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Dynamics of rogue waves on multisoliton background in the Benjamin Ono equation

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Abstract. For the Benjamin Ono equation, the Hirota bilinear method and long wave limit method are applied to obtain the breathers and the rogue wave solutions. Bright and dark rogue waves exist in the Benjamin Ono equation, and their typical dynamics are analysed and illustrated. The semirational solutions possessing rogue waves and solitons are also obtained, and demonstrated by the three-dimensional figures. Furthermore, the hybrid of rogue wave and breather solutions are also found in the Benjamin Ono equation.

Keywords. Benjamin Ono equation; rogue wave; Hirota bilinear method; hybrid solution.

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1. Introduction

In recent years, the phenomena of rogue waves, that *appear from nowhere and disappear without a trace* [1], have generated a hot debate on both experimental observation and theoretical analysis in different fields including Bose–Einstein condensates [2,3], optical system [4–6], ocean [7], superfluids [8], plasma [9,10] and so on. The simplest rogue wave (i.e., the first-order rogue waves) solution of the NLS equation was first discovered by Peregrine [11]. The amplitude of the first-order rogue waves reaches three times the height of the background, and then decays algebraically to the background finally. Recently, different kinds of rogue waves in the NLS equation have been shown in other articles [12–16]. What is more, the hierarchy of rogue wave solutions for other soliton equations have also been reported in refs [17–22], demonstrating that the higher-order rogue waves are also localized in both space and time and can exhibit higher main peak in fundamental patterns. For example, the maximum amplitude of the n-order rogue waves of the NLS equation is $2n+1$ times the background. Two recent articles [23,24] have provided a good review on rogue waves from the physical point of view. Besides, the research about the rogue wave solutions of generating nonlinear evolution equations in different branches of science is an interesting topic. Especially, the interaction between rogue waves and solitons or breathers is a very interesting and important topic, and great research has been done on them [25–28]. But most of them are complex systems, and to the author's best knowledge, research on the interaction between rogue waves and solitons or breathers has not been done before.

In this paper, we focus on the Benjamin Ono (BO) equation

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0, \tag{1}$$

where β and γ are real non-zero constants. The BO equation is one of the important nonlinear equation in physics [29,30]. By means of travelling wave method, the exact solutions of the BO equation were obtained. Based on an improved projective Riccati equation method, the travelling wave solutions of single variable were found [31]. Applying the F-expansion method and the Jacobi elliptic function expansion method to the BO equation, a series of periodic wave solutions was obtained [32]. Applying the bilinear method and extended homoclinic test approach [33–36], first-order rogue waves, periodic solitary waves and doubly periodic solutions for the BO equation were obtained [37].

The outline of the paper is as follows: In §2, the bright rogue waves and dark rogue waves of the BO equation are obtained by the Hirota linear method and long wave limit, and their typical dynamics are analysed and illustrated. In §3, the dynamics of the interaction between rogue waves and solitons have been demonstrated, and the hybrids of rogue waves and breathers have been shown. Section 4 contains summary and discussion.

2. The rogue wave solutions of the Benjamin Ono equation

In this section, we present rational solutions of the Benjamin Ono equation. Before that, we have to obtain Nth-order solitons and breather solutions of the Benjamin Ono equation. Equation (1) can be transformed into the bilinear form

$$(D_t^2 + 2\beta u_0 D_x^2 + \gamma D_x^4) f \cdot f = 0, \tag{2}$$

through the dependent variable transformation

$$u = u_0 + \frac{6\gamma}{\beta}(\ln f)_{xx}. \tag{3}$$

Here u_0, β and γ are real constants, f is a real function with respect to variables x and t, and the operator D is the Hirota's bilinear differential operator [38] defined by $P(D_x, D_y, D_t)F(x, y, t, \ldots) \times G(x, y, t, \ldots) = P(\partial_x - \partial_{x'}, \partial_y - \partial_{y'}, \partial_t - \partial_{t'}, \ldots) \times F(x, y, t, \ldots)G(x', y', t', \ldots)|_{x'=x, y'=y, t'=t}$, where P is a polynomial of $D_x, D_y, D_t, \ldots$.

