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Phase-space treatment of the driven quantum harmonic oscillator

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Abstract. A recent phase-space formulation of quantum mechanics in terms of the Glauber coherent states is applied to study the interaction of a one-dimensional harmonic oscillator with an arbitrary time-dependent force. Wave functions of the simultaneous values of position q and momentum p are deduced, which in turn give the standard position and momentum wave functions, together with expressions for the ηth derivatives with respect to q and p, respectively. Afterwards, general formulae for momentum, position and energy expectation values are obtained, and the Ehrenfest theorem is verified. Subsequently, general expressions for the cross-Wigner functions are deduced. Finally, a specific example is considered to numerically and graphically illustrate some results.

Keywords. Phase-space quantum mechanics; coherent states; harmonic oscillator; Husimi distribution; cross-Wigner functions.

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1. Introduction

In a previous paper [1] a formulation of quantum mechanics in phase space was proposed in which pq and qp representations are introduced in terms of coherent states $|\theta, q,\ p\rangle$, which differ from the normalized Glauber coherent states $|z\rangle$ by phase factors [2] and θ is a parameter with values 0 and 1, respectively. In order to illustrate the application of the method described in [1], it is useful to have a system that can be solved explicitly. This is the case of the driven quantum harmonic oscillator [3–10], which is important in quantum physics, in particular for describing small oscillations about equilibrium positions.

Consider a one-dimensional ($f = 1$) quantum harmonic oscillator of mass m_0 and frequency ω_0, with Hamiltonian $\hat{K} = \hat{p}^2/(2m_0) + \frac{1}{2}m_0\omega_0^2\,\hat{q}^2$ and energy-eigenvalue equation $\hat{K}|n\rangle = (n+\frac{1}{2})\hbar\omega_0|n\rangle$, for integers $n \geq 0$. When the system is driven by an arbitrary force $F(t)$ turned on at $t = t_0$, the Hamiltonian of the perturbed harmonic oscillator becomes $\hat{H}(t) = H(\hat{q}, \hat{p}, t) = \hat{K} + \hat{V}(\hat{q}, t)$, with $\hat{V}(\hat{q}, t) = -F(t)\hat{q}$. The time evolution of the state, which is described either by the ket $|\Psi(t)\rangle$ or the density operator $\hat{\rho}(t)$, is governed by the Schrödinger equation or the quantum Liouville equation, depending on whether the system is in a pure state or in a mixed state.

Now, let us include a short review of [1] on the topics and equations required as basis for this work. *Apropos*, the notation given in [1] is used, and the equations in that article are quoted in the form [1, (#)], where (#) is the equation number. Furthermore, from now on $q_0 = \sqrt{\hbar/(m_0\omega_0)}$ and $p_0 = \sqrt{\hbar m_0\omega_0}$ are natural units of length and momentum, and $\kappa_0 := 1/(q_0\sqrt{2})$ and $\chi_0 := 1/(p_0\sqrt{2})$ are auxiliary quantities.

If the system is in a normalized Glauber coherent state $|z\rangle$, then the position and momentum expectation values are q and p, and one may write $|z\rangle = |\kappa_0 q + i\chi_0 p\rangle = \hat{D}(q, p)\,|0\rangle$, where [1,2,11]

$$\hat{D}(q, p) := \exp\left(\frac{i}{\hbar}[p\hat{q} - q\hat{p}]\right) \tag{1}$$

is the Weyl operator assigned to the phase-space point $(q, p) \in \mathbb{R}^2$. In the context of coherent states, one associates two phase-space wave functions to a pure state $|\Psi(t)\rangle$, namely [1, (7)]: $\Psi(\theta|q, p, t) :=$

$w((2\theta-1)\frac{1}{2}q,\ p)\langle\kappa_0 q+i\chi_0 p|\Psi(t)\rangle$, with the phase factor

$$w(q,p) := \exp\left(\frac{i}{\hbar}qp\right). \tag{2}$$

Sometimes it is convenient to use an alternative notation by writing

$$\Psi(\theta|q,p,t)=\begin{cases}\Psi_+(q,p,t),\\ \quad \text{if } \theta=1\ (qp \text{ representation}),\\ \Psi_-(q,p,t),\\ \quad \text{if } \theta=0\ (pq \text{ representation}),\end{cases} \tag{3}$$

and noting that the phase-space wave functions $\Psi_\pm(q,p,t)$ are linked with each other by the relation $\Psi_+(q,p,t)=w(q,p)\Psi_-(q,p,t)$.

