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Solitary wave solution to a singularly perturbed generalized Gardner equation with nonlinear terms of any order

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Abstract. This paper is concerned with the existence of travelling wave solutions to a singularly perturbed generalized Gardner equation with nonlinear terms of any order. By using geometric singular perturbation theory and based on the relation between solitary wave solution and homoclinic orbits of the associated ordinary differential equations, the persistence of solitary wave solutions of this equation is proved when the perturbation parameter is sufficiently small. The numerical simulations verify our theoretical analysis.

Keywords. Perturbed generalized Gardner equation; solitary wave solution; geometric singular perturbation theory.

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1. Introduction

In this paper, we consider a singularly perturbed generalized Gardner equation with nonlinear terms of any order

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{1.1}$$

where $\alpha, \beta, \gamma, \varepsilon$ are positive parameters, and p is a positive integer. Actually, it is a hard task to seek its explicit travelling wave solutions. So, in this paper, we resort to the geometrical perturbation theory [1,2] to establish the existence of solitary wave solutions to it when the perturbation parameter ε is sufficiently small.

We note that, when the parameters in eq. (1.1) take different values, several celebrated equations can be derived. When $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$, and $\varepsilon = 0$, eq. (1.1) becomes the generalized Gardner equation

$$u_t + \alpha u^p u_x + \beta u^{2p} u_x + \gamma u_{xxx} = 0, \tag{1.2}$$

which is one model in plasma physics and solid physics [3]. Hamdi *et al* [4] obtained an exact solitary wave solution to eq. (1.2). They also derived three conservation laws and three invariants of motion for eq. (1.2) [5]. Antonova and Biswas [6] exploited the soliton perturbation theory to eq. (1.2) with $\gamma = 1$.

When $\alpha \neq 0, \beta \neq 0, \gamma \neq 0, \varepsilon = 0$ and $p = 1$, eq. (1.1) is the classical Gardner equation

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1.3}$$

which is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory [7–9]. It also describes a variety of wave phenomena in plasma and solid state [10,11]. Exact travelling wave solutions to the Gardner equation were given by several researchers [12–19].

When $\alpha \neq 0, \beta = 0, \gamma \neq 0, \varepsilon = 0$ and $p = 1$, eq. (1.1) becomes the famous KdV equation

$$u_t + \alpha u u_x + \gamma u_{xxx} = 0, \tag{1.4}$$

which arises in physics as a model of propagation of dispersive long waves, as was pointed out by Russel in 1834 [20]. The KdV equation has solitary wave solutions of the form [21]

$$u(x,t) = \frac{12\gamma k^2}{\alpha}\operatorname{sech}^2 k(x - 4\gamma k^2 t). \tag{1.5}$$

If $\alpha = 0, \beta \neq 0, \gamma \neq 0, \varepsilon = 0$ and $p = 1$, eq. (1.1) reduces to the so-called mKdV equation

$$u_t + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1.6}$$

which arises in many of the same physical contexts as the KdV equation, such as water waves and plasma physics, but in different parameter regimes [21]. The mKdV equation admits solitary wave solutions as well as periodic solutions [22,23].

It is known that solitary wave solution is a type of solution to nonlinear wave equations, which can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. So, one of the important issues about nonlinear wave equations is to search for their solitary wave solutions. There have been a variety of powerful methods in the literature, such as the Bäcklund transformation [24], the Riccati equation expansion [25,26], the ansatz method [25–32], the Jacobian elliptic equation expansion [30,33], the bifurcation method [34], the method of solution in series [35], the extended G'/G-expansion method [36] and the first integral method [37], to derive solitary wave solutions. However, when considering singularly perturbed wave equations, such as eq. (1.1), it is generally difficult to work out the explicit travelling wave solutions. In such a case, the first question is the existence of travelling wave solutions, such as solitary waves, travelling fronts, or periodic waves. Compared to traditional methods, geometric singular perturbation method [1,2] plays a special role in giving a first picture of the perturbed solutions. In recent years, geometric singular perturbation method [1,2,38] has been extended to some perturbed nonlinear dispersive equations. For example, Ogawa [39] proved the existence of solitary wave solution to a perturbed Korteweg–de Vries equation. Fan and Tian [40] showed that solitary wave solution to a singularly perturbed mKdV–KS equation persists when the perturbation parameter is suitably small. Mansour [41] established the existence of travelling wave solutions to a singularly perturbed Burgers–KdV equation. Tang and Xu [42] showed the persistence of solitary wave solutions of singularly perturbed Gardner equation. Zhuang *et al* [43] considered the persistence of solitary wave solution to the singularly perturbed higher-order KdV equation.

