \qquad & b^*b + d^*d &= 1 \\ aa^* + cc^* &= 1, \qquad & bb^* + dd^* &= 1 \\ a^*a + b^*b &= 1, \qquad & c^*c + d^*d &= 1. \end{aligned}$$

By premultiplying by $u \otimes 1$, where $u \in M_n$ is a suitable unitary matrix (i.e. by working with an equivalent biunitary matrix), we may assume, without loss of generality, that $a$ is positive. Then it follows from the above equations that $0 \leq a \leq 1$. Let $C = a$. So there exist a unique positive matrix $S \in M_n$ such that, $0 \leq S \leq 1$ and $C^2 + S^2 = 1$. Then from the above equations we can conclude that $b, c, d$ are normal and also that $bb^* = cc^* = S^2$ and $dd^* = C^2$. So there exist unitary matrices $U, V, T \in M_n$ such that $b = US, c = VS$ and $d = TC$, and such that $U$ and $V$ commute with $S$ (hence, also with $C$) and $T$ commutes with $C$ (hence, also with $S$).

So we find that $W$ is equivalent to the biunitary matrix

$$\begin{pmatrix} C & US \\ VS & TC \end{pmatrix},$$

where $C, S, U, V$ and $T$ are as above.

The biunitarity of $W$ (i.e. the relation $WW^* = \tilde{W}\tilde{W}^* = 1$) also implies the following equations:

$$SC(V + TU^*) = 0, \qquad SC(U + V^*T) = 0,$$
$$SC(U + TV^*) = 0, \qquad SC(V + U^*T) = 0. \tag{2.3}$$

Since $U, V$ and $T$ leave the eigenspaces $\{H_i\}_{i\in I}$ of $C$ invariant, we may, by conjugating $W$ by a unitary matrix of the form $\Gamma \otimes 1$ (where $\Gamma$ is a unitary matrix which diagonalises $C$), assume that

$$C = \oplus_{i\in I} c_i 1_{H_i}, \quad S = \oplus_{i\in I} s_i 1_{H_i}$$
$$U = \oplus_{i\in I} U_i, \quad V = \oplus_{i\in I} V_i, \quad T = \oplus_{i\in I} T_i,$$

where $1_{H_i}$ denotes the identity in $\mathcal{L}(H_i), 0 \leq c_i, s_i \leq 1$ and $U_i, V_i, T_i \in \mathcal{L}(H_i)$.

Thus we see that $W = \oplus_{i\in I} W_i$, where $W_i$ is a biunitary matrix in $M_{n_i} \otimes M_2$ and $n_i$ is the dimension of $H_i$, and that

$$W_i = \begin{pmatrix} c_i I_{n_i} & U_i s_i \\ s_i V_i & c_i T_i \end{pmatrix}.$$

Note that in order to complete the proof of this proposition it is enough if we prove that each of this $W_i$ is equivalent to a biunitary matrix of the form presented in the proposition by pre- and post-multiplying by unitary matrices of the form $u_i \otimes 1$ (then by pre- and post-multiplying $W$ by matrices of the form $(\oplus_{i\in I} u_i) \otimes 1$ one may prove that $W$ is equivalent to a matrix of the required form). To prove this we now consider two cases depending on whether $c_i$ is zero or non-zero.

Suppose $c_i = 0$, then, by pre-multiplying $W_i$ by the unitary matrix $U_i^* \otimes 1$, we may assume that $U_i = 1_{H_i}$. Now, by conjugating $W_i$ by a unitary matrix $p \otimes 1$, where $p$ is a unitary matrix which diagonalises $V_i$, we can conclude that $W_i$ is equivalent to a matrix of the desired form.

Suppose that $c_i \neq 0$. If $s_i = 0$ we can assume, by conjugating by a unitary matrix $p \otimes 1$ (where $p$ is a unitary matrix which diagonalises $T_i$), that the matrix $W_i$ is in the required form. If $s_i$ is also non-zero, then first conclude from the set of equations (2.3) that $T_i = -U_i V_i = -V_i U_i$. Now, by conjugating $W_i$ by a unitary matrix of the form $p \otimes 1$, where $p$ is a unitary matrix which simultaneously diagonalises the commuting unitaries $U_i, V_i, T_i$, we may conclude that $W_i$ is equivalent to a matrix of the desired form.

Hence, in any case, we find that $W$ is equivalent to a matrix of the form

$$\begin{pmatrix} C & US \\ VS & -UVC \end{pmatrix},$$

where $U, V$ are diagonal unitary matrices and $C, S$ are positive diagonal matrices such that $C^2 + S^2 = 1$. □

It is proved in [USC] that when $n = k = 2$, any $W \in B(2, 2)$ is equivalent to a biunitary matrix of the form

$$W(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix},$$

where $\omega \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Further neither the vertical nor the horizontal subfactor is irreducible. We explicitly point out the ambiguity in such a representation.

PROPOSITION 2.2

*$W(\omega)$ is equivalent to $W(\omega')$ if and only if* $\mathrm{Re}(\omega) = \mathrm{Re}(\omega')$.

*Proof.* Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then it can be easily verified that $(U \otimes A)W(\omega) = W(\bar{\omega})(U \otimes A')$.

Conversely suppose

$$(U \otimes A)\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D' \end{pmatrix}(U' \otimes A'),$$

where $D = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, D' = \begin{pmatrix} 1 & 0 \\ 0 & \omega' \end{pmatrix}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then the following equations hold:

$$aU = a'U', \tag{2.4}$$
$$bUD = b'U', \tag{2.5}$$
$$cU = c'D'U', \tag{2.6}$$
$$dUD = d'D'U'. \tag{2.7}$$

Suppose $a \neq 0$. Then $a' \neq 0$ by eq. (2.4), and so also $d' \neq 0$ (as $D, D', U, U'$ are unitary matrices). From eqs (2.4) and (2.7), we see that $d'^{-1}da^{-1}a'U'DU'^{*} = D'$. Now by comparing the eigenvalues (as $U'$ is an unitary matrix), we conclude that $\{d'^{-1}da^{-1}a', d'^{-1}da^{-1}a'\omega\} = \{1, \omega'\}$. Hence either $\omega = \omega'$ or $\omega = \bar{\omega}'$. Suppose that $a = 0$, then as $A$ is an unitary matrix, it is the case that $b \neq 0$. Exactly in a similar way, using eqs (2.5) and (2.6), we can again conclude that either $\omega = \omega'$ or $\omega = \bar{\omega}'$. □

PROPOSITION 2.3

*Any $W \in B(2, n)$ is equivalent to a biunitary matrix of the form*

$$W(\omega, \theta, \phi, C) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & \theta S \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & \phi S & 0 & 0 & -\theta\phi C \end{pmatrix},$$

*where* $\theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_{n-2})$, $\phi = \mathrm{diag}(\phi_1, \phi_2, \ldots, \phi_{n-2})$, $C = \mathrm{diag}(C_1, C_2, \ldots, C_{n-2})$, $S = \mathrm{diag}(S_1, S_2, \ldots, S_{n-2})$, $\theta_i, \phi_i, \omega \in \{z \in \mathbb{C} : |z| = 1\}$, $\mathrm{Im}(\omega) \geq 0, 0 \leq C_i, S_i \leq 1$, *and* $C_i^2 + S_i^2 = 1$.

*Proof.* From Proposition 2.1, we may assume that $W = \sum_{i=1}^{n} E_{ii} \otimes W_i$ (where $W_i$ is a $2 \times 2$ unitary matrix and $\{E_{ij} : 1 \leq i, j \leq n\}$ denotes – here and elsewhere – the usual system of matrix units in $M_n$), and that $W_i$ has the form

$$W_i = \begin{pmatrix} C_i & \theta_i S_i \\ \phi_i S_i & -\theta_i \phi_i C_i \end{pmatrix},$$

where $\theta_i, \phi_i$ are complex numbers of unit modulus, and $0 \leq C_i, S_i \leq 1$ and $C_i^2 + S_i^2 = 1$.

Note next that if $D = \mathrm{diag}(d_1, \ldots, d_n) \in M_n$ is a diagonal unitary matrix, and if $W, W_i$ are as above, and if $V_1, V_2 \in M_2$ are unitary, then

$$(D \otimes V_1) W (1 \otimes V_2) = \sum_{i=1}^{n} d_i (E_{ii} \otimes V_1 W_i V_2). \tag{$*$}$$

Set $V_1 = 1, V_2 = W_1^*$; if the $(1,1)$ entry of $W_i W_1^*$ is $\omega_i \tilde{C}_i$, with $\tilde{C}_i \geq 0$ and $|\omega_i| = 1$, define $d_i = \bar{\omega}_i$. We may now deduce from equation $(*)$ that we may reduce to the case where $W_1$ is the identity matrix, and $W_i$ are as above.