In the case of a mixed state, the density operator describing the mixture is defined as

$$\hat{\rho}(t) := \sum_{k=1}^{\mathcal{N}} W_{[k]}|\Psi_{[k]}(t)\rangle\langle\Psi_{[k]}(t)|, \tag{4}$$

where the notation $[k]$ indicates that the state $|\Psi_{[k]}(t)\rangle$ is an element of a statistical mixture of $\mathcal{N}$ independent states, with statistical weight $0 < W^{(k)} \le 1$ such that $\sum_{k=1}^{\mathcal{N}} W^{(k)}=1$, and each state $|\Psi_{[k]}(t)\rangle$ is normalized to unity.

For any pair of phase-space points, (q_a,p_a) and (q_b,p_b), complex variables $z_a=\kappa_0 q_a+i\chi_0 p_a$ and $z_b=\kappa_0 q_b+i\chi_0 p_b$, and Glauber coherent states $|z_a\rangle$ and $|z_b\rangle$, the Liouville equation in the phase space is expressed in terms of the function

$$\rho(\theta|q_a,p_a,q_b,p_b,t)=\sum_{k=1}^{\mathcal{N}} W_{[k]}\big[\Psi_{[k]}(\theta|q_a,p_a,t)\big]^\star \times \Psi_{[k]}(\theta|q_b,p_b,t), \tag{5}$$

where $\star$ means complex conjugate. Thus, the density operator $\hat{\rho}(t)$ allows for the pq ($\theta=0$) and qp ($\theta=1$) representations. One also notes that $\rho(\theta|q_a,p_a,q_b,p_b,t)$ is a two-point phase-space function because it depends, besides the time t, on two sets of independent variables, (q_a,p_a) and (q_b,p_b).

In the next step, one introduces the mathematical transform

$$(q_b,p_b)=(q',\ p')+\frac{1}{2}(q,\ p),$$
$$(q_a,p_a)=(q',\ p')-\frac{1}{2}(q,\ p), \tag{6}$$

and considers the set $\{(q_a,p_a),(q_b,p_b)\}$ of all points (q_a,p_a) and (q_b,p_b) having the same middle point (q',p'), so that this point can be treated as a parameter. Under this condition, the pq and qp representations are described by the one-point reduced function [1, (66)]

$$\rho(\theta,q',p'|q,p,t) := \rho\left(\theta|q'-\frac{1}{2}q,\ p'-\frac{1}{2}p,\ q'+\frac{1}{2}q,\ p'+\frac{1}{2}p,\ t\right). \tag{7}$$

Thus, for an 'observer' located at the parametric point (q',p'), $\rho(\theta,q',p'|q,p,t)$ describes the state of the system over phase-space (q,p), for any given value of time t, as a weighted sum of the products of probability amplitudes arising from a pair of points symmetrically located with respect to the central point (q',p'), i.e., $\rho(\theta,q',p'|q,p,t)$ is a measure of the interference effects associated with those points.

Notwithstanding that each wave function $\Psi_{[k]}(\theta|q_a,p_a,t)$ in (5) is normalized to one, the functions $\rho(\theta|q_a,p_a,q_b,p_b,t)$ and $\rho(\theta,q',p'|q,p,t)$ are neither real nor normalized, except when the points (q_a,p_a) and (q_b,p_b) coalesce. As commented after equation [1, (73)], which defines generalized phase-space functions $W(\theta|q',p',t)$, the indicated coalescence is at the basis of the relation between $\rho(\theta,q',p'|q,p,t)$ and the Wigner function (see eqs (68), (69) and (80)).

The pq and qp equations of motion for $\Psi(\theta|q,p,t)$, $\rho(\theta|q_a,p_a,q_b,p_b,t)$ and $\rho(\theta,q',p'|q,p,t)$ have been deduced in [1]: eqs (54), (59) and (64), respectively. In this paper, instead of solving these equations directly, the definitions of these quantities and the time evolution operator are used to build the above functions. So, one gets the functions and can verify their compliance with the equations of motion, if desired.