The remainder of this paper is organized as follows. In §2, we introduce the geometric singular perturbation theory. In §3, we investigate the relation between solitary wave solution and homoclinic orbits. When $\varepsilon > 0$, we show that homoclinic orbit of eq. (3.4) persists by using geometric singular perturbation theory, and then the solitary wave solution to eq. (1.1) persists. Here, we use a method different from the methods used in [40,42,43] and our result generalizes the work in [42]. In §4, we give the numerical results to verify our theoretical analysis. A short conclusion is made in §5.

2. Geometric singular perturbation theory

In this section, we introduce the following result on invariant manifolds which is due to Fenichel [1] and will be used in §3 for our purpose. For convenience, we use a version of this theory due to Jones [2].

For the system

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^l$ and ε is a real parameter, f, g are C^∞ on the set $V \times I$ where $V \in \mathbb{R}^{n+1}$ and I is an open interval, containing 0. When $\varepsilon = 0$, the system has a compact, normally hyperbolic manifold of critical points M_0 which is contained in the set $\{(x, y) : f(x, y, 0) = 0\}$. Then for any $0 < r < +\infty$, if $\varepsilon > 0$, but sufficiently small, there exists a manifold M_ε:

(I) which is locally invariant under the flow of system (2.1);
(II) which is C^r in x, y and ε;
(III) $M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\}$ for some C^r function $h^\varepsilon(y)$ and y in some compact K.
(IV) there exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ and $W^u(M_\varepsilon)$ that lie within $O(\varepsilon)$ of, and are diffeomorphic to, $W^s(M_0)$ and $W^u(M_0)$, respectively.

3. Persistence of solitary waves

A travelling wave solution $u(x, t) = \varphi(x - ct) = \varphi(\xi)$ to eq. (1.1) is called a solitary wave solution if $\varphi(\xi)$ has a well-defined limit, which is zero when ξ approaches $\pm\infty$, that is, $\varphi(\xi) \to 0$ as $\xi \to \pm\infty$.

Assume that eq. (1.1) has a travelling wave solution in the form $u(x, t) = \varphi(\xi)$, $\xi = x - ct$ $(c > 0)$, then we get an ordinary differential equation

$$-c\varphi' + \alpha\varphi^p\varphi' + \beta\varphi^{2p}\varphi' + \gamma\varphi''' + \varepsilon(\varphi'' + \varphi'''') = 0, \tag{3.1}$$

where $'$ denotes the derivative with respect to ξ.

Integrating (3.1) once yields

$$-c\varphi + \frac{\alpha}{p+1}\varphi^{p+1} + \frac{\beta}{2p+1}\varphi^{2p+1} + \gamma\varphi'' + \varepsilon(\varphi' + \varphi''') = 0, \tag{3.2}$$

where we have taken the constant of integration as zero.

Make the transformation

$$\varphi = \sqrt[2p]{\frac{c}{\beta}}u, \quad z = \sqrt{c}\xi, \varphi'_\xi = \sqrt[2p]{\frac{c}{\beta}} \cdot \sqrt{c} \cdot u'_z,$$

$$\varphi''_{\xi\xi} = \sqrt[2p]{\frac{c}{\beta}} \cdot c \cdot u''_{zz}, \quad \varphi'''_{\xi\xi\xi} = \sqrt[2p]{\frac{c}{\beta}} \cdot c\sqrt{c} \cdot u'''_{zzz},$$

then eq. (3.2) is written as

$$-cu+\frac{\alpha}{p+1}\sqrt{\frac{c}{\beta}}u^{p+1}+\frac{c}{2p+1}u^{2p+1}+c\gamma u''_{zz}+\varepsilon(\sqrt{c}u'_z+c\sqrt{c}u'''_{zzz})=0. \quad (3.3)$$