Next, let $U$ be the unitary matrix which diagonalises (the new) $W_2$. Then, by setting $d_i = \overline{\omega_i}$ if $\omega_i \tilde{C}_i'$ is the $(1,1)$ entry of $U^* W_i U$, with $\tilde{C}_i' \geq 0$ and $|\omega_i| = 1$, and by setting $U = V_1^* = V_2$, we find that we may reduce to the case where $W$ is as above, and in addition, $W_1 = 1$ and $W_2 = \mathrm{diag}(1, \omega)$, where $\omega$ is a complex number of unit modulus.

If $\mathrm{Im}(\omega) \geq 0$, the proof of the Proposition is complete. If $\mathrm{Im}(\omega) < 0$, then set

$$V_1 = V_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $d_1 = 1$, $d_2 = \bar{\omega}, d_i = -\overline{\theta_i \phi_i}\ \forall i = 3, \ldots, n$, to conclude that $W$ is indeed equivalent to a biunitary matrix of the prescribed form. □

## 3. Classification of $B(2, n)$

We shall use the notation $\Omega(2,n) = \mathbb{T}^+ \times \mathbb{T}^{n-2} \times \mathbb{T}^{n-2} \times [0,1]^{n-2}$, where $\mathbb{T}$ is the unit circle in the complex plane and $\mathbb{T}^+ = \{\omega \in \mathbb{T} : \mathrm{Im}(\omega) \geq 0\}$; we shall denote a typical pair of points of $\Omega(2,n)$ by $P = (\omega, \theta, \phi, C)$ and $P' = (\omega', \theta', \phi', C')$ and the corresponding biunitary matrices by $W$ and $W'$. Also we shall denote the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \omega' \end{pmatrix}$ by $D$ and $D'$ respectively.

We isolate a simple assertion as a lemma, since we will need to repeatedly use it.

*Lemma* 3.1. *Suppose $a, b, c, d$ are non-zero complex numbers, and suppose $\theta, \phi, \omega_j$, $j = 0, 1, 2$ are complex numbers of unit modulus, and suppose $C$ and $S$ are non-negative real numbers satisfying $C^2 + S^2 = 1$. Assume that $\mathrm{Im}(\omega_2) > 0$ and that the following equations are satisfied:*

$$a(C - \omega_0) + b\phi S = 0, \tag{3.8}$$

$$a\theta S - b(\theta\phi C + \omega_0\omega_1) = 0, \tag{3.9}$$

$$c(C - \omega_0\omega_2) + d\phi S = 0, \tag{3.10}$$

$$c\theta S - d(\theta\phi C + \omega_0\omega_1\omega_2) = 0. \tag{3.11}$$

*Then,* $S \neq 0, C \neq 1$ *and*

$$\mathrm{Re}(\omega_2) = -[S^2 + \mathrm{Re}(\bar{\omega}_1\theta\phi)C^2]. \tag{3.12}$$

*Proof.* If $S = 0$, then the eqs (3.8) and (3.10) would imply that $\omega_0 = \omega_0\omega_2 = 1$. Hence, as we have assumed that $\omega_2 \neq 1$, conclude that $S \neq 0$. Now, deduce from (3.8) and (3.9) that

$$(\omega_0 - C)(\theta\phi C + \omega_0\omega_1) = \theta\phi S^2; \tag{3.13}$$

similarly, deduce from eqs (3.10) and (3.11) that

$$(\omega_0\omega_2 - C)(\theta\phi C + \omega_0\omega_1\omega_2) = \theta\phi S^2.$$

These equations may be re-written as

$$\begin{aligned}
&\omega_0^2\omega_1 + \omega_0(\theta\phi - \omega_1)C = \theta\phi S^2,\\
&\omega_0^2\omega_2^2\omega_1 + \omega_0\omega_2(\theta\phi - \omega_1)C = \theta\phi S^2,
\end{aligned}$$

from which we may deduce that

$$\omega_0^2(1 - \omega_2^2)\omega_1 + \omega_0(1 - \omega_2)(\theta\phi - \omega_1)C = 0. \tag{3.14}$$

As we have assumed that $\mathrm{Im}(\omega_2) > 0$, we may infer from eq. (3.14) that

$$\omega_0 = \frac{(\omega_1 - \theta\phi)C}{\omega_1(1 + \omega_2)}.$$

Substituting this expression for $\omega_0$ into eq. (3.13), we find that

$$\begin{aligned}
(\omega_0 - C)(\theta\phi C + \omega_0\omega_1) &= \frac{C}{\omega_1(1 + \omega_2)}(\omega_1 - \theta\phi - \omega_1(1 + \omega_2))\\
&\quad\times \frac{C}{\omega_1(1 + \omega_2)}(\theta\phi\omega_1(1 + \omega_2) + \omega_1(\omega_1 - \theta\phi))\\
&= -\frac{C^2}{(1 + \omega_2)^2}(\overline{\omega}_1\theta\phi + \omega_2)(\omega_1 + \theta\phi\omega_2);
\end{aligned}$$

and hence,

$$\theta\phi S^2 - (\omega_0 - C)(\theta\phi C + \omega_0\omega_1) = \theta\phi\left[S^2 + \frac{C^2}{(1 + \omega_2)^2}(\overline{\omega}_1\theta\phi + \omega_2)(\omega_1\overline{\theta\phi} + \omega_2)\right].$$

Thus we find that the equation $(\omega_0 - C)(\theta\phi C + \omega_0\omega_1) = \theta\phi S^2$ will be satisfied precisely when

$$\begin{aligned}
0 &= (1 + \omega_2)^2 S^2 + C^2(\overline{\omega}_1\theta\phi + \omega_2)(\omega_1\overline{\theta\phi} + \omega_2)\\
&= \omega_2^2(S^2 + C^2) + \omega_2(2S^2 + 2C^2\,\mathrm{Re}(\overline{\omega}_1\theta\phi)) + (S^2 + C^2)\\
&= \omega_2^2 - 2\alpha\omega_2 + 1, (\text{say}),
\end{aligned}$$

where $\alpha = -(S^2 + C^2\mathrm{Re}(\overline{\omega}_1\theta\phi))$.

On the other hand, it is clear that if a complex number $\omega$ satisfies the equation $\omega^2 - 2\alpha\omega + 1 = 0$, where $\alpha$ is real and $|\alpha| \leq 1$, then $\omega = \alpha \pm i\sqrt{1 - \alpha^2}$, so that $\mathrm{Re}(\omega) = \alpha$, and hence eq. (3.12) is satisfied. □

In the next proposition we give a partial classification of $B(2,n)$. $B(2,3)$ is completely classified in [Sr].

PROPOSITION 3.2

*Let n be arbitrary. Assume that* $\mathrm{Im}(\omega),\ \mathrm{Im}(\omega') > 0$, *and* $C_i, S_i, C'_i, S'_i \neq 0$ *for all i.*

(a) *If* $W(\omega,\theta,\phi,C)$ *is equivalent to* $W(\omega',\theta',\phi',C')$*, then one of the following relations holds:*

(0) $\omega = \omega'$*, and there exists a permutation* $\sigma \in S_{n-2}$ *such that*

$$\mathrm{Ad}(P_\sigma)(C) = C', \quad \mathrm{Ad}(P_\sigma)(\theta) = \zeta\theta', \quad \textit{and} \quad \mathrm{Ad}(P_\sigma)(\phi) = \overline{\zeta}\phi',$$

*where* $\zeta$ *is some complex number of unit modulus,* $P_\sigma$ *denotes the permutation matrix corresponding to* $\sigma$*, and we write* $\mathrm{Ad}(P_\sigma) = P_\sigma(\cdot)P_\sigma^{-1}$.

(1) $\omega = \omega'$*, and there exists a permutation* $\sigma \in S_{n-2}$ *such that*

$$\mathrm{Ad}(P_\sigma)(C) = C', \quad \mathrm{Ad}(P_\sigma)(\theta) = \zeta\theta'^*, \quad \textit{and} \quad \mathrm{Ad}(P_\sigma)(\phi) = \omega\overline{\zeta}\phi'^*,$$

*where* $\zeta$ *is some complex number of unit modulus.*

(2) *There exist i, i′ such that* $(\mathrm{Re}(\omega'),\ \mathrm{Re}(\omega)) \in \Lambda_i \times \Lambda'_{i'}$, *where* $\Lambda_i = \{-(S_i^2 + \mathrm{Re}(\theta_i\phi_i)C_i^2), -(S_i^2 + \mathrm{Re}(\bar{\omega}\,\theta_i\phi_i)C_i^2)\}$ *and* $\Lambda'_i = \{-(S_i'^2 + \mathrm{Re}(\theta'_i\phi'_i)C_i'^2), -(S_i'^2 + \mathrm{Re}(\bar{\omega}'\theta'_i\phi'_i)C_i'^2)\}$.

