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Kac's ring: The case of four colours

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Abstract. We present an instance from nonequilibrium statistical mechanics which combines increase in entropy and finite Poincaré recurrence time. The model we consider is a variation of the well-known Kac's ring where we consider balls of four colours. As is known, Kac introduced this model where balls arranged between lattice sites, in each time step, move one step clockwise. The colour of the balls change as they cross marked sites. This very simple example rationalize the increase in entropy and recurrence. In our variation, the interesting quantity which counts the difference in the number of balls of different colours is shown to reduce to a set of linear equations if the probability of change of colour is symmetric among a pair of colours. The transfer matrix turns out to be non-Hermitian with real eigenvalues, leading to all colours being equally likely for long times, and a monotonically varying entropy. The new features appearing due to four colours is very instructive.

Keywords. Stosszahlansatz; H-theorem; nonequilibrium statistical mechanics.

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1. Introduction

Kac's ring is a simple and instructive model illustrating subtle aspects of nonequilibrium statistical mechanics, particularly the approach to equilibrium. One of the most significant aspects of its dynamics is that it makes us understand the explicit role played by the finite Poincaré recurrence time. This model, proposed by Kac [1] has been studied thoroughly by many people. It has been shown that the Boltzmann H function recurs every time interval of length $2N$ where N is the number of sites for the usual version where there are balls of only two colours. The model consists of N balls, each residing between successive lattice sites. Out of a total of N sites, there are a fraction, μ of randomly chosen sites that are marked. The discrete-time dynamics of the system consists of moving all the balls to the next site, each step in a clockwise manner. As the balls cross a marked site, the colour of the ball is changed. For the model with balls of two colours, the difference between their number fluctuates randomly about a fixed value. It was shown that the statistical fluctuations also recur with the time period, $2N$ [2]. Also, a recent pedagogical account $3N$ [3] was given which explained coarse-graining of Boltzmannian stosszahlansatz, ensemble averages, the difference between ensemble averaged and typical system behaviour, and the notion of entropy. This and many other studies make it very interesting to try to generalize the discussion to more than two colours. Although the generalization seems nearly trivial, it will be shown that there appear interesting, unanticipated technical difficulties or surprises when we consider four colours.

In this work, we consider the Kac's ring with balls of four colours, calling them red (R), yellow (Y), green (G), and blue (B). We shall consider a symmetric system in which the probabilities of a ball changing its colour from i to j is the same as for change from j to i. Let $R(t)$, $Y(t)$, $G(t)$, $B(t)$ denote the functions giving the number of balls of each colour at a time t; $r(t)$, $y(t)$, $g(t)$, $b(t)$ denote the functions giving the number of balls of each colour at a time t in front of a marker (which will change the colour). Further, we define six variables p, q, k, m, c and d denoting the probabilities of the process (change of colour):

$$p\colon Y \to R \text{ and } R \to Y,\quad q\colon G \to R \text{ and } R \to G$$
$$k\colon B \to R \text{ and } R \to B,\quad m\colon G \to Y \text{ and } Y \to G$$
$$c\colon B \to Y \text{ and } Y \to B,\quad d\colon G \to B \text{ and } B \to G.$$

The number of balls of different colours change with time according to the following equations:

$$
\begin{aligned}
R(t+1) &= R(t)+py(t)+qg(t)+kb(t)-r(t),\\
Y(t+1) &= Y(t)+cb(t)+mg(t)+pr(t)-y(t),\\
G(t+1) &= G(t)+qr(t)+my(t)+db(t)-g(t),\\
B(t+1) &= B(t)+cy(t)+kr(t)+dg(t)-b(t). \qquad (1)
\end{aligned}
$$

As indicated in the brief discussion made above, the differences in the number of balls of different colours is a quantity of great interest. In order to calculate $\Delta(t)$, we can make an assumption similar to stosszahlansatz, also employed in the problem with two colours. We suppose that the fraction of red, yellow, green or blue balls that change colour at a given time step is equal to the probability μ that a lattice site has a marker on it, where μ is equal to the number of markers divided by the number of sites. That is,

$$r/R = y/Y = g/G = b/B = \mu. \qquad (2)$$

Using (2), we can transform (1) as

$$
\begin{aligned}
R(t+1) &= R(t)(1-\mu)+\mu[pY(t)+qG(t)+kB(t)],\\
Y(t+1) &= Y(t)(1-\mu)+\mu[cB(t)+mG(t)+pR(t)],\\
G(t+1) &= G(t)(1-\mu)+\mu[qR(t)+mY(t)+dB(t)],\\
B(t+1) &= B(t)(1-\mu)+\mu[cY(t)+kR(t)+dG(t)]. \qquad (3)
\end{aligned}
$$