Let $u'_z=v$, $v'_z=w$, then eq. (3.3) can be rewritten as a system of ordinary differential equations

$$\begin{cases} u'_z=v,\\ v'_z=w,\\ \varepsilon w'_z=\dfrac{1}{\sqrt{c}}\left(u-\dfrac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}-\dfrac{1}{2p+1}u^{2p+1}\right.\\ \qquad\qquad\left.-\gamma w-\dfrac{\varepsilon v}{\sqrt{c}}\right). \end{cases} \quad (3.4)$$

It is known that a solitary wave solution to eq. (1.1) corresponds to a homoclinic orbit of system (3.4). So, the existence of solitary wave solution to eq. (1.1) is transformed to the existence of homoclinic orbit of system (3.4). For this, let $z=\varepsilon\eta$, then system (3.4) turns into

$$\begin{cases} u'_\eta=\varepsilon v,\\ v'_\eta=\varepsilon w,\\ w'_\eta=\dfrac{1}{\sqrt{c}}\left(u-\dfrac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}-\dfrac{1}{2p+1}u^{2p+1}\right.\\ \qquad\qquad\left.-\gamma w-\dfrac{\varepsilon v}{\sqrt{c}}\right). \end{cases} \quad (3.5)$$

Systems (3.4) and (3.5) can be called the slow system and the fast system, respectively.

Let

$$G(Z)=\begin{pmatrix} \varepsilon v\\ \varepsilon w\\ \dfrac{1}{\sqrt{c}}\left(u-\dfrac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}-\dfrac{1}{2p+1}u^{2p+1}-\gamma w-\dfrac{\varepsilon v}{\sqrt{c}}\right)\end{pmatrix},$$

where $Z=(u,v,w)^T$. For $\varepsilon\neq 0$, (3.4) and (3.5) are equivalent. When $p=2k+1, k\in N$, the equilibrium points are $Z_0=(0,0,0)^T$, $Z_1=(\sqrt[p]{Y},0,0)^T$ and $Z_2=(\sqrt[p]{X},0,0)^T$. When $p=2k, k\in N$, the equilibrium points are $Z_0=(0,0,0)^T$, $Z^+=(\sqrt[p]{X},0,0)^T$ and $Z^-=(-\sqrt[p]{X},0,0)^T$, where

$$X=-\frac{2p+1}{2}\left(\frac{\alpha}{\sqrt{\beta c}(p+1)}-\sqrt{\frac{\alpha^2}{\beta c(p+1)^2}+\frac{4}{2p+1}}\right)$$

and

$$Y=-\frac{2p+1}{2}\left(\frac{\alpha}{\sqrt{\beta c}(p+1)}+\sqrt{\frac{\alpha^2}{\beta c(p+1)^2}+\frac{4}{2p+1}}\right).$$

Now, we investigate the coefficient matrix of the linearization system of (3.5) at Z_0. Let

$$A_0=DG(Z_0)=\begin{pmatrix} 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\\ \dfrac{1}{\sqrt{c}} & -\dfrac{\varepsilon}{c} & -\dfrac{\gamma}{\sqrt{c}}\end{pmatrix},$$

then the characteristic equation for A_0 is

$$\det(A_0-\lambda I):=-\lambda^3-\frac{\gamma}{\sqrt{c}}\lambda^2-\frac{\varepsilon^2}{c}\lambda+\frac{\varepsilon^2}{\sqrt{c}}=0.$$

The discriminant of $\det(A_0-\lambda I)$ is

$$D(c,\varepsilon):=\frac{\varepsilon^2}{c^3}(4\varepsilon^4+(27c^2+18c\gamma-\gamma^2)\varepsilon^2-4c\gamma^3).$$

Define

$$D_1(c,\varepsilon^2):=4\varepsilon^4+(27c^2+18c\gamma-\gamma^2)\varepsilon^2-4c\gamma^3,$$

then $D_1(c,0)=-4c\gamma^3<0$ for any $\gamma, c>0$. Therefore, $D_1(c,\varepsilon^2)<0$ for any $\gamma, c>0$ and sufficiently small ε. This implies $D(c,\varepsilon)<0$ for any positive c,γ and sufficiently small ε. Thus, the matrix A_0 has three real eigenvalues, noted by $\lambda_1,\lambda_2,\lambda_3$. By Viete theorem, we see that $\lambda_1,\lambda_2,\lambda_3$ satisfy