(3) *There exist* $i, j, i', j'$ *such that* $(\mathrm{Re}(\omega'),\ \mathrm{Re}(\omega)) = (-m_{i,j}, -m'_{i',j'})$ *where* $m_{i,j} = 1-(1 + \mathrm{Re}(\theta_i\phi_i\bar{\theta}_j\bar{\phi}_j))C_i^2C_j^2 - (1 + \mathrm{Re}(\theta_i\phi_j\bar{\theta}_j\bar{\phi}_i))S_i^2S_j^2 - 2(\mathrm{Re}(\theta_i\overline{\theta}_j) + \mathrm{Re}(\phi_i\bar{\phi}_j))C_iC_jS_iS_j$, *and* $m'_{i,j}$ *is the corresponding 'primed' expression.*

(b) *In* (a), *conditions* (0) *and* (1) *are also sufficient conditions for* $W(\omega,\theta,\phi,C)$ *to be equivalent to* $W(\omega,\theta',\phi',C')$.

(c) (i) *The vertical subfactor associated with* $W(\omega,\theta,\phi,C)$ *is always reducible.*

(ii) *The horizontal subfactor associated with* $W(\omega,\theta,\phi,C)$ *is reducible if and only if either of the following two conditions holds:*

(1) $S = 0$;

(2) $\omega = 1$ *and there exists scalars* $\lambda_1 \in \mathbb{T}$ *and* $\lambda_2 \in \mathbb{C}$ *such that* $\phi S = \lambda_1\theta S$ *and* $(1 + \theta\phi)C = \lambda_2\theta S$.

*Proof.* First we write the condition for $P, P' \in \Omega(2,n)$ to afford equivalent connections, as a set of equations. Thus, in order for $W$ to be equivalent to $W'$, i.e. $(U \otimes A)W(\omega,\theta,\phi,C) = W(\omega',\theta',\phi',C')(U' \otimes A')$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U(2),$$

$$U = \begin{pmatrix} u_{1,1} & u_{1,2} & P^t \\ u_{2,1} & u_{2,2} & Q^t \\ X & Y & Z \end{pmatrix}, \quad U' = \begin{pmatrix} u'_{1,1} & u'_{1,2} & P'^t \\ u'_{2,1} & u'_{2,2} & Q'^t \\ X' & Y' & Z' \end{pmatrix} \in U(n),$$

$P, Q, X, Y, P', Q', X', Y' \in M_{(n-2)\times 1}$ and $Z, Z' \in M_{n-2}$, where $X^t$ denotes the matrix transpose of $X$, it is necessary and sufficient that the following set of equations holds.

$$u_{1,1}A = u'_{1,1}A', \tag{3.15}$$

$$u_{1,2}AD = u'_{1,2}A', \tag{3.16}$$

$$P^t(aC + b\phi S) = a'P'^t, \tag{3.17}$$

$$
\begin{aligned}
P^t(a\theta S - b\theta\phi C) &= b'P'^t, && (3.18)\\
P^t(cC + d\phi S) &= c'P'^t, && (3.19)\\
P^t(c\theta S - d\theta\phi C) &= d'P'^t, && (3.20)\\
u_{2,1}A &= u'_{2,1}D'A', && (3.21)\\
u_{2,2}AD &= u'_{2,2}D'A', && (3.22)\\
Q^t(aC + b\phi S) &= a'Q'^t, && (3.23)\\
Q^t(a\theta S - b\theta\phi C) &= b'Q'^t, && (3.24)\\
Q^t(cC + d\phi S) &= c'\omega' Q'^t, && (3.25)\\
Q^t(c\theta S - d\theta\phi C) &= d'\omega'^t Q'^t, && (3.26)\\
aX &= (a'C' + c'\theta'S')X', && (3.27)\\
bX &= (b'C' + d'\theta'S')X', && (3.28)\\
cX &= (a'\phi'S' - c'\theta'\phi'C')X', && (3.29)\\
dX &= (b'\phi'S' - d'\theta'\phi'C')X', && (3.30)\\
aY &= (a'C' + c'\theta'S')Y', && (3.31)\\
b\omega Y &= (b'C' + d'\theta'S')Y', && (3.32)\\
cY &= (a'\phi'S' - c'\theta'\phi'C')Y', && (3.33)\\
d\omega Y &= (b'\phi'S' - d'\theta'\phi'C')Y', && (3.34)\\
Z(aC + b\phi S) &= (a'C' + c'\theta'S')Z', && (3.35)\\
Z(a\theta S - b\theta\phi C) &= (b'C' + d'\theta'S')Z', && (3.36)\\
Z(cC + d\phi S) &= (a'\phi'S' - c'\theta'\phi'C')Z', && (3.37)\\
Z(c\theta S - d\theta\phi C) &= (b'\phi'S' - d'\theta'\phi'C')Z'. && (3.38)
\end{aligned}
$$

Now we consider cases depending on whether various entries of $U$ are zero or non-zero.

*Case* (1): $u_{1,1} \neq 0$. The unitarity of $A$ and $A'$, together with eq. (3.15) imply that $|u_{1,1}| = |u'_{1,1}|$. Let $u_{1,1} = zu'_{1,1}$ where $|z| = 1$; it follows that $A' = zA$. So (by replacing the pair $(A, U)$ by $(zA, z^{-1}U)$, in case $z \neq 1$) we may assume, without loss of generality, that $A = A'$ and $u_{1,1} = u'_{1,1}$.

Since $A$ is a unitary matrix deduce from (3.16) that $u_{1,2}D = u'_{1,2}I_2$, where $I_2$ denotes the identity matrix in $M_2$. The assumption $\omega \neq 1$ now implies that $u_{1,2} = u'_{1,2} = 0$.

Similarly eq. (3.21) implies that $u_{2,1}I_2 = D'u'_{2,1}$. Again the assumption that $\omega' \neq 1$ implies that $u_{2,1} = u'_{2,1} = 0$.

We consider two sub-cases depending upon whether the entry $u_{2,2}$ is non-zero or zero.

*Case* (1.1): $u_{2,2} \neq 0$. As $\mathrm{Im}(\omega)$, $\mathrm{Im}(\omega') > 0$. First we deduce from eq. (3.22) that either $a = 0$ or $b = 0$. But $a = d = 0$ implies that $\omega = \bar{\omega}'$. But as we have assumed that $\mathrm{Im}(\omega)$, $\mathrm{Im}(\omega') > 0$, it is the case that $b = c = 0$, and that $\omega = \omega'$, and $u_{2,2} = u'_{2,2}$. We will show that the relation (0) is satisfied in this case.

As $S_i \neq 0$ for all $i$ (by the assumption in the statement of the Proposition), we find from eq. (3.18) and (3.24) that $P^t = Q^t = 0$. At the same time eqs (3.17) and (3.23) imply that $P'^t = Q'^t = 0$. Also by the assumption $S'_i \neq 0$ for all $i$, we find from eqs (3.28) and (3.32) that $X' = Y' = 0$, while the eqs (3.27) and (3.31) imply that $X = Y = 0$. The unitarity of $U$ and $U'$ is now seen to imply that $Z$ and $Z'$ also are unitary.

Equations (3.35) and (3.38) may be rewritten as

$$ZC = C'Z', \tag{3.39}$$

$$ZC\theta\phi = \theta'\phi'C'Z'. \tag{3.40}$$

Since $C$ and $C'$ are invertible positive operator (as follows from the assumption in the statement of the Proposition that $C_i \neq 0$ for all $i$) and since eq. (3.39) may be re-written as

$$(ZZ'^*)(Z'CZ'^*) = C',$$

we may deduce from the uniqueness of polar decomposition that $Z = Z'$.

Next, we may deduce from eqs (3.39) and (3.40) – using the invertibility of the matrix $C'$ – that $Z\theta\phi Z^* = \theta'\phi'$.

Thus,

$$ZC = C'Z \quad \text{(hence also } ZS = S'Z\text{) and } Z\theta\phi = \theta'\phi'Z.$$

Notice now that

$$\begin{aligned} Z\theta S &= Z(\theta\phi)\phi^* S \\ &= (\theta'\phi')Z\phi^* S \\ &= (\theta'\phi')(Z\phi^* Z^*)ZS \\ &= (\theta'\phi')(Z\phi^* Z^*)S'Z. \end{aligned}$$

Hence, we may deduce from (3.36) that

$$a(\theta'\phi')(Z\phi^*Z^*)S'Z = d\theta'S'Z;$$

deduce from the invertibility of $S'Z$ that

$$Z\phi^*Z^* = \zeta\phi'^*,$$

where $\zeta = d/a$. Since $Z\theta\phi Z^* = (\theta'\phi')$, we thus find that

$$Z\theta Z^* = \zeta\theta'.$$

Let $\mathcal{A}$ (resp., $\mathcal{A}'$) denote the $^*$-subalgebra of $M_{n-2}$ generated by $\{\theta, \phi, C\}$ (resp., $\{\theta', \phi', C'\}$). The preceding analysis shows that the map $\mathrm{Ad}(Z) = Z(\cdot)Z^*$ maps $\mathcal{A}$ onto $\mathcal{A}'$ (since it carries the generators to non-zero multiples of the generators).