The phase-space representations used in this paper provide advantages and differences when they are contrasted with methods developed in other contributions [3–10]. Here are some of them: (a) In this work, the evolution operator is used to construct the complex-valued wave functions $\Psi(\theta|q,p,t)$, which describe the state of the system in the phase space, whereas some other treatments deal with the coordinate representation of the Schrödinger equation: e.g., the Kerner–Treanor method [8] assumes that $\Psi(q,t)=\phi(q,t)\exp(qg(t))$, where $\phi(q,t)$ and $g(t)$ are general, unspecified functions. (b) Here, after having $\Psi(\theta|q,p,t)$, one gets not only the coordinate and momentum wave functions but also expressions for their derivatives of any order, whereas conventional methods start from the coordinate representation for obtaining the momentum representation. (c) As a consequence of the foregoing point, it is possible to obtain analytical expressions not only for the momentum, position and energy expectation values but also for the

Husimi distribution and the cross-Wigner functions. (d) The last ones include the familiar Wigner function as a particular case and, to the best of my knowledge, they are calculated for the first time for the system studied in this contribution. (e) Finally, the functions $W(\theta|q', p', t)$ given by eqs (69) and (80) are at the origin of the cross-Wigner functions, a fact that can imply a fresh point of view about the formulation of quantum mechanics in phase space.

This work is arranged as follows. In §2, the evolution operator for a driven quantum harmonic oscillator is deduced by using the interaction picture and the Magnus expansion. In §3, the wave functions $\Psi_\pm(q, p, t)$ of the simultaneous values of position q and momentum p are constructed in terms of pq and qp coherent states which differ from the Glauber coherent states and each other by well-defined phase factors. In this section, the Husimi function associated with the state $|\Psi(t)\rangle$ is also constructed. In §4, the standard position and momentum wave functions, $\Psi(q, t)$ and $\tilde{\Psi}(p, t)$, are deduced, together with expressions for the ηth derivatives ($\eta = 0, 1, 2, \ldots$) of these functions evaluated at the points q and p, respectively. In §5, general formulae for momentum, position and energy expectation values are obtained, and they are applied to verify the Ehrenfest theorem. In §6, general expressions for the cross-Wigner functions are produced. Finally, in §7 the theoretical results are illustrated numerically and are depicted for the case of harmonic oscillator in an oscillating electric field, when the initial state is chosen as a linear combination of the ground state and the first excited state.

2. Evolution operator for a driven quantum harmonic oscillator

In the Schrödinger picture, the state of the system at time t is connected to a given initial state at time t_0 by the relation $|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi(t_0)\rangle$, where the evolution operator $\hat{U}(t, t_0)$ is obtained by solving the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t, t_0) = \hat{H}(t)\hat{U}(t, t_0), \quad \lim_{t\to t_0}\hat{U}(t, t_0) = \hat{1}. \tag{8}$$

Here, one uses the Hamiltonian splitting $\hat{H}(t) = \hat{K} + \hat{V}(t)$, with $\hat{K} = \hat{p}^2/(2m_0) + \frac{1}{2}m_0\omega_0^2\hat{q}^2$ and $\hat{V}(\hat{q}, t) = -F(t)\hat{q}$, and $\hat{U}_0(t, t_0) := \exp(-i\tau\hat{K}/\hbar)$ is the time-evolution operator associated with the time-independent Hamiltonian $\hat{K}$, where $\tau := t - t_0$. Thus, with the help of the interaction picture and the Magnus expansion [12], one obtains the expression

$$\hat{U}(t, t_0) = \exp\left(\frac{i}{\hbar}\gamma(t, t_0)\right)\exp\left(-\frac{i}{\hbar}\tau\hat{K}\right) \times\hat{D}(Q(t, t_0), P(t, t_0)), \tag{9}$$

which involves the Weyl operator assigned to the point $(Q(t, t_0), P(t, t_0))$ in phase-space.