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3=-\dfrac{\gamma}{\sqrt{c}},\\ \lambda_1\lambda_2\lambda_3=\dfrac{\varepsilon^2}{\sqrt{c}}. \end{cases}$$

It is obvious that one of the three real eigenvalues is positive and the other two are all negative. So, the stable subspace at Z_0, named as $E^S(Z_0)$, is two-dimensional for positive γ, c and ε and $E^U(Z_0)$, the unstable subspace at Z_0 is one-dimensional. Let $W^S(Z_0)$ and $W^U(Z_0)$ be the local stable and unstable manifold, respectively. By Hartman-Grobman theorem, we have $\mathrm{Dim}(W^S(Z_0))=2$ and $\mathrm{Dim}(W^U(Z_0))=1$.

Because $\mathrm{Dim}(W^S(Z_0)) + \mathrm{Dim}(W^U(Z_0)) = 3$ in $\mathbb{R}^3$, it is not so obvious to ensure a homoclinic orbit here.

Next, we shall prove that system (3.5) has a homoclinic orbit for sufficiently small ε. When $\varepsilon = 0$, we get the zero-order slow manifold

$$M_0 = \left\{ (u, v, w) \in \mathbb{R}^3 : w = \frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right)\right\},$$

which is suitably restricted to any compact domain K of (u, v) space. Then, the linearization of the fast system (3.5), restricted to M_0, has the matrix

$$A^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{c}}\left(1 - \frac{\alpha}{\sqrt{\beta c}}u^p - u^{2p}\right) & 0 & -\frac{\gamma}{\sqrt{c}} \end{pmatrix}.$$

The matrix A^* has eigenvalues 0, 0 and $-\gamma/\sqrt{c}$. Thus, M_0 is normally hyperbolic. By geometric singular perturbation theory described in §2, we know that there exists a perturbed manifold M_ε which lies within $O(\varepsilon)$ of M_0 and is diffeomorphic to M_0.

To determine the dynamics on M_ε, we write

$$M_\varepsilon = \{(u, v, w) \in \mathbb{R}^3 : w = h(u, v, \varepsilon)\},$$

where $h(u, v, \varepsilon)$ depends smoothly on ε and satisfies

$$h(u, v, 0) = \frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right).$$

Substituting $h(u, c, \varepsilon)$ into (3.4), we obtain a system of ordinary differential equations on M_ε

$$\begin{cases} u'_\eta = v, \\ v'_\eta = h(u, v, \varepsilon), \\ \varepsilon w'_\eta = \frac{1}{\sqrt{c}}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1} \right. \\ \qquad\qquad \left. -\gamma h(u, v, \varepsilon) - \frac{\varepsilon v}{\sqrt{c}}\right). \end{cases} \tag{3.6}$$

The limit form of eq. (3.6) restricted to M_0 can be written as

$$\begin{cases} u'_\eta = v, \\ v'_\eta = h(u, v, 0), \\ = \frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right), \end{cases} \tag{3.7}$$

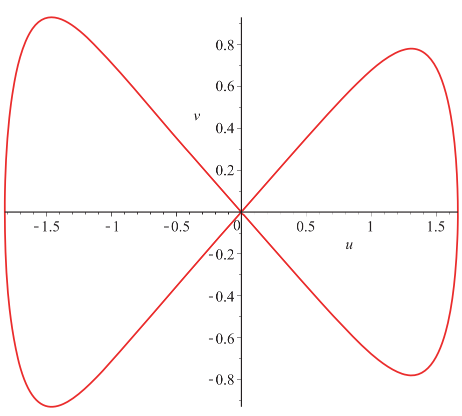
which is the dynamical form of the generalized Gardner equation (1.2) and its Hamiltonian function is

$$H(u, v) = \frac{v^2}{2} + \frac{\alpha}{\gamma\sqrt{\beta c}(p+1)(p+2)}u^{p+2} + \frac{1}{\gamma(2p+1)(2p+2)}u^{2p+2} - \frac{1}{2\gamma}u^2.$$