Note now that $\mathcal{A}$ and $\mathcal{A}'$ are contained in the algebra of diagonal matrices. If $\{e_\alpha : \alpha \in \Lambda\}$ denotes the set of minimal projections in the abelian $C^*$-algebra $\mathcal{A}$, and if $Ze_\alpha Z^* = e'_\alpha$, then clearly $\{e'_\alpha : \alpha \in \Lambda\}$ is the set of minimal projections in $\mathcal{A}'$. The fact that some unitary matrix i.e., $Z$ – simultaneously conjugates each $e_\alpha$ into $e'_\alpha$, clearly implies now that we can find some permutation $\sigma \in S_{n-2}$ such that $\mathrm{Ad}(P_\sigma)$ maps each $e_\alpha$ into $e'_\alpha$.

It follows easily now from the construction that

$$\mathrm{Ad}(P_\sigma)(\theta) = \zeta\theta', \mathrm{Ad}(P_\sigma)(\phi) = \bar{\zeta}\phi', \quad \text{and} \quad \mathrm{Ad}(P_\sigma)(C) = C'.$$

*Case* (1.2): $u_{2,2} = 0$. We will prove that the relation (2) is satisfied in this case. From the eq. (3.22) we conclude that $u'_{2,2} = 0$. Suppose $Q^t = (q_1, q_2, \ldots, q_{n-2})$. As $u_{2,1} = u_{2,2} = 0$, we find that $Q$ is a unit vector; hence there exists an index $i$ such that $q_i \neq 0$. Then, we find from eqs (3.23)–(3.26) that

$$q_i(aC_i + b\phi_i S_i) = aq_i', \tag{3.41}$$
$$q_i(a\theta_i S_i - b\theta_i\phi_i C_i) = bq_i', \tag{3.42}$$
$$q_i(cC_i + d\phi_i S_i) = c\omega' q_i', \tag{3.43}$$
$$q_i(c\theta_i S_i - d\theta_i\phi_i C_i) = d\omega' q_i'. \tag{3.44}$$

The unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ would imply necessarily that $|q_i'| = |q_i| \neq 0$. Also, as $S_i \neq 0$, we may infer from eqs (3.41) and (3.42) that $a, b, c, d \neq 0$. Let $Y^t = (y_1, y_2, \ldots, y_{n-2})$. Since $u_{1,2} = u_{2,2} = 0$, we find that $Y$ is a unit vector; hence there exists an index $i'$ such that $y_{i'} \neq 0$. Then, we find from eqs (3.31)–(3.34), that the following equations hold:

$$\begin{aligned} ay_{i'} &= (aC_{i'}' + c\theta_{i'}' S_{i'}')y_{i'}', \\ b\omega y_{i'} &= (bC_{i'}' + d\theta_{i'}' S_{i'}')y_{i'}', \\ cy_{i'} &= (a\phi_{i'}' S_{i'}' - c\theta_{i'}'\phi_{i'}' C_{i'}')y_{i'}', \\ d\omega y_{i'} &= (b\phi_{i'}' S_{i'}' - d\theta_{i'}'\phi_{i'}' C_{i'}')y_{i'}'. \end{aligned} \tag{3.45}$$

Again, using the unitarity of the matrix $\begin{pmatrix} C_{i'}' & \phi_{i'}' S_{i'}' \\ \theta_{i'}' S_{i'}' & -\theta_{i'}'\phi_{i'}' C_{i'}' \end{pmatrix}$, deduce that $y_{i'}$ and $y_{i'}'$ have the same absolute value.

Let $\omega_0 = q_i' q_i^{-1}$ and $\omega_0' = y_{i'}' y_{i'}^{-1}$. Now, first by re-writing the above two equations in the form as in eqs (3.8)–(3.11), and then by applying Lemma 3.1 separately to the two sets of equations above conclude that

$$(\mathrm{Re}(\omega'), \mathrm{Re}(\omega)) = (-(S_i^2 + \mathrm{Re}(\theta_i\phi_i)C_i^2), -(S_{i'}'^2 + \mathrm{Re}(\theta_{i'}'\phi_{i'}')C_{i'}'^2)).$$

Hence the relation (2) is satisfied in this case.

*Case* (2): $u_{1,1} = 0$.

*Case* (2.1): $u_{1,2} \neq 0$. Using (3.16) and the unitarity of $A$ and $A'$, we can assume without loss of generality that $AD = A'$ and $u_{1,2} = u_{1,2}'$. Also as $\omega' \neq 1$, we find from (3.22) that $u_{2,2} = u_{2,2}' = 0$. There are two cases now, depending on whether $u_{2,1}$ is not or is 0, which we consider separately.

*Case* (2.1.1): $u_{2,1} \neq 0$. As $\mathrm{Im}(\omega), \mathrm{Im}(\omega') > 0$ we may deduce from (3.21) that $a = d = 0$, $\omega = \omega'$, and $u_{2,1} = \omega u_{2,1}'$. We will show that the relation (1) is satisfied in this case.

As $S$ is invertible, (i.e. $S_i \neq 0$ for all $i$) we find from (3.17) and (3.23) that $P^t = Q^t = 0$. From (3.18) and (3.24), we get $P'^t = Q'^t = 0$. As $S'$ is invertible, we find from (3.27) and (3.31) that $X' = Y' = 0$, and then from (3.28) and (3.32) we get $X = Y = 0$. Now it follows that $Z$ and $Z'$ are unitary.

From (3.36) and (3.37) we have

$$\begin{aligned} -Z\theta\phi C &= \omega C' Z', \\ ZC &= -\theta'\phi' C' Z'. \end{aligned}$$

It follows (as before, from the uniqueness of polar decomposition and the invertibility of the positive operators $C, C'$) that $Z\theta\phi = -\omega Z'$ and $Z = -\theta'\phi' Z'$. These equations together with the equation $bZ\phi S = c\theta' S' Z'$ (which is a consequence of (3.35)) are seen to imply (after some minor manipulations) that

$$\mathrm{Ad}(Z')(C) = C', \quad \mathrm{Ad}(Z')(\theta) = \zeta\theta'^*, \quad \text{and} \quad \mathrm{Ad}(Z')(\phi) = \omega\bar{\zeta}\phi'^*,$$

where $\zeta$ is some scalar of unit modulus.

Arguing exactly as in the proof of Case (1.1), we may deduce the existence of a permutation $\sigma \in S_{n-2}$ such that

$$\mathrm{Ad}(P_\sigma)(C) = C', \quad \mathrm{Ad}(P_\sigma)(\theta) = \zeta\theta'^*, \quad \text{and} \quad \mathrm{Ad}(P_\sigma)(\phi) = \omega\bar{\zeta}\phi'^*.$$

*Case* (2.1.2): $u_{2,1} = 0$. It follows from (3.21) that $u'_{2,1} = 0$. Using the unitarity of $U$ and the fact that $(u_{2,1}, u_{2,2}) = 0$, deduce that $Q^t(= (q_1, \ldots, q_{n-2}))$ is a unit vector and hence there exists an index $i$ such that $q_i \neq 0$. Similarly the unitarity of $U$ and the fact that $(u_{1,1}, u_{1,2}) = 0$, implies that the vector $X(= (x_1, \ldots, x_{n-2}))$ is a unit vector and hence there exists an index $i'$ such that $x_{i'} \neq 0$.

Then, we find from eqs (3.23)–(3.26) that

$$\begin{aligned} q_i(aC_i + b\phi_i S_i) &= aq'_i, \\ q_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= b\omega q'_i, \\ q_i(cC_i + d\phi_i S_i) &= c\omega' q'_i, \\ q_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\omega\omega' q'_i. \end{aligned} \tag{3.46}$$

Also we find from (3.27)–(3.30) that the following equations hold:

$$\begin{aligned} ax_{i'} &= (aC'_{i'} + c\theta'_{i'}S'_{i'})x'_{i'}, \\ b\bar{\omega}x_{i'} &= (bC'_{i'} + d\theta'_{i'}S'_{i'})x'_{i'}, \\ cx_{i'} &= (a\phi'_{i'}S'_{i'} - c\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}, \\ d\bar{\omega}x_{i'} &= (b\phi'_{i'}S'_{i'} - d\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}. \end{aligned}$$

Now using the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ $\left(\text{resp. the matrix } \begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}\right)$ deduce that $|q'_i| = |q_i| \neq 0$ (resp. $|x_i| = |x'_i|$). Also, as $S_i \neq 0$, we may infer from the set of equations (3.46) that $a, b, c, d \neq 0$.