In (9), one has auxiliary quantities of position and momentum,

$$Q(t, t_0) = -\frac{1}{m_0\omega_0}\int_{t_0}^{t} F(t')\sin(\omega_0(t' - t_0))\mathrm{d}t', \tag{10}$$

$$P(t, t_0) := \int_{t_0}^{t} F(t')\cos(\omega_0(t' - t_0))\mathrm{d}t' \tag{11}$$

and the phase-factor

$$\gamma(t, t_0) = \frac{1}{2m_0\omega_0}\int_{t_0}^{t}\mathrm{d}t''\int_{t_0}^{t''}\mathrm{d}t' F(t'')F(t') \times\sin(\omega_0(t'' - t')). \tag{12}$$

Equation (9) can be reorganized by moving the operator $\hat{U}_0(t, t_0) := \exp(-i\tau\hat{K}/\hbar)$ to the right of the Weyl operator $\hat{D}(Q(t), P(t))$. Thus, one finds the time-dependent operators

$$\begin{aligned}\hat{Q}(\tau) &:= \hat{U}_0(t, t_0)\,\hat{q}\,\hat{U}_0^{+}(t, t_0) = \cos(\omega_0\tau)\hat{q} - \frac{1}{m_0\omega_0}\sin(\omega_0\tau)\hat{p},\\ \hat{P}(\tau) &:= \hat{U}_0(t, t_0)\,\hat{p}\,\hat{U}_0^{+}(t, t_0) = m_0\omega_0\sin(\omega_0\tau)\hat{q} + \cos(\omega_0\tau)\hat{p},\end{aligned} \tag{13}$$

which satisfy the commutation relation $[\hat{Q}(\tau), \hat{P}(\tau)] = i\hbar$. As a result, one obtains

$$\hat{U}(t, t_0) = \exp\left(\frac{i}{\hbar}\gamma(t, t_0)\right)\hat{D}(\mathbb{Q}(t, t_0), \mathbb{P}(t, t_0)) \times\exp\left(-\frac{i}{\hbar}\tau\hat{K}\right), \tag{14}$$

where $(\mathbb{Q}(t, t_0), \mathbb{P}(t, t_0))$ is a point in the phase space with coordinates

$$\begin{aligned}\mathbb{Q}(t, t_0) &:= \cos(\omega_0\tau)Q(t, t_0) + \frac{1}{m_0\omega_0}\sin(\omega_0\tau)P(t, t_0),\\ \mathbb{P}(t, t_0) &:= -m_0\omega_0\sin(\omega_0\tau)Q(t, t_0) + \cos(\omega_0\tau)P(t, t_0).\end{aligned} \tag{15}$$

In what follows, if there is no ambiguity, the notation can be simplified by writing $\mathbb{Q}(t)$ and $\mathbb{P}(t)$ instead of $\mathbb{Q}(t, t_0)$ and $\mathbb{P}(t, t_0)$, respectively.

The transform (14) allows to write the state of the system at time t as

$$|\Psi(t)\rangle = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right)\hat{D}(\mathbb{Q}(t),\mathbb{P}(t)) \times \exp\left(-\frac{i}{\hbar}\tau\hat{K}\right)|\Psi(t_0)\rangle. \tag{16}$$

Thus, notwithstanding that (9) and (16) are equivalent, (16) has the merit that $\exp(-(i/\hbar)\tau\hat{K})\,|\Psi(t_0)\rangle$ can be interpreted as the propagation of the initial state $|\Psi(t_0)\rangle$ under the action of the evolution operator associated with the unperturbed system. The so-generated state is then moved in the phase space by the action of the Weyl operator, $\hat{D}(\mathbb{Q}(t),\mathbb{P}(t))$.

At this point, it is to be noted that (15), (10) and (11) imply the equations of motion

$$\frac{\mathrm{d}\mathbb{Q}(t)}{\mathrm{d}t}=\frac{\mathbb{P}(t)}{m_0},\quad \frac{\mathrm{d}\mathbb{P}(t)}{\mathrm{d}t}=-m_0\omega_0^2\,\mathbb{Q}(t)+F(t), \tag{17}$$

with the initial condition $Q(t_0,t_0)=P(t_0,t_0)=0$.

One wonders if the method described in this section can be generalized to other systems beyond the driven harmonic oscillator. Appendix E is included in order to shed light on this theme.