By the Hirota direct method [38], the Nth-order soliton solutions of eq. (1) can be obtained as

$$u = u_0 + \frac{6\gamma}{\beta}(\ln f)_{xx}, \tag{4}$$

where

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{N} \mu_i \mu_j A_{ij} + \sum_{i=1}^{N} \mu_i \eta_i\right) \tag{5}$$

and

$$\exp(A_{ij}) = -\frac{\gamma(p_i - p_j)^4 + 2u_0\,\beta(p_i - p_j)^2 + (k_i - k_j)^2}{\gamma(p_i + p_j)^4 + 2u_0\,\beta(p_i + p_j)^2 + (k_i + k_j)^2},$$

$$\eta_i = p_i x + k_i t + \eta_{0i}, \quad k_i = \sqrt{-p_i^2\,\gamma - 2\beta\mu_0\, p_i} \tag{6}$$

Here N, p_i and η_{0i} are arbitrary constants.

To get first-order breather solutions in eq. (1), applying similar transformation to the soliton solutions in the BO equation as in ref. [39], and taking parameter constraints

$$N = 2, \quad p_1 = p_2^*, \quad \eta_{01} = \eta_{02}^*, \tag{7}$$

two types of corresponding breather solutions (i.e., bright breathers and dark breathers) are obtained which are shown in figure 1.

As first-order rogue waves can be treated as the limitation of first-order breathers, two types of rogue waves (i.e., bright rogue waves and dark rogue waves) should exist in eq. (1). To get the first-order rational solutions, we use the long-wave limit on second-order soliton solutions in (4) and (5). Putting $N = 2$, $p_1 = \alpha_1\epsilon$, $p_2 = \alpha_2\epsilon$, $\eta_{01} = i\pi$, $\eta_{02} = -i\pi$ and taking the limits $\epsilon \to 0$, we obtain the following rational solutions of eq. (1):

$$u = u_0 - \frac{24u_0\gamma(-4u_0^2\beta t^2 + 2u_0\beta x^2 + 3\gamma)}{(4u_0^2\beta^2 t^2 + 2u_0\beta x^2 - 3\gamma)^2}. \tag{8}$$

It is easy to find that the rational solutions u (8) is nonsingular when $\gamma < 0$, $u_0\beta > 0$. To discuss those nonsingular rational solutions u (8) further, we present the critical points of u:

$$A_1 = (x_1, t_1) = (0, 0), \quad A_2 = (x_2, t_2) = \left(3\sqrt{-\frac{\gamma}{2\beta u_0}}, 0\right),$$

$$A_3 = (x_3, t_3) = \left(-3\sqrt{-\frac{\gamma}{2\beta u_0}}, 0\right).$$

Setting

$$H(x, t) = \frac{\partial^2 u^2}{\partial x^2}, \quad \Delta(x, t) = \frac{\partial^2 u^2}{\partial x^2}\frac{\partial^2 u^2}{\partial t^2} - \left(\frac{\partial^2 u^2}{\partial x \partial t}\right)^2,$$

then

$$H(x_1, t_1) = -\frac{32u_0^2\beta}{\gamma}, \quad \Delta(x_1, t_1) = \frac{2048}{3}\frac{u_0^5\beta^3}{\gamma^2},$$

$$H(x_2, t_2) = H(x_3, t_3) = \frac{u_0^2\beta}{\gamma},$$

$$\Delta(x_2, t_2) = \Delta(x_3, t_3) = \frac{8}{3}\frac{u_0^5\beta^3}{\gamma^2}.$$