Let $\omega_0 = q'_i q_i^{-1}$ and $\omega'_0 = x'_{i'}x_{i'}^{-1}$. Now, by applying Lemma 3.1 twice to the two sets of equations above (exactly as before), conclude that

$$(\mathrm{Re}(\omega'), \mathrm{Re}(\omega)) = (-(S_i^2 + \mathrm{Re}(\bar{\omega}\theta_i\phi_i)C_i^2), -(S_{i'}'^2 + \mathrm{Re}(\theta'_{i'}\phi'_{i'})C_{i'}'^2)).$$

Hence the relation (2) is satisfied in this case.

*Case* (2.2): $u_{1,2} = 0$. We break this into cases depending on whether $u_{2,1}$ vanishes or not.

*Case* (2.2.1): $u_{2,1} \neq 0$. As before using the unitarity of $A, A'$ and (3.21) we may assume that $u_{2,1} = u'_{2,1}$ and $A = D'A'$. Using the unitarity of $U$ and the fact that $(u_{1,1}, u_{1,2}) = 0$, deduce that $P^t(= (p_1, \ldots, p_{n-2}))$ is a unit vector and hence that there exists an index $i$ such that $p_i \neq 0$. As $\omega \neq 1$, the matrix $D$ is linearly independent from the identity matrix. Hence using (3.22) conclude that $u_{2,2} = u_{2,2} = 0$. Now the unitarity of $U$ and the fact that $(u_{1,2}, u_{2,2}) = 0$, implies that the vector $Y(= (y_1, \ldots, y_{n-2}))$ is a unit vector and hence that there exists an index $i'$ such that $y_{i'} \neq 0$.

Then, we find from (3.17)–(3.20) that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= ap_i', \\ p_i(a\theta_i S_i) - b\theta_i\phi_i C_i) &= bp_i', \\ p_i(cC_i + d\phi_i S_i) &= c\bar{\omega}' p_i', \\ p_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\bar{\omega}' p_i'. \end{aligned}$$

Also we find from the (3.31)–(3.34) that the following equations hold:

$$\begin{aligned} ay_{i'} &= (aC'_{i'} + c\bar{\omega}'\theta'_{i'}S'_{i'})y'_{i'}, \\ b\omega y_{i'} &= (bC'_{i'} + d\bar{\omega}'\theta'_{i'}S'_{i'})y'_{i'}, \\ cy_{i'} &= (a\phi'_{i'}S'_{i'} - c\bar{\omega}'\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'}, \\ d\omega y_{i'} &= (b\phi'_{i'}S'_{i'} - d\bar{\omega}'\theta'_{i'}\phi'_{i'}C'_{i'})y'_{i'}. \end{aligned}$$

Again the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ $\left(\text{resp. the matrix } \begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}\right)$ implies that $|p_i'| = |p_i| \neq 0$ (resp. $|y_i| = |y_i'|$). Also, as $S_i \neq 0$, we may infer from the above set of equations that $a, b, c, d \neq 0$.

Let $\omega_0 = p_i' p_i^{-1}$ and $\omega_0' = y'_{i'} y_{\bar{i}'}^{-1}$. Now, by applying Lemma 3.1 twice to the two sets of equations above (exactly as before), conclude that

$$(\mathrm{Re}(\omega'),\ \mathrm{Re}(\omega)) = (-(S_i^2 + \mathrm{Re}(\theta_i\phi_i)C_i^2), -(S_{i'}'^2 + \mathrm{Re}(\bar{\omega}'\theta'_{i'}\phi'_{i'})C_{i'}'^2)).$$

Hence the relation (2) is satisfied in this case.

*Case* (2.2.2): $u_{2,1} = 0$. First, suppose $u_{2,2} \neq 0$. Then, using the unitarity of $A$ and equation (3.22), we may assume without loss of generality that $u_{2,2} = u'_{2,2}$ and $AD = D'A'$. As before using the unitarity of $U$ and the fact that $(u_{1,1}, u_{1,2}) = 0$, deduce that $P^t(= (p_1, \ldots, p_{n-2}))$ is a unit vector and hence there exists an index $i$ such that $p_i \neq 0$. Also the unitarity of $U$ and the fact that $(u_{1,1}, u_{2,1}) = 0$, implies that the vector $X(= (x_1, \ldots, x_{n-2}))$ is a unit vector and hence that there exists an index $i'$ such that $x_{i'} \neq 0$.

Then, we find from eqs (3.17)–(3.20) that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= ap_i', \\ p_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= b\omega p_i', \\ p_i(cC_i + d\phi_i S_i) &= c\bar{\omega}' p_i', \\ p_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d\omega\bar{\omega}' p_i'. \end{aligned}$$

Also we find from (3.27)–(3.30) that the following equations hold:

$$\begin{aligned} ax_{i'} &= (aC'_{i'} + c\bar{\omega}'\theta'_{i'}S'_{i'})x'_{i'}, \\ b\bar{\omega}x_{i'} &= (bC'_{i'} + d\bar{\omega}'\theta'_{i'}S'_{i'})x'_{i'}, \\ cx_{i'} &= (a\phi'_{i'}S'_{i'} - c\bar{\omega}'\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}, \\ d\bar{\omega}x_{i'} &= (b\phi'_{i'}S'_{i'} - d\omega'\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}. \end{aligned}$$

Again the unitarity of the matrix $\begin{pmatrix} C_i & \phi_i S_i \\ \theta_i S_i & -\theta_i\phi_i C_i \end{pmatrix}$ $\left(\text{resp. the matrix } \begin{pmatrix} C'_{i'} & \phi'_{i'}S'_{i'} \\ \theta'_{i'}S'_{i'} & -\theta'_{i'}\phi'_{i'}C'_{i'} \end{pmatrix}\right)$ implies that $|p_i'| = |p_i| \neq 0$ (resp. $|x_i| = |x_i'|$). Also, as $S_i \neq 0$, we may infer from the above set of equations that $a, b, c, d \neq 0$.

Let $\omega_0 = p'_i p_i^{-1}$ and $\omega'_0 = x'_{i'} x_{i'}^{-1}$. Now, by applying Lemma 3.1 twice to the two sets of equations above, conclude exactly as before that

$$(\mathrm{Re}(\omega'),\ \mathrm{Re}(\omega)) = (-(S_i^2 + \mathrm{Re}(\bar{\omega}\theta_i\phi_i)C_i^2), -(S_{i'}^{\prime 2} + \mathrm{Re}(\bar{\omega}'\theta'_{i'}\phi'_{i'})C_{i'}^{\prime 2})).$$

Hence the relation (2) is satisfied in this case also.

Next we consider the final case when $u_{2,2}$ is also zero.

*Case* 2.2.3: $\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} = 0$. First note that $P, Q, X$ and $Y$ are all unit vectors. So, there exist indices $i, j$ such that $p_i \neq 0 \neq q_j$. We see from (3.17)–(3.20) and (3.23)–(3.26) that

$$\begin{aligned} p_i(aC_i + b\phi_i S_i) &= a' p'_i, \\ p_i(a\theta_i S_i - b\theta_i\phi_i C_i) &= b' p'_i, \\ p_i(cC_i + d\phi_i S_i) &= c' p'_i, \\ p_i(c\theta_i S_i - d\theta_i\phi_i C_i) &= d' p'_i, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} q_j(aC_j + b\phi_j S_j) &= a' q'_j, \\ q_j(a\theta_j S_j - b\theta_j\phi_j C_j) &= b' q'_j, \\ q_j(cC_j + d\phi_j S_j) &= c'\omega' q'_j, \\ q_j(c\theta_j S_j - d\theta_j\phi_j C_j) &= d'\omega' q'_j. \end{aligned} \tag{3.48}$$

Arguing exactly as in the proof of Case (1.2) (of this proposition), we find that $|p'_i| = |p_i|$ and $|q'_j| = |q_j|$. Further, the fact that $S_i, S_j \neq 0$ implies (as before) that $a, b, c, d \neq 0$. Setting $\omega_0 = p_i p_i'^{-1} q_j^{-1} q'_j$, we see that equations (3.47) and (3.48) imply the following identities:

$$\begin{aligned} a(\omega_0 C_i - C_j) + b(\omega_0\phi_i S_i - \phi_j S_j) &= 0, \\ a(\omega_0\theta_i S_i - \theta_j S_j) - b(\omega_0\theta_i\phi_i C_i - \theta_j\phi_j C_j) &= 0, \\ c(\omega_0\omega' C_i - C_j) + d(\omega_0\omega'\phi_i S_i - \phi_j S_j) &= 0, \\ c(\omega_0\omega'\theta_i S_i - \theta_j S_j) - d(\omega_0\omega'\theta_i\phi_i C_i - \theta_j\phi_j C_j) &= 0. \end{aligned}$$