3. Phase-space wave functions, $\Psi_-(q,p,t)$ and $\Psi_+(q,p,t)$

By following [1, (12)], the pq and qp coherent states are defined by

$$|\theta,q,p\rangle=\begin{cases}|p,q\rangle:=\hat{D}(0,p)\hat{D}(q,0)|0\rangle, & \text{if } \theta=0\\ |q,p\rangle=\hat{D}(q,0)\hat{D}(0,p)\,|0\rangle, & \text{if } \theta=1,\end{cases} \tag{18}$$

which differ from the Glauber coherent state $|z\rangle$, and each other, by a phase factor. In particular, it follows that $|p,q\rangle = w(q,p)|q,p\rangle$ and the wave functions $\Psi(\theta|q,p,t)$ defined in (3) can be rewritten as $\Psi(\theta|q,p,t)=\langle\theta,q,\ p|\Psi(t)\rangle$.

According to Glauber [2,13], to each complex number $z=\kappa_0 q+i\chi_0 p$ corresponds a normalized coherent state $|z\rangle=|\kappa_0 q+i\chi_0 p\rangle$ given by

$$|z\rangle=\hat{D}(q,p)|0\rangle=[(\pi\hbar)^{f/2}\tilde{M}(p)M(q)]^{1/2} \times\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \tag{19}$$

where $|n\rangle$ are the eigenkets of the number operator $\hat{N}=\hat{a}^+\hat{a}$, and the Gaussian functions

$$M(q)=(q_0\sqrt{\pi})^{-1/2}\exp(-(\kappa_0 q)^2),$$
$$\tilde{M}(p)=(p_0\sqrt{\pi})^{-1/2}\exp(-(\chi_0 p)^2), \tag{20}$$

are related to each other by Fourier transforms [1, (4) and (5)]. It is seen that

$$M^2(q)=(q_0\sqrt{\pi})^{-1}\exp(-(q/q_0)^2)$$

and

$$\tilde{M}^2(p)=(p_0\sqrt{\pi})^{-1}\exp(-(p/p_0)^2).$$

Inserting (16) into the relation $\Psi(\theta|q,p,t)=\langle\theta,q,p|\Psi(t)\rangle=\Psi_\pm(q,p,t)$, and using the rule for the product of two Weyl operators

$$\hat{D}(q',p')\hat{D}(q'',p'')=w\left(\frac{1}{2}q'',p'\right)w^\star\left(\frac{1}{2}q',p''\right) \times\hat{D}(q'+q'',p'+p''), \tag{21}$$

one gets

$$\Psi_\pm(q,p,t)=\exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right)w\left(\pm\frac{1}{2}q,p\right) \times w^\star\left(\frac{1}{2}\mathbb{Q}(t),p\right)w\left(q,\frac{1}{2}\mathbb{P}(t)\right) \times\langle Z(t)|\exp\left(-\frac{i}{\hbar}\tau\hat{K}\right)|\Psi(t_0)\rangle, \tag{22}$$

with $Z(t):=\kappa_0[q-\mathbb{Q}(t)]+i\chi_0[p-\mathbb{P}(t)]$, and the Glauber coherent state

$$|Z(t)\rangle=\hat{D}(q-\mathbb{Q}(t),p-\mathbb{P}(t))\,|0\rangle = (\pi\hbar)^{1/4}\,\tilde{M}^{1/2}(p-\mathbb{P}(t))\,M^{1/2}(q-\mathbb{Q}(t)) \times\sum_{n=0}^{\infty}\frac{Z^n(t)}{\sqrt{n!}}|n\rangle. \tag{23}$$

Notwithstanding that $\Psi_+(q,p,t)$ and $\Psi_-(q,p,t)$ only differ from each other by the factors $w(\pm\frac{1}{2}q,p)$, their overall phase factors can be written in terms of the auxiliary quantities

$$A_\pm(q,p,t):=w\left(\pm\frac{1}{2}q,p\right)w^\star\left(\frac{1}{2}\mathbb{Q}(t),p\right)w\left(q,\frac{1}{2}\mathbb{P}(t)\right),$$

which are given by

$$A_+(q,p,t) =w\left(q-\frac{1}{2}\mathbb{Q},\mathbb{P}(t)\right)w\left(\frac{1}{2}(q-\mathbb{Q}),p-\mathbb{P}(t)\right),$$
$$A_-(q,p,t) =w^\star\left(\mathbb{Q}(t),p-\frac{1}{2}\mathbb{P}\right)w^\star\left(q-\mathbb{Q},\frac{1}{2}(p-\mathbb{P}(t))\right). \tag{24}$$