Now, we shall find all the differences between the number of balls of different colours and how these differences evolve. For instance, we have the difference between red and yellow balls at time, $(t+1)$, $\Delta_{RY}(t+1) = R(t+1)-Y(t+1)$ which on substitutions of $R(t+1)$ and $Y(t+1)$ gives

$$
\begin{aligned}
\Delta_{RY}(t+1) &= \Delta_{RY}(t)[1-\mu(1+p)]\\
&\quad + \mu[G(t)(q-m)+B(t)(k-c)]. \qquad (4)
\end{aligned}
$$

Similarly, for the other colours,

$$
\begin{aligned}
\Delta_{YG}(t+1) &= \Delta_{YG}(t)[1-\mu(1+m)]\\
&\quad + \mu[B(t)(c-d)+R(t)(p-q)],\\
\Delta_{GB}(t+1) &= \Delta_{GB}(t)[1-\mu(1+d)]\\
&\quad + \mu[R(t)(q-k)+Y(t)(m-c)],\\
\Delta_{BR}(t+1) &= \Delta_{BR}(t)[1-\mu(1+k)]\\
&\quad + \mu[Y(t)(c-p)+G(t)(d-q)],\\
\Delta_{RG}(t+1) &= (1-\mu)\Delta_{RG}(t)+\mu(p-m)Y(t)\\
&\quad + \mu(q+1)G(t)+\mu(k-d)B(t). \qquad (5)
\end{aligned}
$$

$\Delta_{YB}(t+1)$ will also involve three more terms in addition to $(1-\mu)\Delta_{YB}(t)$. To solve for all the differences, one of the ways is to try to write the equations as linear equations in the differences alone. In the next section, we show how to do this.

2. Exact coupled linear equations for Δ's

First, we need to rewrite the above equations in a different form and manipulate them further. Adding and subtracting $B(t)(q-m)$ in the expression of $\Delta_{RY}(t+1)$, we have

$$
\begin{aligned}
\Delta_{RY}(t+1) &= [1-\mu(1+p)]\Delta_{RY}(t)\\
&\quad + \mu(q-m)\Delta_{GB}(t)\\
&\quad + b(t)(q-m+k-c). \qquad (6)
\end{aligned}
$$

Similarly, we need to add and subtract $G(t)(c-p)$ in the expression of $\Delta_{BR}(t+1)$, $R(t)(m-c)$ in the expression of $\Delta_{GB}(t+1)$, $B(t)(p-q)$ in the expression of $\Delta_{YG}(t+1)$. Thus,

$$
\begin{aligned}
\Delta_{BR}(t+1) &= [1-\mu(1+k)]\Delta_{BR}(t)\\
&\quad + \mu(c-p)\Delta_{YG}(t)\\
&\quad + g(t)(c-p+d-q), \qquad (7)\\
\Delta_{GB}(t+1) &= [1-\mu(1+d)]\Delta_{GB}(t)\\
&\quad - \mu(m-c)\Delta_{RY}(t)\\
&\quad + r(t)(q-k+m-c), \qquad (8)\\
\Delta_{YG}(t+1) &= [1-\mu(1+m)]\Delta_{YG}(t)\\
&\quad - \mu(p-q)\Delta_{BR}(t)\\
&\quad + b(t)(c-d+p-q). \qquad (9)
\end{aligned}
$$

Now, the equations for $\Delta_{RY}(t+1)$, $\Delta_{BR}(t+1)$, $\Delta_{GB}(t+1)$ and $\Delta_{YG}(t+1)$ can be further rewritten as follows:

$$
\begin{aligned}
\Delta_{BR}(t+1) &= [1-\mu(1+k)]\Delta_{BR}(t)\\
&\quad + \mu(c-d)\Delta_{YG}(t)+y(t)(c-p+d-q),\\
\Delta_{RY}(t+1) &= [1-\mu(1+p)]\Delta_{RY}(t)\\
&\quad - \mu(k-c)\Delta_{GB}(t)+g(t)(q-m+k-c),\\
\Delta_{GB}(t+1) &= [1-\mu(1+d)]\Delta_{GB}(t)\\
&\quad + \mu(q-k)\Delta_{RY}(t)+y(t)(q-k+m-c),\\
\Delta_{YG}(t+1) &= [1-\mu(1+m)]\Delta_{YG}(t)\\
&\quad + \mu(c-d)\Delta_{BR}(t)+r(t)(c-d+p-q). \qquad (10)
\end{aligned}
$$