The consistency of the above equations demands that

$$\begin{aligned} &(\omega_0 C_i - C_j)(\omega_0\theta_i\phi_i C_i - \theta_j\phi_j C_j) + (\omega_0\phi_i S_i - \phi_j S_j)(\omega_0\theta_i S_i - \theta_j S_j) = 0, \\ &(\omega_0\omega' C_i - C_j)(\omega_0\omega'\theta_i\phi_i C_i - \theta_j\phi_j C_j) + (\omega_0\omega'\phi_i S_i - \phi_j S_j)(\omega_0\omega'\theta_i S_i - \theta_j S_j) = 0. \end{aligned} \tag{3.49}$$

The fact that $\omega' \neq \pm 1$ enables us to derive the following consequence of the two equations above:

$$\omega_0 = \frac{(\theta_i\phi_i + \theta_j\phi_j)C_i C_j + (\theta_i\phi_j + \theta_j\phi_i)S_i S_j}{\theta_i\phi_i(1 + \omega')}.$$

Substituting this value for $\omega_0$ in eq. (3.49), we get

$$\omega'^2 + 2m_{i,j}\,\omega' + 1 = 0,$$

where, of course, $m_{i,j}$ is as in the statement of relation (3) in the proposition. It follows that $\mathrm{Re}(\omega') = -m_{i,j}$.

As $X, Y \neq 0$, in a similar way to the previous cases, it follows (from equations (3.27)–(3.34)) that there exist indices $i'$, $j'$ such that $|x_{i'}| = |x'_i| \neq 0 \neq |y'_{i'}| = |y_{i'}|$ and

$$\begin{aligned} ax_{i'} &= (a'C'_{i'} + c'\theta'_{i'}S'_{i'})x'_{i'}, \\ bx_{i'} &= (b'C'_{i'} + d'\theta'_{i'}S'_{i'})x'_{i'}, \\ cx_{i'} &= (a'\phi'_{i'}S'_{i'} - c'\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}, \\ dx_{i'} &= (b'_{i'}\phi'_{i'}S'_{i'} - d'\theta'_{i'}\phi'_{i'}C'_{i'})x'_{i'}, \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} ay_{j'} &= (a'C'_{j'} + c'\theta'_{j'}S'_{j'})y'_{j'}, \\ b\omega y_{j'} &= (b'C'_{j'} + d'\theta'_{j'}S'_{j'})y'_{j'}, \\ cy_{j'} &= (a'\phi'_{j'}S'_{j'} - c'\theta'_{j'}\phi'_{j'}C'_{j'})y'_{j'}, \\ d\omega y_{j'} &= (b'\phi'_{j'}S'_{j'} - d'\theta'_{j'}\phi'_{j'}C'_{j'})y'_{j'}. \end{aligned} \tag{3.51}$$

Again, setting $\omega'_0 = x_{i'}x'^{-1}_{i'}y^{-1}_{j'}y'_{i'}$, we find the following consequence of the above sets of equations:

$$\begin{aligned} a'(\omega'_0 C'_{i'} - C'_{j'}) + c'(\omega'_0\theta'_{i'}S'_{i'} - \theta'_{j'}S'_{j'}) &= 0, \\ a'(\omega'_0\phi'_{i'}S'_{i'} - \phi'_{j'}S'_{j'}) - c'(\omega'_0\theta'_{i'}\phi'_{i'}C'_{i'} - \theta'_{j'}\phi'_{j'}C'_{j'}) &= 0, \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} b'(\omega'_0\omega C'_{i'} - C'_{j'}) + d'(\omega'_0\omega\theta'_{i'}S'_{i'} - \theta'_{j'}S'_{j'}) &= 0, \\ b'(\omega'_0\omega\phi'_{i'}S'_{i'} - \phi'_{j'}S'_{j'}) - d'(\omega'_0\omega\theta'_{i'}\phi'_{i'}C'_{i'} - \theta'_{j'}\phi'_{j'}C'_{j'}) &= 0. \end{aligned} \tag{3.53}$$

The consistency of these two sets of equations implies that

$$(\omega'_0 C'_{i'} - C'_{j'})(\omega'_0\theta'_{i'}\phi'_{i'}C'_{i'} - \theta'_{j'}\phi'_{j'}C'_{j'}) + (\omega'_0\phi'_{i'}S'_{i'} - \phi'_{j'}S'_{j'})(\omega'_0\theta'_{i'}S'_{i'} - \theta'_{j'}S'_{j'}) = 0,$$

and

$$(\omega'_0\omega C'_{i'} - C'_{j'})(\omega'_0\omega\theta'_{i'}\phi'_{i'}C'_{i'} - \theta'_{j'}\phi'_{j'}C'_{j'}) + (\omega'_0\omega\phi'_{i'}S'_{i'} - \phi'_{j'}S'_{j'})(\omega'_0\omega\theta'_{i'}S'_{i'} - \theta'_{j'}S'_{j'}) = 0$$

and we may deduce as before that $\mathrm{Re}(\omega) = -m'_{i',j'}$; i.e., the relation (3) is satisfied. Finally the proof of (a) is complete.

(b) If condition (0) is satisfied, we may define

$$A = A' = \begin{pmatrix} \theta'_1 & 0 \\ 0 & \theta_{\sigma^{-1}(1)} \end{pmatrix}$$

and

$$U = U' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & P_\sigma \end{pmatrix}$$

and verify that eqs (3.15) to (3.38) are satisfied; and thus, it is indeed true that $(U \otimes A)W(\omega, \theta, \phi, C) = W(\omega, \theta', \phi', C')(U' \otimes A')$.

If condition (1) is satisfied, we may define

$$A = \begin{pmatrix} 0 & 1 \\ -\omega\bar{\zeta} & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & \omega \\ -\omega\bar{\zeta} & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 0 & 1 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & -\theta'\phi' P_\sigma \end{pmatrix}, \quad U' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & P_\sigma \end{pmatrix},$$

and verify that (3.15) to (3.38) are satisfied; and thus, it is indeed true that $(U \otimes A)W(\omega, \theta, \phi, C) = W(\omega, \theta', \phi', C')(U' \otimes A')$.

(c) (i) By Ocneanu's compactness result (see [O1] or [JS]), we know that

$$A_0^{\infty'} \cap A_1^\infty = (M_n \otimes 1) \cap W(M_n \otimes 1)W^*.$$

It is easily seen that if $X = E_{11} \otimes 1$, then $WXW^* = X$, and so we see that $A_0^{\infty'} \cap A_1^\infty$ contains a non-trivial projection, thus establishing reducibility of the vertical subfactor.

(ii) In this case, Ocneanu's compactness result says that

$$A_\infty^0 \cap A_\infty^1 = (1 \otimes M_2) \cap W(1 \otimes M_2)W^*.$$

The above algebra does not reduce to the scalars – i.e., the horizontal sub-factor is reducible – precisely when it is possible to find non-scalar matrices

$$X = \begin{pmatrix} x_1 I_n & x_2 I_n \\ x_3 I_n & x_4 I_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 I_n & y_2 I_n \\ y_3 I_n & y_4 I_n \end{pmatrix} \in 1 \otimes M_2,$$

where $x_i, y_i \in \mathbb{C}$ such that $WX = YW$.

Easy calculation shows that this matrix equation is satisfied if and only if the following relations hold:

$$\begin{aligned} x_1 &= y_1, x_4 = y_4, \\ x_2 &= y_2 = \omega y_2, x_3 = y_3 = \omega x_3, \\ x_3\theta S &= x_2 \phi S, \\ x_2(1+\theta\phi)C &= (x_1 - x_4)\theta S, \\ x_3(1+\theta\phi)C &= (x_1 - x_4)\phi S. \end{aligned} \tag{$**$}$$

First we will prove that the conditions (1) and (2) are sufficient for the horizontal subfactor to be reducible.

(1) If $S = 0$, it is readily seen that a non-scalar solution to the above system of equations is provided by

$$x_i = y_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) Suppose $\omega = 1$ and there exists scalar $\lambda_1 \in \mathbb{T}$ and $\lambda_2 \in \mathbb{C}$ such that $\phi S = \lambda_1 \theta S$ and $(1+\theta\phi)C = \lambda_2 \theta S$. Choose scalars $x \neq 0$ and $x_1, x_4$ such that $x_1 - x_4 = \lambda_2 x$. Now if we define

$$X = Y = \begin{pmatrix} x_1 I_n & x I_n \\ \lambda_1 x I_n & x_4 I_n \end{pmatrix},$$

then it is an easy verification to see that the set of equations $(**)$ is satisfied.