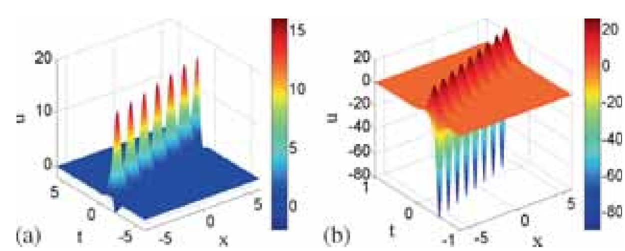


Figure 1. Two types of the first-order breather solutions u of eq. (1): **(a)** The bright breathers with $\eta_{01} = 0$, $\eta_{02} = 0$, $p_1 = 1 + 4i$, $p_2 = 1 - 4i$, $u_0 = 0$, $\beta = -1$, $\gamma = -1$ and **(b)** the dark breathers with $\eta_{01} = 0$, $\eta_{02} = 0$, $p_1 = 1 + 4i$, $p_2 = 1 - 4i$, $u_0 = 0$, $\beta = 1$, $\gamma = -1$.

According to the above discussions on A_i ($1 \le i \le 3$), H and Δ, those nonsingular rational solutions u (8) can be classified into two patterns:

(a) The bright rogue waves: when $u_0 < 0$, $\beta < 0$, $\gamma < 0$, u has one global maximum point (point A_1) and two global minimum points (point A_2 and point A_3), the maximum value of u is $-7u_0$, and the minimum value of u is $2u_0$.

(b) The dark rogue waves: when $u_0 > 0$, $\beta > 0$, $\gamma < 0$, u has two global maximum points (point A_2 and point A_3), and one global minimum point (point A_1), the maximum value of u is $2u_0$, and the minimum value of u is $-7u_0$.

Figures 2 and 3 show two types of rogue waves in eq. (1). Obviously, the parameter u_0 controls the height of the rogue waves and this property is demonstrated vividly in figure 2, and the parameters γ and β also have a close relationship with localization characters, which are shown in figure 3. It is noticed that the height and the proportion of the first-order rogue waves in eq. (1) could be arbitrary, that is very different from the localization character of complex MKDV equation discussed by He *et al* [40] which is constant. This may be a big difference between rogue waves in complex systems and real systems.

3. The hybrid solutions of the Benjamin Ono equation

It may be interesting to examine the interaction between the rogue wave solutions and the solitons. First, we

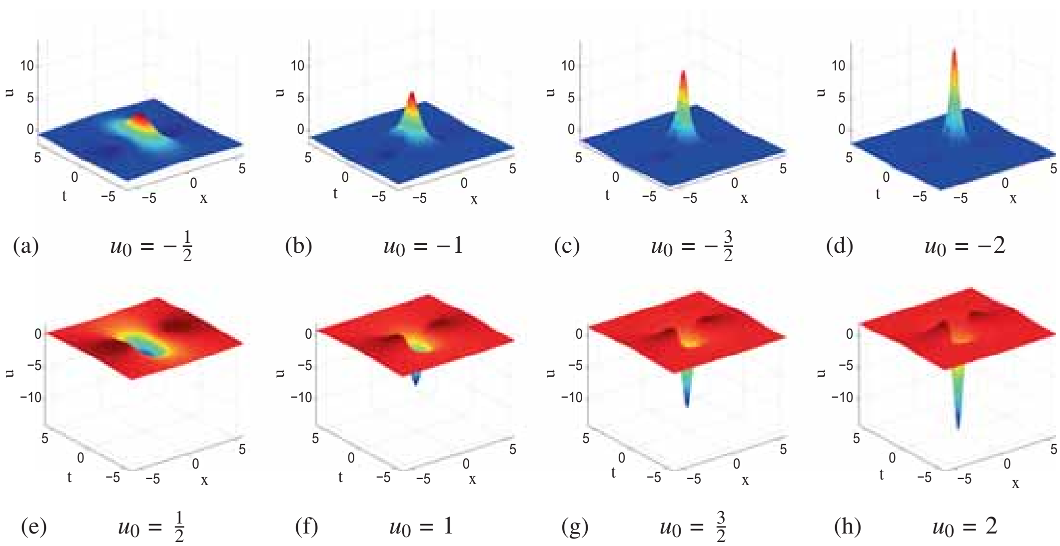


Figure 2. Two types of rogue waves (8) with $\gamma = -1$. Top row: first-order bright rogue waves with $\beta = -1$ and different u_0; bottom row: first-order dark rogue waves with $\beta = 1$ and different u_0.