Equations for $\Delta_{BY}(t+1)$ and $\Delta_{RG}(t+1)$:

$$\begin{aligned}
\Delta_{BY}(t+1) &= \Delta_{BR}(t+1) - \Delta_{RY}(t+1)\\
&= \Delta_{BR}(t) - \mu(1+k)\Delta_{BR}(t)\\
&\quad + \mu(c-p)Y(t) + \mu(d-q)G(t)\\
&\quad - \Delta_{RY}(t) + \mu(1+p)\Delta_{RY}(t)\\
&\quad - \mu(q-m)G(t) - \mu B(t)(k-c)\\
&= \Delta_{BY}(t) - \mu[\Delta_{BR}(t) + k\Delta_{BR}(t)\\
&\quad - \Delta_{RY}(t) - p\Delta_{RY}(t)]\\
&\quad + \mu Y(t)(c-p) - \mu B(t)(k-c)\\
&\quad + \mu G(t)(d-2q+m)\\
&= \Delta_{BY}(t) - \mu[\Delta_{BY}(t) + k\Delta_{BR}(t)\\
&\quad - p\Delta_{RY}(t)] + \mu[Y(t)(c-p)\\
&\quad - B(t)(k-c) + G(t)(d-2q+m)]\\
&= \Delta_{BY}(t) - \mu[B(t) - Y(t) + kB(t)\\
&\quad - kR(t) - pR(t) + pY(t) + Y(t)(c-p)\\
&\quad - B(t)(k-c) + G(t)(d-2q+m)]\\
&= \Delta_{BY}(t) - \mu[B(t)(1+c) - Y(t)(1+c)\\
&\quad - R(t)(k+p) + G(t)(d-2q+m)]\\
&= \Delta_{BY}(t) - \mu(1+c)\Delta_{B}Y(t)\\
&\quad + \mu[R(t)(k+p) - G(t)(d-2q+m)]\\
&= \Delta_{BY}(t)[1-\mu(1+c)] + \mu[R(t)(k+p)\\
&\quad - G(t)(d-2q+m)]. \qquad (11)
\end{aligned}$$

This can be rewritten after simple manipulations in the following form:

$$\begin{aligned}
\Delta_{BY}(t+1) &= \Delta_{BY}(t)[1-\mu(1+c)]\\
&\quad + \mu(d+m-2q)\Delta_{RG}(t)\\
&\quad + r(t)(k+p+d+m-2q). \qquad (12)
\end{aligned}$$

Similar manipulations can be carried out for $\Delta_{RG}(t+1)$, resulting in

$$\begin{aligned}
\Delta_{RG}(t+1) &= \Delta_{RG}(t)[1-\mu(1+q)]\\
&\quad - \mu(d+k-2c)\Delta_{BY}(t)\\
&\quad + y(t)(p+m-d-k+2c)\\
&= \Delta_{RG}(t)[1-\mu(1+q)]\\
&\quad - \mu(m+p)\Delta_{BY}(t)\\
&\quad + b(t)(p+m-d-k+2c). \qquad (13)
\end{aligned}$$

We can invoke the conditions: $k = m$, $p = d$, $c = q$ in $\Delta_{RG}(t+1)$:

$$\begin{aligned}
\Delta_{RG}(t+1) &= \Delta_{RG}(t)[1-\mu(1+c)]\\
&\quad - \mu(p+k-2c)\Delta_{BY}(t) + 2cy(t),\\
&= \Delta_{RG}(t)[1-\mu(1+c)]\\
&\quad - \mu(k+p)\Delta_{BY}(t) + 2cb(t). \qquad (14)
\end{aligned}$$