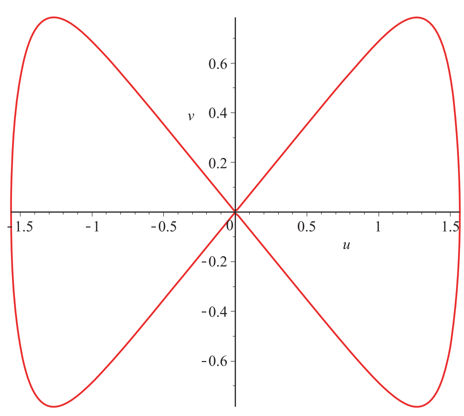
It is clear that the equilibrium point (0, 0) of (3.7) is a saddle point, and there exists a homoclinic Γ_0 passing through (0, 0) (see figure 1)

$$\Gamma_0 : \frac{v^2}{2} + \frac{\alpha}{\gamma\sqrt{\beta c}(p+1)(p+2)}u^{p+2} + \frac{1}{\gamma(2p+1)(2p+2)}u^{2p+2} - \frac{1}{2\gamma}u^2 = 0,$$

which corresponds to a solitary wave solution to eq. (1.2).



(a) $p = 3$



(b) $p = 4$

Figure 1. Homoclinic orbit of system (3.7). $\alpha = 1.0$, $\beta = 2.0$, $\gamma = 2.0$, $c = 2.0$.

As M_0 is smooth, the vector field in (3.6) is smooth and M_ε can be characterized as the graph of a function; $h(u, v, \varepsilon)$ can be expanded in ε when ε is sufficiently small, that is

$$\begin{aligned} w &= h(u, v, \varepsilon) \\ &= \frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right) \\ &\quad + \varepsilon h_1(u, v) + O(\varepsilon^2). \end{aligned} \tag{3.8}$$

We need to calculate the term $h_1(u, v)$ in (3.8), which also likely depends on the parameters γ and c. The only remaining information about M_ε is the local invariance relative to the equation and this can then be used to evaluate h_1. Differentiating (3.8) with respect to η, we get

$$\begin{aligned} w_\eta &= \frac{1}{\gamma}\left(v - \frac{\alpha}{\sqrt{\beta c}}u^p v - u^{2p} v\right) \\ &\quad + \varepsilon\left(\frac{\partial h_1}{\partial u}v + \frac{\partial h_1}{\partial v}\left(\frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right.\right.\right. \\ &\quad \left.\left.\left. - \frac{1}{2p+1}u^{2p+1}\right)\right)\right) + O(\varepsilon^2). \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.4), we obtain

$$\begin{aligned} \varepsilon w_\eta &= \frac{1}{\sqrt{c}}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right. \\ &\qquad \left. - \gamma w - \frac{\varepsilon v}{\sqrt{c}}\right) \\ &= \frac{1}{\sqrt{c}}\left(-\varepsilon\gamma h_1 - \frac{\varepsilon v}{\sqrt{c}}\right) + O(\varepsilon^2) \\ &= \frac{\varepsilon}{\gamma}\left(v - \frac{\alpha}{\sqrt{\beta c}}u^p v - u^{2p} v\right) + \varepsilon^2 \\ &\quad \times\left(\frac{\partial h_1}{\partial u}v + \frac{\partial h_1}{\partial v}\left(\frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right.\right.\right. \\ &\qquad \left.\left.\left. - \frac{1}{2p+1}u^{2p+1}\right)\right)\right) \\ &\quad + O(\varepsilon^2). \end{aligned} \tag{3.10}$$

Comparing coefficients of ε in (3.10), we have

$$h_1(u, v) = \frac{\sqrt{c}}{\gamma^2}\left(u^{2p} + \frac{\alpha}{\sqrt{\beta c}}u^p - \left(1 + \frac{\gamma}{c}\right)\right)v.$$

So, the dynamics on the slow manifold M_ε is given as

$$\begin{cases} u'_\eta = v, \\ v'_\eta = \dfrac{1}{\gamma}\left(u - \dfrac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right. \\ \qquad \left. - \dfrac{1}{2p+1}u^{2p+1}\right) \\ \qquad + \dfrac{\varepsilon\sqrt{c}}{\gamma^2}\left(u^{2p} + \dfrac{\alpha}{\sqrt{\beta c}}u^p - \left(1 + \dfrac{\gamma}{c}\right)\right)v \\ \qquad + O(\varepsilon^2). \end{cases} \tag{3.11}$$