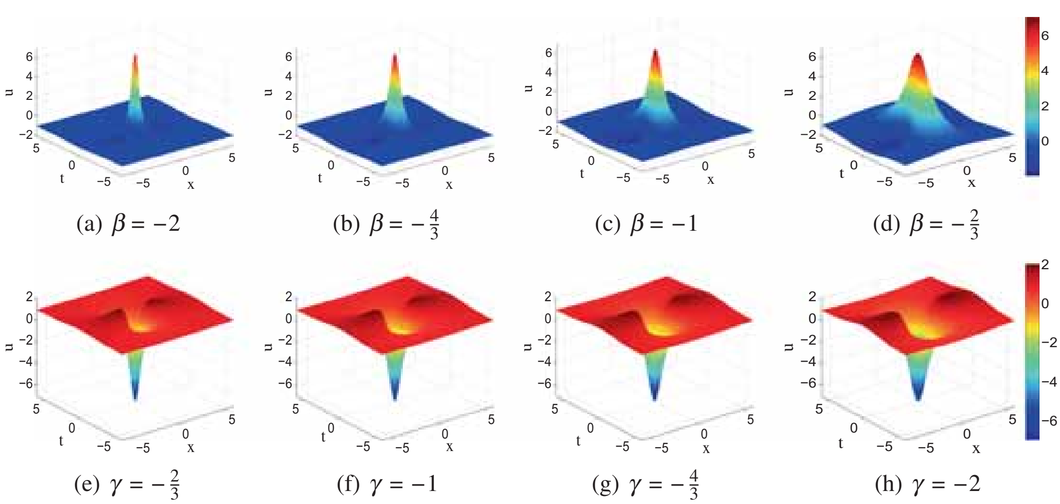


Figure 3. Two types of rogue waves (8). Top row: first-order bright rogue waves with $u_0 = -1$, $\gamma = -1$ and different β; bottom row: first-order dark rogue waves with $u_0 = 1$, $\beta = 1$ and different γ.

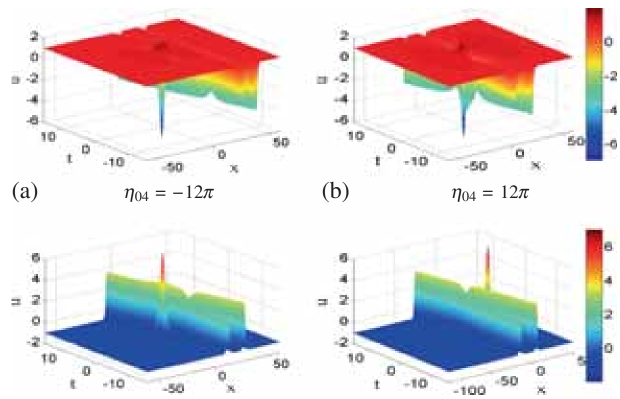
consider a solution consisting of a first-order rogue wave and a first-order soliton obtained from the conventional three-soliton solutions, where p_i, η_{0i} are taken in the same way for $i = 1, 2$ as the derivation of eq. (8). For simplicity, setting $p_3 = \sqrt{-(2\beta u_0/\gamma)}$, $\beta = u_0 = \pm 1$, $\gamma = -1$, we obtain the following equation:

$$f = (4t^2+2x^2+3)+(4t^2+2(x-3\sqrt{2})^2+3)\mathrm{e}^{\sqrt{2}x+\eta_{03}} \tag{9}$$

and the corresponding solutions describe a first-order rogue wave on a first-order soliton background as shown in figure 4. As can be seen, when rogue waves interact with the soliton, amplitudes of the soliton and the rogue wave change. For the bright rogue waves on the bright soliton background, when soliton travels closer to the soliton, a higher peak appears on the soliton (see figure 4b). When the rogue wave is located on the soliton, the amplitudes of the rogue waves become lower while the peaks on the soliton become higher (see figure 4c). When soliton and rogue waves are separated completely, the peak on the soliton disappears keeping the amplitude of the rogue wave constant. In the case of dark rogue waves on the dark soliton background, when rogue wave is located on the dark soliton background, the amplitude of the rogue wave becomes higher and the peak on the soliton also becomes higher (see figure 4g). But, when the rogue wave travels far from the soliton, the soliton and the rogue wave remain in the original state, i.e., peaks on the soliton and amplitude of the rogue wave will not be constant.