In the same manner, other Δ's can be rewritten as

$$\begin{aligned}
\Delta_{BY}(t+1) &= \Delta_{BY}(t)[1-\mu(1+c)]\\
&\quad + \mu(k+p)\Delta_{RG}(t) + 2cg(t),\\
&= \Delta_{BY}(t)[1-\mu(1+c)]\\
&\quad + \mu(k+p-2c)\Delta_{RG}(t)\\
&\quad + 2(k+p-c)r(t), \qquad (15)\\
\Delta_{RY}(t+1) &= \Delta_{RY}(t)[1-\mu(1+p)]\\
&\quad + \mu(c-k)\Delta_{GB}(t), \qquad (16)\\
\Delta_{BR}(t+1) &= \Delta_{BR}(t)[1-\mu(1+k)]\\
&\quad + \mu(c-p)\Delta_{YG}(t), \qquad (17)\\
\Delta_{GB}(t+1) &= \Delta_{GB}(t)[1-\mu(1+p)]\\
&\quad - \mu(k-c)\Delta_{RY}(t), \qquad (18)\\
\Delta_{YG}(t+1) &= \Delta_{YG}(t)[1-\mu(1+k)]\\
&\quad - \mu(p-c)\Delta_{BR}(t). \qquad (19)
\end{aligned}$$

Now, we observe that the extra inhomogeneous terms $2cb(t)$ and $2cg(t)$ still remain in the equations for $\Delta_{RG}(t+1)$ and $\Delta_{BY}(t+1)$. We simplify further by subtracting $\Delta_{BY}(t+1)$ from $\Delta_{RG}(t+1)$:

$$\begin{aligned}
\Delta_{RG}(t+1) - \Delta_{BY}(t+1) &= [\Delta_{RG}(t) - \Delta_{BY}(t)]\\
&\quad \times [1-\mu(1+c+k+p)] + 2c[b(t) - g(t)]. \qquad (20)
\end{aligned}$$

Denoting $\Delta_{RG} - \Delta_{BY} = \beta$ and substituting $b(t) = \mu B(t)$ and $g(t) = \mu G(t)$ in (20), we get

$$\begin{aligned}
\beta(t+1) &= \beta(t)[1-\mu(1+c)] + \beta(t)\mu(p+k)\\
&\quad - 2c\mu\Delta_{GB}(t). \qquad (21)
\end{aligned}$$

We have therefore reduced the equations for the difference in colours into a set of equations which can be cast as a matrix equation:

$$\begin{bmatrix} \Delta_{RY}(t+1)\\ \Delta_{BR}(t+1)\\ \Delta_{GB}(t+1)\\ \Delta_{YG}(t+1)\\ \beta(t+1) \end{bmatrix} = \begin{bmatrix} M_{11} & 0 & M_{13} & 0 & 0\\ 0 & M_{21} & 0 & M_{24} & 0\\ M_{13} & 0 & M_{11} & 0 & 0\\ 0 & M_{24} & 0 & M_{21} & 0\\ 0 & 0 & -2\mu c & 0 & M_{55} \end{bmatrix} \times \begin{bmatrix} \Delta_{RY}(t)\\ \Delta_{BR}(t)\\ \Delta_{GB}(t)\\ \Delta_{YG}(t)\\ \beta(t) \end{bmatrix}, \qquad (22)$$

where $M_{11} = [1-\mu(1+p)]$, $M_{13} = \mu(c-k)$, $M_{21} = [1-\mu(1+k)]$, $M_{24} = \mu(c-p)$, $M_{55} = [1-\mu(1+c-p-k)]$.

Eigenvalues of the matrix are

$$\lambda_1 = 1-\mu[1-c+k+p], \quad \lambda_2 = 1-\mu[1+c+k-p],$$
$$\lambda_3 = 1-\mu[1-c+k+p], \quad \lambda_4 = 1-\mu[1+c-k+p],$$
$$\lambda_5 = 1-\mu[1+c-k-p]. \tag{23}$$

Eigenvectors corresponding to the above eigenvalues are

$$\lambda_{1e} = (0, 1, 0, 1, 0), \qquad \lambda_{2e} = (0, -1, 0, 1, 0),$$
$$\lambda_{3e} = (((p+k)/c) - 1, 0, ((k+p)/c) - 1, 0, 1),$$
$$\lambda_{4e} = (-p/c, 0, p/c, 0, 1), \qquad \lambda_{5e} = (0, 1, 0, 1, 0). \tag{24}$$