Now, we still cannot see a homoclinic orbit merely by adding the $O(\varepsilon)$ term. However, we have other two parameters c and ε that can be used. Adding parameters c and ε into (3.11), we get an extended system in $\mathbb{R}^4$

$$\begin{cases} u'_\eta = v, \\ v'_\eta = \dfrac{1}{\gamma}\left(u - \dfrac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right. \\ \qquad \left. - \dfrac{1}{2p+1}u^{2p+1}\right) \\ \qquad + \dfrac{\varepsilon\sqrt{c}}{\gamma^2}\left(u^{2p} + \dfrac{\alpha}{\sqrt{\beta c}}u^p - \left(1 + \dfrac{\gamma}{c}\right)\right)v \\ \qquad + O(\varepsilon^2), \\ \varepsilon'_\eta = 0, \\ c'_\eta = 0. \end{cases} \tag{3.12}$$

Our purpose is to seek homoclinic orbits for (3.12) with small ε. These will be found at values of c that depends on ε. From the original equation, we can see the origin

$$\begin{aligned} O(u,v) &= (u(c, \varepsilon), v(c, \varepsilon)) \\ &= (u(c, \varepsilon), 0) \\ &= \left(\frac{1}{\gamma}\left(u - \frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1} - \frac{1}{2p+1}u^{2p+1}\right), 0\right) \\ &= (0, 0), \end{aligned}$$

is a critical point for (3.4) and it is still a critical point and must lie on M_ε. We need to look for these orbits homoclinic to the origin O. The critical point O can be construed as a surface of critical point, say S, parametrized by c, ε, i.e., critical point $S = (u(c, \varepsilon), v(c, \varepsilon)) = (0, 0)$. This in turn spans an unstable manifold $W^u_\varepsilon(S)$ and stable manifold $W^s_\varepsilon(S)$ which meet in the curve at $\varepsilon = 0$, namely the homoclinic orbit found previously. Furthermore, by a simple proof, $W^u_\varepsilon(S)$ and $W^s_\varepsilon(S)$ must still cross hyperplane $v = 0$. In the set $v = 0$, we parametrize W^u and

W^s, respectively, near the intersection away from the critical point 0, as $u = h^-(c, \varepsilon)$ and $u = h^+(c, \varepsilon)$.

We next define

$$d(c, \varepsilon) := h^-(c, \varepsilon) - h^+(c, \varepsilon),$$

and observe that zeroes of d render homoclinic orbits. Since there are homoclinic orbits independently of c when $\varepsilon = 0$, we have $d(c, 0) = 0$, and thus let $d(c, \varepsilon) = \varepsilon \bar{d}(c, \varepsilon)$. Then we have

$$\bar{d}(c, 0) = M(c) = \left(\frac{\partial h^-}{\partial \varepsilon} - \frac{\partial h^+}{\partial \varepsilon} \right) \Big|_{\varepsilon=0}.$$

If there exists a (unique) value of $c = c(\varepsilon)$ for ε small, near to $c = c(\varepsilon)$, such that $d(c, \varepsilon) = 0$, that means if at $c = c(0)$,

$$M(c) = 0, \qquad M'(c) \neq 0, \tag{3.13}$$

hold, then it is a simple application of the Implication Function Theorem to see that there is a curve of homoclinic orbit.