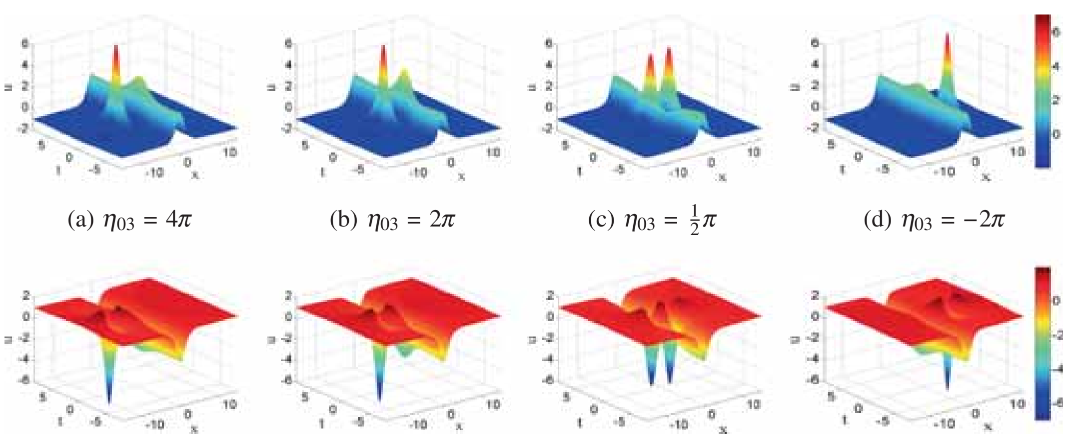
Apparently, the parameter η_{03} controls the shift of the soliton in space and time, and both rogue waves and solitons have the phase shift on the collision area. The existence of the phase shifts is a distinctive phenomenon and their occurrence would be catastrophic in physical systems.

Secondly, taking p_i, η_{0i} in the same way for $i = 1, 2$ as the derivation of eq. (8) and keeping p_3, p_4 real parameters, we obtain the hybrid of a first-order rogue wave and a second-order soliton from the fourth-order soliton solutions. As shown in figure 5, these hybrid solutions have qualitatively similar behaviours, except that more permanent soliton line waves interact with the localized rogue waves, and more complicated wave fronts will be formed in the interaction region.



(a) $\eta_{04} = -12\pi$ (b) $\eta_{04} = 12\pi$ (c) $\eta_{04} = -12\pi$ (d) $\eta_{04} = 12\pi$

Figure 5. The dynamics of the interaction between the rogue waves and the second-order solitons. Top row: the hybrid of bright rogue wave and second-order bright solitons with $p_3 = \sqrt{-(2\beta u_0/\gamma)}$, $p_4 = \sqrt{2}$, $u_0 = -1$, $\beta = -1$, $\gamma = -1$, $\eta_{03} = \eta_{04}$ and different η_{04}; bottom row: the hybrid of dark rogue waves and first-order dark solitons with $p_3 = \sqrt{-(2\beta u_0/\gamma)}$, $p_4 = \sqrt{2}$, $u_0 = 1$, $\beta = 1$, $\gamma = -1$, $\eta_{03} = \eta_{04}$ and different η_{04}.



(a) $\eta_{03} = 4\pi$ (b) $\eta_{03} = 2\pi$ (c) $\eta_{03} = \frac{1}{2}\pi$ (d) $\eta_{03} = -2\pi$

(e) $\eta_{03} = 4\pi$ (f) $\eta_{03} = 2\pi$ (g) $\eta_{03} = \frac{1}{2}\pi$ (h) $\eta_{03} = -2\pi$

Figure 4. Two types of hybrid solutions of eq. (1). Top row: the hybrid of bright rogue waves and first-order bright solitons with $p_3 = \sqrt{-(2\beta u_0/\gamma)}$, $u_0 = -1$, $\beta = -1$, $\gamma = -1$ and different η_{03}; bottom row: the hybrid of dark rogue waves and first-order dark solitons with $p_3 = \sqrt{-(2\beta u_0/\gamma)}$, $u_0 = 1$, $\beta = 1$, $\gamma = -1$ and different η_{03}.