The conclusive effect is that

$$A_\pm(q,p,t)=\exp(\pm i\,\zeta_\pm(q,p,t)/\hbar),$$

with phases

$$\zeta_+(q,p,t)=\frac{1}{2}(q-\mathbb{Q}(t))p+\frac{1}{2}\mathbb{P}(t)q,$$
$$\zeta_-(q,p,t)=\frac{1}{2}(p-\mathbb{P}(t))q+\frac{1}{2}\mathbb{Q}(t)p. \tag{25}$$

To conclude, the phase-space wave functions can be written in a unified way, as

$$\Psi_\pm(q,p,t) = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right) A_\pm(q,p,t) \times M^{1/2}(q-\mathbb{Q}(t))\tilde{M}^{1/2}(p-\mathbb{P}(t)) \times \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\{\kappa_0[q-\mathbb{Q}(t)]-i\chi_0[p-\mathbb{P}(t)]\}^n \times a_n(t-t_0). \tag{26}$$

Here, the probability amplitude, at time t, of finding the system in state $|n\rangle$ of the unperturbed harmonic oscillator is given by the coefficient

$$a_n(\tau) := \exp\left(-\frac{i}{\hbar}\tau\varepsilon_n\right)\langle n|\Psi(t_0)\rangle, \quad \tau := t-t_0. \tag{27}$$

In contrast with the standard formulation of quantum mechanics where one has wave functions $\Psi(q,t)$ and $\Psi(p,t)$ in position space q and momentum space p, respectively, the phase-space wave functions $\Psi_\pm(q,p,t)$ display the probability amplitudes simultaneously in the q and p variables.

From (26), the squared magnitude of the phase-space wave functions $\Psi_\pm(q,p,t)$ gives the Husimi function associated with the state $|\Psi(t)\rangle$,

$$\rho_{\mathrm{H}}(q,p,t) = |\Psi_\pm(q,p,t)|^2 = M(q-\mathbb{Q}(t))\tilde{M}(p-\mathbb{P}(t)) \times \left|\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\{\kappa_0[q-\mathbb{Q}(t)] - i\chi_0[p-\mathbb{P}(t)]\}^n a_n(\tau)\right|^2, \tag{28}$$

which can be explicitly written as a double sum, with sum indices m and n.

From what follows, it is useful to write $Z(t) := \kappa_0[q-\mathbb{Q}(t)] + i\chi_0[p-\mathbb{P}(t)] = x+iy$ and $w(q-\mathbb{Q}, \frac{1}{2}(p-\mathbb{P})) = \exp(ixy)$, with the auxiliary dimensionless quantities

$$x := \kappa_0[q-\mathbb{Q}(t)], \qquad y = \chi_0[p-\mathbb{P}(t)],$$
$$\mathcal{Q}(t) := \kappa_0\mathbb{Q}(t), \qquad \mathcal{P}(t) := \chi_0\mathbb{P}(t). \tag{29}$$

4. Position and momentum wave functions

Adopting the notation $\Psi^{(\eta)}(q,t) := \partial^\eta\Psi(q,t)/\partial q^\eta$, the ηth derivatives of the position ($\theta=1$) and momentum ($\theta=0$) wave functions evaluated at points q and p, respectively, are given by [1, (22) and (25)]

$$\Psi^{(\eta)}(q,t) = (2p_0\sqrt{\pi})^{-1/2}(-iq_0\sqrt{2})^{-\eta}(2\pi\hbar)^{-1/2} \times \int H_\eta(\chi_0 p)\Psi_+(q,p,t)\mathrm{d}p \tag{30}$$

and

$$\tilde{\Psi}^{(\eta)}(p,t) = (2q_0\sqrt{\pi})^{-1/2}(+ip_0\sqrt{2})^{-\eta}(2\pi\hbar)^{-1/2} \times \int H_\eta(\kappa_0 q)\Psi_-(q,p,t)\mathrm{d}q. \tag{31}$$

The idea here is to use eqs (26) and (29), and the binomial theorem [14a]

$$(x\pm iy)^n = \frac{1}{2^n}\sum_{m=0}^{n}(\pm i)^m\binom{n}{m}H_{n-m}(x)H_m(y) = \frac{1}{2^n}\sum_{m=0}^{n}(\pm i)^{n-m}\binom{n}{m}H_m(x)H_{n-m}(y), \tag{32}$$

where $H_m(x)$ are the Hermite polynomials of order m.