Now, eigenvalue equations are as follows:

$$\begin{aligned} \psi_1(t) &= \Delta_{BR}(t) + \Delta_{YG}(t), \\ \psi_2(t) &= \Delta_{YG}(t) - \Delta_{BR}(t), \\ \psi_3(t) &= [((p+k)/c) - 1]\Delta_{RY}(t) \\ &\quad + [((p+k)/c) - 1]\Delta_{GB}(t) + \beta(t), \\ \psi_4(t) &= (p/c)[\Delta_{GB}(t) - \Delta_{RY}(t)] + \beta(t), \\ \psi_5(t) &= \beta(t) = \Delta_{RG}(t) - \Delta_{BY}(t). \end{aligned} \tag{25}$$

Adding $\psi_1(t)$ and $\psi_2(t)$, we have

$$\Delta_{YG}(t) = [\psi_1(t) + \psi_2(t)]/2. \tag{26}$$

Subtracting $\psi_2(t)$ from $\psi_1(t)$, we have

$$\Delta_{BR}(t) = [\psi_1(t) - \psi_2(t)]/2. \tag{27}$$

Adding $\psi_3(t)$ and $\psi_4(t)$, we have

$$\begin{aligned} \psi_3(t) + \psi_4(t) &= \Delta_{RY}(t)[(k/c) - 1] \\ &\quad + \Delta_{GB}(t)[((2p+k)/c) - 1] + 2\beta(t). \end{aligned} \tag{28}$$

Subtracting $\psi_4(t)$ from $\psi_3(t)$, we have

$$\begin{aligned} &\Delta_{RY}(t)[((2p+k)/c) - 1] + \Delta_{GB}(t)[(k/c) - 1] \\ &\quad = \psi_3(t) - \psi_4(t). \end{aligned} \tag{29}$$

To see how $\psi_i(t)$, $i = 1, 2, ..., 5$ behaves with time, we need to specify the initial conditions in terms of initial numbers of coloured balls. Specifically, we have

$$\begin{aligned} \psi_1(0) &= B(0) - R(0) + Y(0) - G(0), \\ \psi_2(0) &= Y(0) - G(0) + R(0) - B(0), \\ \psi_3(0) &= \left(\frac{p+k}{c}\right)[R(0) - B(0)] \\ &\quad - \left[\left(\frac{p+k}{c}\right) - 2\right][G(0) - Y(0)], \\ \psi_4(0) &= \left(1 - \frac{p}{c}\right)[R(0) - G(0)] \\ &\quad + \left(1 + \frac{p}{c}\right)[Y(0) - B(0)], \\ \psi_5(0) &= R(0) - G(0) + Y(0) - B(0). \end{aligned} \tag{30}$$

The behaviour of $\psi_i(t)$ as a function of time for different values of parameters is shown in figure 1.

So we have established that beginning with balls of different colours, and some probabilities for changes of colours occurring across the marked sites, eventually, the number of balls of different colours become equal. We can define the Boltzmann H-function which varies monotonically with time. For the cases in figure 1, entropy is defined as

$$S(t) = -\sum_j \psi_j(t) \log \psi_j(t). \tag{31}$$

The monotonic variation of $S(t)$ for the cases considered in figure 1 is shown convincingly in figure 2.