Now to prove the necessity of one of the conditions (1) and (2) to hold for the horizontal subfactor to be reducible, we will prove that the horizontal subfactor is irreducible if both the conditions (1) and (2) are not satisfied. So assume $S \neq 0$.

If $\omega \neq 1$, it follows at once from the second and fourth lines of $(**)$ that the equations above are satisfied if and only if $x_2 = y_2 = x_3 = y_3 = 0$, and $x_1 = y_1 = x_4 = y_4$ i.e., if and only if $X = Y = \zeta I_{2n}$ for some $\zeta \in \mathbb{C}$. Hence the horizontal subfactor is irreducible in this case.

Suppose $\theta S$ and $\phi S$ are not scalar multiples of one another. Then we may deduce from the third line of $(**)$ (as $S \neq 0$) that $x_3 = x_2 = 0$; since $S \neq 0$, either of the last two lines then forces $x_1 = x_4$.

Suppose $\theta S$ and $(1 + \theta\phi)C$ are not scalar multiples of one another (in particular $(1 + \theta\phi)C \neq 0$). Then we may deduce from the last two lines of $(**)$ (also as $S \neq 0$) that $x_3 = x_2 = 0$ and $x_1 = x_4$. □

We end this section with the following Proposition, which asserts the existence of a continuous $(3n - 6)$-parameter family of pairwise inequivalent connections in $B(2, n)$. It also asserts that the number $(3n - 6)$ is sharp. What we mean by the sharpness of the number $(3n - 6)$ is that there does not exist a subset $\mathcal{B} \subset B(2, n)$ with the following two properties: (i) no two distinct elements of $\mathcal{B}$ are equivalent (as connections); and (ii) $\mathcal{B}$ is homeomorphic to an open subset of Euclidean space of dimension $(3n - 5)$.

PROPOSITION 3.3

*There exist non-empty open sets $\Omega \subset \mathbb{T}$, $\Theta \subset \mathbb{T}^{n-2}$, $\Phi_0 \subset \mathbb{T}^{n-3}$, $\Gamma \subset (0, 1)^{n-2}$ such that if $(\omega, \theta, \phi, C), (\omega', \theta', \phi' C') \in \Omega \times \Theta \times \Phi \times \Gamma$, where $\Phi = \{1\} \times \Phi_0$ and $(\omega, \theta, \phi, C) \neq (\omega', \theta', \phi' C')$ then $W(\omega, \theta, \phi, C)$ is not equivalent to $W(\omega', \theta', \phi', C')$. Thus, there exist a $(3n - 6)$ parameter family of pairwise inequivalent connections and that is the best possible number.*

*Further, we may assume that $1 \notin \Omega \cup \Gamma$; hence all these connections have the property that the associated vertical subfactor is reducible and has index $n^2$, and the horizontal subfactor is irreducible and has index 4.*

*Proof.* Fix $0 < x_1 < x_2 < \pi/4$, and define $\Omega_0 = \{e^{ix} \in \mathbb{T} : x_1 \leq x \leq x_2\}$. Fix $\pi/2 < y_1 < y_2 < 3\pi/4$, such that $0 < y_2 - y_1 < x_1$, and let $\Theta_0 = \{e^{ix} \in \mathbb{T} : y_1 \leq x \leq y_2\}$.

The definitions have the following (easily verified) consequences. Suppose $\omega, \omega' \in \Omega_0$ and $\zeta, \zeta' \in \Theta_0$ are arbitrary. Then,

- $\zeta\zeta' \notin \Omega_0$;
- $\mathrm{Re}(\omega) + \mathrm{Re}(\zeta) \neq 0$;
- $\mathrm{Re}(\omega) + \mathrm{Re}(\bar{\omega}'\zeta) \neq 0$;
- $\mathrm{Re}(\omega) - \mathrm{Re}(\zeta\bar{\zeta}') \neq 0$.

Define $f : [0, 1] \times \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ by $f(C, \zeta, \omega) = \mathrm{Re}(\omega) + (1 - C^2) + C^2\,\mathrm{Re}(\zeta)$. Then $0 \notin f(\{1\} \times \Theta_0 \times \Omega_0)$. The compactness of $(\{1\} \times \Theta_0 \times \Omega_0)$ and continuity of $f$ imply the existence of an $\epsilon > 0$ such that for all $C \in (1 - \epsilon, 1), \omega \in \Omega_0, \zeta \in \Theta_0$, we have $\mathrm{Re}(\omega) + (1 - C^2) + C^2\mathrm{Re}(\zeta) \neq 0$. In a similar way, by considering suitable continuous functions, we can see that if $\epsilon$ is chosen sufficiently small, then the following relations are also valid.

Suppose $\omega \in \Omega_0$, and $\theta, \phi, \theta', \phi' \in \mathbb{T}$ are such that $\theta\phi, \theta'\phi' \in \Theta_0$, and suppose $C, C' \in (1 - \epsilon, 1)$. Then we simultaneously have

$$\mathrm{Re}(\omega) + (1 - C'^2) + C'^2\,\mathrm{Re}(\bar{\omega}'\theta\phi) \neq 0,$$

and

$$\mathrm{Re}(\omega) + m \neq 0,$$

where $m = 1 - (1 + \mathrm{Re}(\theta\phi\bar{\theta}'\bar{\phi}'))C^2C'^2 - (1 + \mathrm{Re}(\theta\phi'\bar{\theta}'\bar{\phi}))S^2S'^2 - 2(\mathrm{Re}(\theta\bar{\theta}') + \mathrm{Re}(\phi\bar{\phi}'))$ $CC'SS'$.

Let $\Omega$ denote the interior of $\Omega_0$, and $\Gamma_0 = (1-\epsilon, 1)$. Let $\{\Theta_i' : 1 \leq i \leq n-2\}$ be a collection of pairwise disjoint open subsets of $\Theta_0$. Define $\Gamma = \{\mathrm{diag}(C_1, \ldots, C_{n-2}) : C_i \in \Gamma_0 \forall i\}$, and $\Theta' = \{\mathrm{diag}(\zeta_1, \ldots, \zeta_{n-2}) : \zeta_i \in \Theta_i' \forall i\}$.

Define $\Theta_1 = \Theta_1', \Phi_1 = \{1\}$ and for $1 < i \leq n-2$, choose non-empty open subsets $\Theta_i, \Phi_i \subset \mathbb{T}$ such that $\Theta_i\Phi_i \subset \Theta_i'$. Let $\Theta = \{\mathrm{diag}(\theta_1, \ldots, \theta_{n-2}) \in M_{n-2} : \theta_i \in \Theta_i \forall i\}$ and $\Phi = \{\mathrm{diag}(\phi_1, \ldots, \phi_{n-2}) \in M_{n-2} : \phi_i \in \Phi_i \forall i\}$.

Suppose now that $W(\omega, \theta, \phi, C)$ is equivalent to $W(\omega', \theta', \phi', C')$, where $(\omega, \theta, \phi, C)$, $(\omega', \theta', \phi', C') \in \Omega \times \Theta \times \Phi \times \Gamma$.

First notice that if $\zeta, \zeta' \in \Theta'$ and if $\sigma \in S_{n-2}$ are such that $(\mathrm{Ad}(P_\sigma))(\zeta) = \zeta'$, then necessarily $\zeta = \zeta'$ and $\sigma$ is the identity permutation.

Our choice of ($\epsilon$ and consequently of) $\Gamma$ ensures that neither of the relations (2) or (3) of Proposition 3.2(a) can occur. Suppose the relation (1) were to hold; this would imply that (in the notation of the proposition) $(\mathrm{Ad}(P_\sigma))(\theta\phi) = \omega(\theta'\phi')^*$; in particular, looking at any one diagonal entry of this matrix equation, we would be able to produce elements $\zeta_1, \zeta_2 \in \Theta_0$ such that $\omega = \zeta_1\zeta_2$, which we have already observed to be impossible. Thus the relation (1) can also not hold.

Thus, by Proposition 3.2(a), the relation (0) must necessarily hold. Then the permutation $\sigma$ (whose existence is the content of (0)) must satisfy the condition $(\mathrm{Ad}(P_\sigma))(\theta\phi) = (\theta'\phi')$, which can only happen when $\sigma$ is the identity permutation (by the discussion in the paragraph preceding the last one). Hence $\phi = \bar{\zeta}\phi'$, where $\zeta$ is as in the statement of Proposition 3.2(a) (0); since $\phi_1 = \phi_1' = 1$, we see that necessarily $\zeta = 1$; but relation (0), when $\sigma = id$ and $\zeta = 1$, then just says that $(\omega, \theta, \phi, C) = (\omega', \theta', \phi', C')$.