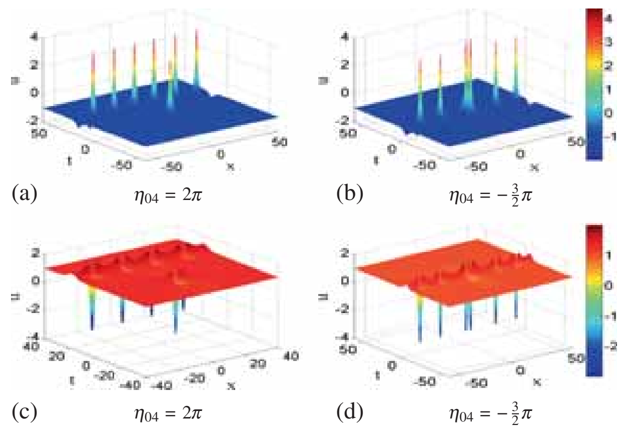


Figure 6. The dynamics of the interaction between rogue waves and breathers. Top row: the hybrid of bright rogue waves and first-order bright breathers with $p_3 = \frac{i}{2}$, $p_4 = -\frac{i}{2}$, $u_0 = -1$, $\beta = -1$, $\gamma = -1$, $\eta_{03} = \eta_{04}$ and different η_{04}; bottom row: the hybrid of dark rogue waves and first-order dark breathers with $p_3 = \frac{i}{2}$, $p_4 = -\frac{i}{2}$, $u_0 = 1$, $\beta = 1$, $\gamma = -1$, $\eta_{03} = \eta_{04}$ and different η_{04}.

Lastly, taking p_i, η_{0i} in the same way for $i = 1, 2$ as the derivation of eq. (8) and keeping the parameters p_3, p_4 satisfied ($p_3^* = p_4$, $\eta_{03} = \eta_{04}$), we can obtain the hybrid solutions consisting of first-order rogue waves and first-order breathers from the fourth-order soliton solutions. For example, with parameters $p_3 = \frac{i}{2}$, $p_4 = -\frac{i}{2}$, the corresponding solutions are shown in figure 6. As can be seen, the interaction between the dark rogue waves and the dark breathers and interaction between the bright rogue waves and the bright breathers are demonstrated, and the transient solution patterns become more intricate and funny. Specially, when the breather moves closer to the rogue wave, the rogue wave is completely immersed into the breather, and these solutions feature a breather possessing different wave patterns (see figures 6b and 6d).

These hybrid solutions may have been reported in many multicomponent systems, but to the author's best knowledge, was never reported in a single-component real system before. It is a pity that we cannot obtain higher-order rogue waves in eq. (1). We hope we can get higher-order rogue waves interacting with higher-order solitons in future.

4. Summary and discussion

In summary, the dynamics of both bright rogue waves and dark rogue waves on multisoliton background in the Benjamin Ono equation are demonstrated in this paper. Firstly, N-th order soliton solutions of the Benjamin Ono equation are obtained by the Hirota bilinear method. And then, by the long wave limit on the second-order soliton solutions, the bright and dark rogue waves are obtained, and typical dynamics of the obtained rogue waves are analysed and illustrated. Furthermore, the hybrid solution of rogue waves and first-order solitons can also be obtained through long wave limit method, and the interaction between the rogue waves and higher-order solitons have also been shown. To the author's best knowledge, the interaction of rogue waves with multisoliton in real systems have not been reported before. Finally, from the fourth-order soliton solutions, the hybrid solutions of the rogue waves and breathers are obtained. What is more, typical dynamics of these hybrid solutions are also analysed and demonstrated.

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