4.1 *Position wave function, $\Psi(q,t)$*

After inserting the function $\Psi_\pm(q,p,t)$ given by (26) into (30), one defines the integral

$$B_{m\eta}(x,\mathcal{P}) := \int_{-\infty}^{\infty}\exp(ixy)\,\tilde{M}^{1/2}(y)H_m(y)H_\eta(\chi_0 p)\mathrm{d}p, \tag{33}$$

and uses (29) for changing the integration variable from p to y. Thus, with the help of formulae (A.1), (A.2) and (A.4), the integral $B_{m\eta}(x,\mathcal{P})$ is turned into

$$B_{m\eta}(x,\mathcal{P}) = \chi_0^{-1}(p_0\sqrt{\pi})^{-1/4}\sqrt{2\pi}\,\underbrace{(q_0\sqrt{\pi})^{1/4}M^{1/2}(x)} \times \sum_{k=0}^{\eta}(+i)^{m+k}\binom{\eta}{k}(2\mathcal{P})^{\eta-k}\sum_{r=0}^{\min(m,k)}(-2)^r \times r!\binom{m}{r}\binom{k}{r}H_{m+k-2r}(x), \tag{34}$$

where, according to (20), the quantity within the brace is equal to $\exp\left(-\frac{1}{2}(\kappa_0 q)^2\right)$.

Consequently, $\Psi^{(\eta)}(q,t)$ can be written as

$$\Psi^{(\eta)}(q,t) = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right)w\left(q-\frac{1}{2}\mathbb{Q},\,\mathbb{P}(t)\right) \times M(q-\mathbb{Q}(t))\sum_{n=0}^{\infty}\frac{a_n(t-t_0)}{2^n\sqrt{n!}}\mathbb{X}_{n,\eta}(q,t), \tag{35}$$

where

$$\mathbb{X}_{n,\eta}(q,t) := (-iq_0\sqrt{2})^{-\eta}\sum_{k=0}^{\eta}(+i)^k\binom{\eta}{k}[2\chi_0\mathbb{P}(t)]^{\eta-k} \times \Pi_{n,k}(\kappa_0[q-\mathbb{Q}(t)]) \tag{36}$$

and $\Pi_{n,k}(x)$ is the polynomial (see Appendix B)

$$\Pi_{n,k}(x) := \sum_{\lambda=0}^{n} \binom{n}{\lambda} H_{n-\lambda}(x) \sum_{r=0}^{\min(\lambda,k)} (-2)^r r! \binom{\lambda}{r} \binom{k}{r} \times H_{\lambda+k-2r}(x). \quad (37)$$

Here, please note that the sum over r on the right-hand side of (37) only differs by the sign factor $(-2)^r$ from the formula (A.2) for the product of two Hermite polynomials.

If $\eta = 0$, with the help of summation formula (A.3), one finds that

$$\Pi_{n,0}(x) = \sum_{\lambda=0}^{n} \binom{n}{\lambda} H_{n-\lambda}(x) H_\lambda(x) = 2^{n/2} H_n(x\sqrt{2}) \quad (38)$$

and

$$\mathbb{X}_{n,0}(q,t) = \Pi_{n,0}(\kappa_0[q - \mathbb{Q}(t)]) = 2^{n/2} H_n\left(\frac{q - \mathbb{Q}(t)}{q_0}\right). \quad (39)$$

Therefore, the position wave function $\Psi(q,t) = \Psi^{(0)}(q,t)$ can be rewritten as

$$\Psi(q,t) = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right) w\left(q - \frac{1}{2}\mathbb{Q},\ \mathbb{P}(t)\right) \times M(q - \mathbb{Q}(t)) \sum_{n=0}^{\infty} \frac{a_n(t-t_0)}{2^{n/2}\sqrt{n!}} H_n\left(\frac{q - \mathbb{Q}(t)}{q_0}\right), \quad (40)$$

where $M(p)$ is the function defined by (20). As a check on the results, if the initial state is an eigenstate of the harmonic oscillator, $|\Psi(t_0)\rangle = |\nu\rangle$, and the system is not driven, $F(t) = 0$, then (40) reproduces the coordinate wave function for the νth state of the harmonic oscillator (see e.g., [1, (26)]).