The function $M(c)$ can be calculated explicitly as in [44] as

$$\begin{aligned} M(c) &= \frac{\partial h^-}{\partial \varepsilon} - \frac{\partial h^+}{\partial \varepsilon} \\ &= \frac{\sqrt{c}}{\gamma^2} \int_{-\infty}^{+\infty} \left(u^{2p} + \frac{\alpha}{\sqrt{\beta c}} u^p - \left(1 + \frac{\gamma}{c}\right) \right) v^2 \mathrm{d}\xi \\ &= \frac{\sqrt{c}}{\gamma^2} \int_{-\infty}^{+\infty} \left(u^{2p} + \frac{\alpha}{\sqrt{\beta c}} u^p \right) v^2 \mathrm{d}\xi \\ &\quad - \frac{\sqrt{c}}{\gamma^2} \left(1 + \frac{\gamma}{c}\right) \int_{-\infty}^{+\infty} v^2 \mathrm{d}\xi \\ &= \frac{\sqrt{c}}{\gamma^2} \left(\frac{u^{2p+1}}{2p+1} + \frac{\alpha}{\sqrt{\beta c}(p+1)} u^{p+1} \right) v \Big|_{-\infty}^{+\infty} \end{aligned}$$

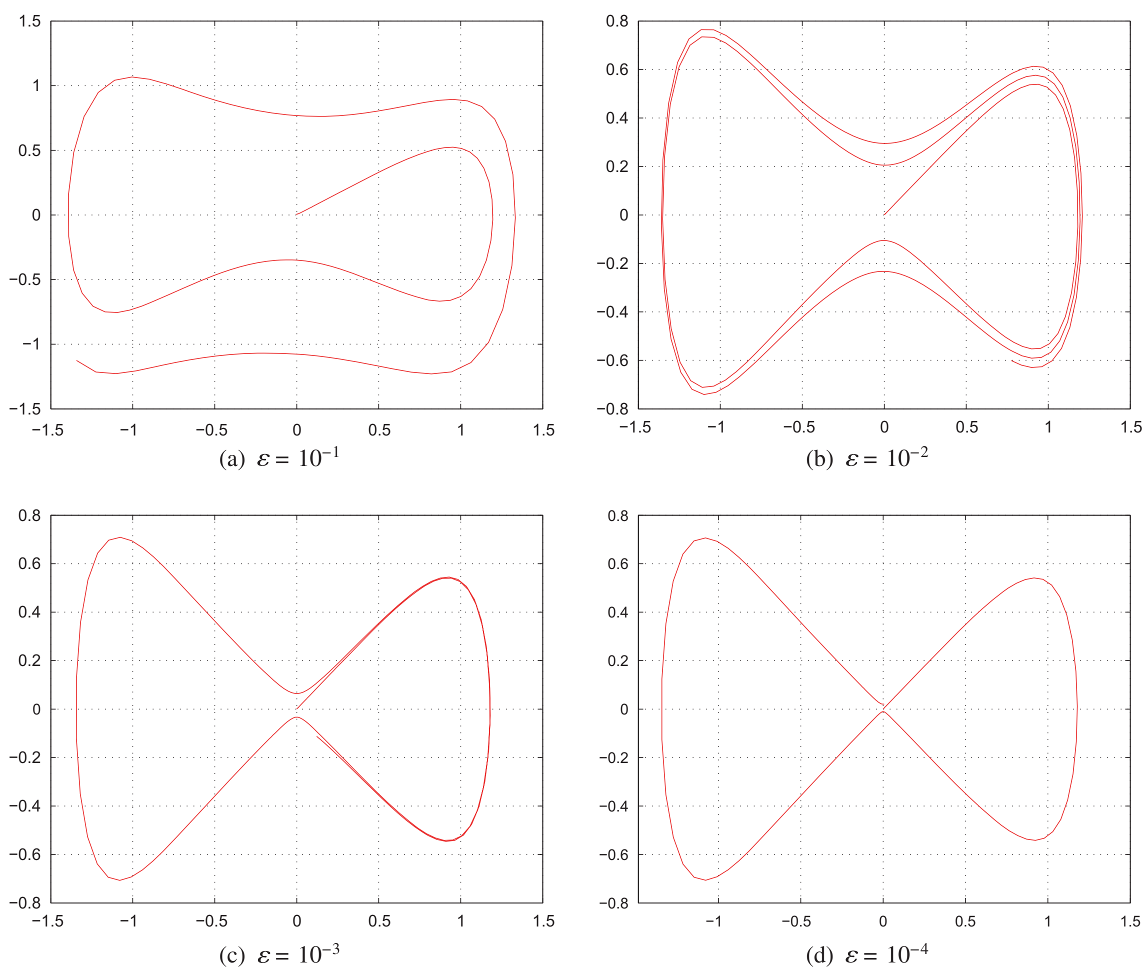


Figure 2. Persistence of homoclinic orbit of system (3.4) when ε varies. $\alpha = 1.0$, $\beta = 2.0$, $\gamma = 2.0$, $p = 3$, $c = 2.0$ and initial data $(u(0), v(0), w(0)) = (0, 0, 0.1)$.