Now we will prove that the number $(3n-6)$ is sharp.

For this, let $F : \mathbb{T} \times \mathbb{T}^{n-2} \times \mathbb{T}^{n-2} \times [0,1]^{n-2} \to B(2,n)$ denote the (obviously continuous) mapping given by $F(\omega, \theta, \phi, C) = W(\omega, \theta, \phi, C)$. Suppose now that there exists a subset $\mathcal{B}$ with the following two properties: (i) no two distinct elements of $\mathcal{B}$ are equivalent (as connections); and (ii) $\mathcal{B}$ is homeomorphic to an open subset of Euclidean space of dimension $(3n-5)$. Then $F^{-1}(\mathcal{B})$ is a subset of $\mathbb{T}^{2n-3} \times [0,1]^{n-2}$ which is homeomorphic to an open subset of $\mathbb{T}^{2n-3} \times [0,1]^{n-2}$, and is consequently itself open (see, for instance [Spa], Th. 4.8.16).

So it suffices to show that any open subset of $\mathbb{T}^{3n-5} \times (0,1)^{n-2}$ contains two distinct points whose images under $F$ are equivalent, as connections. It clearly suffices to establish this assertion when the open subset is a product $\Omega \times \Theta \times \Phi \times \Gamma$ with open factors.

So, suppose $\Omega \subset \mathbb{T}, \Theta, \Phi \subset \mathbb{T}^{n-2}, \Gamma \subset (0,1)^{n-2}$ are open subsets, Let $\phi \in \Phi$, $\theta \in \Theta$ be arbitrary. As $\Phi$ is assumed to be open, for all $\epsilon > 0$ there exists a $\phi'' \in \Phi$ such that $\phi_1 \neq \phi_1''$ and $\mathrm{Arg}(\phi_1\bar{\phi}_1'') < \epsilon$, where $\phi_1$ and $\bar{\phi}_1''$ are the (1,1)th entry of $\phi$ and $\phi''$ respectively. Let $\zeta = \phi_1\phi_1''$. Now define $\theta_i' = \theta_i\zeta$ for $i = 1, 2, \ldots, n-2$ and $\phi_i' = \phi_i\bar{\zeta}$ for $1 = 2, 3, \ldots, n-2$. Now choose $\epsilon$, as $\Theta$ and $\phi$ are open, so that $\theta' \in \Theta$ and $\phi' \in \Phi$. Now it is easily seen that the pair $(\omega, \theta, \phi, C)$ and $(\omega, \theta', \phi', C)$ satisfies the relation (0) in Proposition 3.2, and hence, using (b) of the same proposition, we conclude that $W(\omega, \theta, \phi, C)$ is equivalent to $W(\omega, \theta', \phi', C)$. Finally, the proof is complete. $\square$

## 4. The principal graph of the horizontal subfactor

In [P] it is shown that for finite-depth subfactors of index 4, the principal graph has to be one of the extended Dynkin diagrams. We will show that all those diagrams can be obtained from vertex models coming from $B(2,n)$ for some $n$.

**Theorem 4.1 (Popa).** *Let $N \subset M$ be an inclusion of $II_1$ factors, with finite depth and $[M:N]=4$. Then the principal graph for the inclusion $N \subset M$ is one of the following diagrams: $A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$.*

For a group $G \subset U(N)$, let $\pi$ denotes the standard (or identity) representation of $G$ in $U(N)$, and let $C(\hat{G},\pi)$ denote the bipartite graph obtained as follows: let $\mathcal{G}$ denote the bipartite graph with the set of even (respectively odd) vertices being given by $\mathcal{G}^{(0)} = \hat{G} \times \{0\}$ (respectively $\mathcal{G}^{(1)} = \hat{G} \times \{1\}$), where $\hat{G}$ denotes the (unitary) dual of $G$, and the number of bonds joining $(\rho,0)$ and $(\sigma,1)$ is given by $(\rho \otimes \pi, \sigma)$; finally, let $C(\hat{G},\pi)$ denote the connected component in $\mathcal{G}$ containing $(\mathrm{tr},0)$, where tr denotes the trivial representation of $G$.

The following theorem is proved in [USC] (also, see [BHJ] and [JS]).

**Theorem 4.2.** *Let $\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ be any collection of $k \times k$ unitary matrices, and define $W_{\beta b}^{\alpha a} = \delta_\beta^\alpha (\gamma_\alpha)_b^a$; then $W$ is a biunitary and the principal graph of the horizontal subfactor given by the vertex model corresponding to $W$ is $C(G,\pi)$, where $G$ is the group generated by $\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$.*

Suppose $H$ is a finite subgroup of $\mathrm{SO}(3)$. Let $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ be the 2-fold covering map (i.e., surjective homomorphism such that $\ker\phi = \{+I,-I\}$); let $\pi_n$ be the (unique, up to isomorphism) irreducible representation of $\mathrm{SU}(2)$ of dimension $n+1$. Let $G = \phi^{-1}(H)$, and let $\pi = \pi_1|_G$. The following lemma can be easily seen to be true.

*Lemma* 4.3. *Let $\rho \in \hat{G}$. Then* (i) $(\rho,0) \in C(\hat{G},\pi)^{(0)}$ *if and only if* $\pi(-1)=1$ *if and only if* $\pi = \pi_0 \circ \phi$ *for some* $\pi_0 \in \hat{H}$. (ii) $(\rho,1) \in C(\hat{G},\pi)^{(1)}$ *if and only if* $\pi(-1) \neq 1$ *if and only if $\pi$ does not factor through $H$.*

PROPOSITION 4.4

*For all $\mathcal{G} \in \{A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}\}$ there exists an $n \in \mathbb{N}$ and $W \in B(2,n)$ such that the principal graph of the horizontal subfactor given by the vertex model corresponding to $W$ is $\mathcal{G}$.*

*Proof.* It is enough to show that there exists $G \subset SU(2)$ with the property that $C(\hat{G},\pi) = \mathcal{G}$. Note that $\pi$ is self-contragredient and faithful. Using the lemma and some combinatorial arguments one can see, without too much difficulty, that if we let $H$ be the group $Z_n, D_n, A_4, S_4$ or $A_5$, then the corresponding Cayley graph $C(\hat{G},\pi)$ turns out to be the extended Coxeter graph $A_{2n}^{(1)}, D_{n+2}^{(1)}, E_6^{(1)}, E_7^{(1)}$, or $E_8^{(1)}$ respectively. $\square$

## Acknowledgement

I would like to acknowledge my sincere thanks to Prof V S Sunder for suggesting this problem, patiently going through the manuscript, pointing out mistakes, and encouraging me at every stage. I also would like to thank Dr Vijay Kodiyalam for helping in the combinatorial argument used to sort out the graphs in the last section and Dr D S Nagaraj for telling about a Theorem in [Spa] which is used in the proof of Proposition 3.3.

## References

[BHJ] Bacher Roland, Harpe de la Pierre and Jones V F R, Carres commutatifs et invariants de structures combinatoires, *Comptes Rendus Acad. Sci. Paris*, *Serie* 1 (1995) 1049–1054

[GHJ] Goodman F, Harpe de la Pierre and Jones V F R, *Coxeter graphs and towers of algebras* (New York: MSRI Publ., 14, Springer) (1989)

[HS] Haagerup U and Schou J, *Some new subfactors of the hyperfinite $II_1$ factor*, preprint (1989)

[J] Jones V F R, Index for subfactors, *Invent. Math.* **71** (1983) 1–25

[JS] Jones V and Sunder V S, Introduction to subfactors, *LMS Lect. Note Ser. 123* (Cambridge) (1997)

[O1] Ocneanu A, Quantized groups, String algebras and Galois theory for algebras, *Operator Algebras and Appl. (Warwick) vol. 2 (1987); London Math. Soc. Lecture Notes Ser.* (Cambridge University Press) (1988) 136, pp. 119–172

[P] Popa S, Sur la classification sous-facteurs d'indice fini du facteurs hyperfini, *C R Acad. Sciences*, Serie I Mathematiques 311 **95** (1990) 95–110

[P1] Popa S, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994) 163–255

[Spa] Spanier E H, *Algebraic Topology* (New York: Mc-Graw Hill) (1966)

[Sr] Srnivasan R, *Connections on small vertex models*, thesis, submitted to the Indian Statistical Institute (1998)

[USC] Uma Krishnan, Sunder V S and Varughese Cherian, On some subfactors arising from vertex models, *J. Funct. Anal.* **140** (1996) 449–471