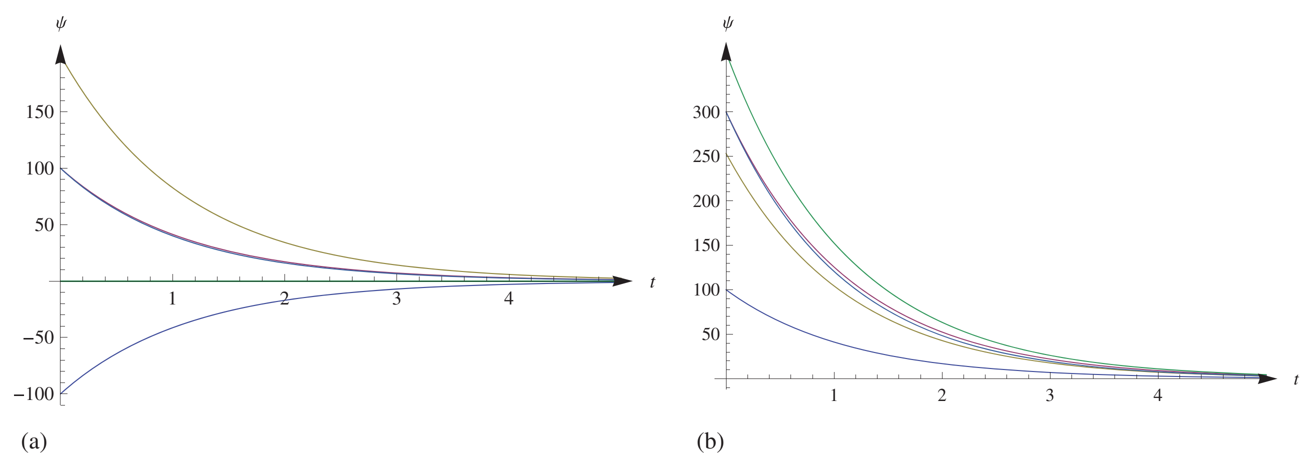

Figure 1. From the left, we present the following cases: (**a**) The probabilities are equal, p, k, c, q, m, d are equal to $1/6$, the fraction of marked sites is $\mu = 1/10$. Initial values are $R(0) = 100$, $Y(0) = G(0) = B(0) = 0$; (**b**) The probabilties are $p = 1/6$, $k = 1/5$, $c = 1/4$, $\mu = 1/10$. Balls are initially $R(0) = 100$, $Y(0) = 200$, $G(0) = B(0) = 0$. In both cases, we see that all the colours become equally likely with time.

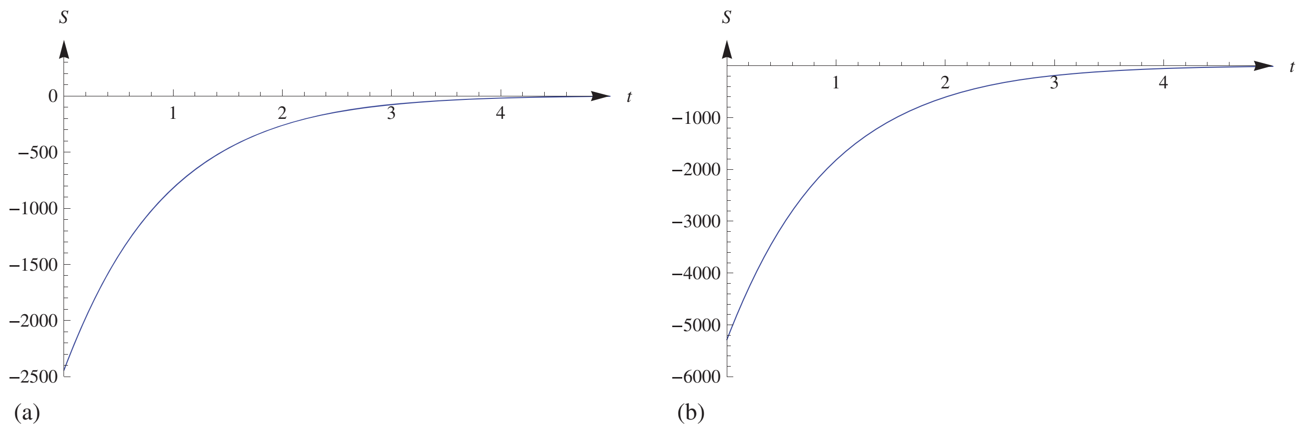

Figure 2. Entropy is plotted as a function of time. The monotonic variation of entropy is seen although with four colours and finite number of sites, there must be Poincaré recurrence.

For the case of two colours, it is clear that the Poincaré recurrence time is two times the number of sites, N. As recurrence in the case of two colours is exact, the entropy becomes a periodic function of time with period, $2N$. However, in the present case where we have four colours, the recurrence time will be much longer. What is indeed instructive is the fact that by just adding two more colours, the degree of complexity is increased many-fold. Recently, time evolution of entanglement $4N$ [4] in a quantum version of the Kac's ring was studied where they have replaced classical markers by two spin chains and quantum gates. In this model, the entanglement evolution was understood by considering the ensemble of Kac's rings. The model thus eludicated the relation between distribution of measurement results in classical and quantum systems. It would be interesting to develop a quantum version for the case of four colours. It is worth recalling that for this case the transfer matrix turned out to be non-Hermitian, possessing real eigenvalues. This mathematical twist is also owing to the increase in colours. To conclude, we have seen an interesting development of the classical states under Boltzmann equation with stosszahlansatz for Kac's ring with four colours.

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