$$
\begin{aligned}
&-\frac{\sqrt{c}}{\gamma^2}\int_{-\infty}^{+\infty}\left(\frac{u^{2p+1}}{2p+1}+\frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right)\ddot{u}\mathrm{d}\xi\\
&-\frac{\sqrt{c}}{\gamma^2}\left(1+\frac{\gamma}{c}\right)\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\\
&=\frac{\sqrt{c}}{\gamma^2}\int_{-\infty}^{+\infty}\left(u-\frac{u^{2p+1}}{2p+1}-\frac{\alpha}{\sqrt{\beta c}(p+1)}u^{p+1}\right)\ddot{u}\mathrm{d}\xi\\
&-\frac{\sqrt{c}}{\gamma^2}\int_{-\infty}^{+\infty}u\ddot{u}\mathrm{d}\xi-\frac{\sqrt{c}}{\gamma^2}\left(1+\frac{\gamma}{c}\right)\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\\
&=\frac{\sqrt{c}}{\gamma}\int_{-\infty}^{+\infty}\ddot{u}^2\mathrm{d}\xi-\frac{\sqrt{c}}{\gamma^2}\left(uv\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\right)\\
&-\frac{\sqrt{c}}{\gamma^2}\left(1+\frac{\gamma}{c}\right)\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\\
&=\frac{\sqrt{c}}{\gamma}\int_{-\infty}^{+\infty}\ddot{u}^2\mathrm{d}\xi-\frac{1}{\gamma\sqrt{c}}\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\\
&=\frac{\sqrt{c}}{\gamma}\left(\int_{-\infty}^{+\infty}\ddot{u}^2\mathrm{d}\xi-\frac{1}{c}\int_{-\infty}^{+\infty}\dot{u}^2\mathrm{d}\xi\right). \qquad (3.14)
\end{aligned}
$$

From (3.14), we can easily see that (3.13) holds at a unique value of c.

Therefore, we have the following existence result.

Theorem 3.1. *If $\varepsilon > 0$ is sufficiently small, the singularly perturbed generalized Gardner equation with nonlinear terms of any order, that is, eq.* (1.1) *has solitary wave solution.*

4. Numerical results

In this section, we numerically investigate the persistence of solitary wave solution to eq. (1.1) by solving an initial-value problem associated with system (3.4).

We take the initial data $(u(0), v(0), w(0)) = (0, 0, 0.1)$. The parameters $\alpha = 1.0$, $\beta = 2.0$, $\gamma = 2.0$, $p = 3$, the wave speed $c = 2.0$ and ε take the values 10^{-1}, 10^{-2}, 10^{-3} and 10^{-4}, respectively. Using the software *MATLAB* 7.0, we numerically investigate system (3.4) and present the results in figure 2. We can see from figure 2 that the homoclinic orbit persists when ε is small, while it breaks when ε becomes larger. This verifies our theoretical analysis in §3.

5. Conclusion

The solitary wave, as a typical nonlinear phenomenon, always plays an important role in the nature. In this paper, we have proved theoretically for the first time the existence of such elegantly coherent structure in the singularly perturbed generalized Gardner equation (1.1) when the perturbation parameter ε is sufficiently small. On the other hand, we have carried out numerical investigations to verify our theoretical analysis.

Our results indicate that eq. (1.1) with sufficiently small perturbation parameter ε can describe the propagation of ion acoustic waves or explain the propagation of thermal pulse through single crystal, just like the unperturbed generalized Gardner equation (1.2). Our work, from theoretical and numerical aspects, make it possible to find the solitary wave to (1.1) in experiment. We believe that model (1.1) is of much interest to the physicists in the field of plasma physics or solid physics.

Our results give a picture of the perturbed solution to model (1.1) for the first time. To work out such solution analytically or numerically will be a significant advance in the study of this nonlinear weakly dispersive model.

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