Here it is important to recall that $\Psi_+(q,p,t)$ can be reconstructed as [1, (20) and (21)]

$$\Psi_+(q,p,t) = (2\pi\hbar)^{1/2} \sum_{\eta=0}^{\infty} \frac{1}{\eta!} \Psi^{(\eta)}(q,t) \tilde{J}_\eta(p) \quad (41)$$

with p-dependent coefficients

$$\tilde{J}_\eta(p) = \left(-i\frac{q_0}{\sqrt{2}}\right)^{\eta} H_\eta(\chi_0 p) \tilde{M}(p). \quad (42)$$

4.2 *Momentum wave function,* $\tilde{\Psi}(p,t)$

For the calculation of $\tilde{\Psi}^{(\eta)}(p,t)$, one rearranges the function $\Psi_-(q,p,t)$ given by (26) by using the identity $(x - iy)^n = (-i)^n (y + ix)^n$, and inserts the resulting expression into (31). Following a procedure similar to the one in §4.1, one defines the integral

$$\tilde{B}_{m\eta}(y,\mathcal{Q}) := \kappa_0^{-1} \int_{-\infty}^{\infty} \exp(-ixy) M^{1/2}(x/\kappa_0) H_m(x) \times H_\eta(x + \mathcal{Q}(t)) \mathrm{d}x, \quad (43)$$

that can be written as

$$\tilde{B}_{m\eta}(y,\mathcal{Q}) = \kappa_0^{-1} (q_0\sqrt{\pi})^{-1/4} \sqrt{2\pi} \underbrace{(p_0\sqrt{\pi})^{1/4} \tilde{M}^{1/2}(y)} \times \sum_{k=0}^{\eta} (-i)^{m+k} \binom{\eta}{k} (2\mathcal{Q})^{\eta-k} \times \sum_{r=0}^{\min(m,k)} (-2)^r r! \binom{m}{r} \binom{k}{r} \times H_{m+k-2r}(y), \quad (44)$$

where, according to (20), the quantity within the brace is equal to $\exp\left(-\frac{1}{2}(\chi_0 p)^2\right)$.

As a result, the ηth derivative $\tilde{\Psi}^{(\eta)}(p,t)$ of the momentum wave function becomes

$$\tilde{\Psi}^{(\eta)}(p,t) = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right) w^\star\left(\mathbb{Q}(t),\ p - \frac{1}{2}\mathbb{P}\right) \times \tilde{M}(p - \mathbb{P}(t)) \sum_{n=0}^{\infty} \frac{(-i)^n a_n(t-t_0)}{2^n\sqrt{n!}} \times \mathbb{Y}_{n,\eta}(p,t), \quad (45)$$

where

$$\mathbb{Y}_{n,\eta}(p,t) := (+ip_0\sqrt{2})^{-\eta} \sum_{k=0}^{\eta} (-i)^k \binom{\eta}{k} \times [2\kappa_0 \mathbb{Q}(t)]^{\eta-k} \Pi_{n,k}(\chi_0[p - \mathbb{P}(t)]), \quad (46)$$

and $\Pi_{n,k}(x)$ is the polynomial defined by (37).

For the particular case $\eta = 0$, one gets the momentum wave function

$$\tilde{\Psi}(p,t) = \exp\left(\frac{i}{\hbar}\gamma(t,t_0)\right) w^\star\left(\mathbb{Q}(t),\ p - \frac{1}{2}\mathbb{P}\right) \times \tilde{M}(p - \mathbb{P}(t)) \sum_{n=0}^{\infty} \frac{(-i)^n a_n(t-t_0)}{2^{n/2}\sqrt{n!}} \times H_n\left(\frac{p - \mathbb{P}(t)}{p_0}\right), \quad (47)$$

where the function $\tilde{M}(p)$ is given by (20).

Notice also that one can reconstruct the phase-space wave function [1, (53) and (54)]

$$\Psi_-(q,p,t) = (2\pi\hbar)^{1/2} \sum_{\eta=0}^{\infty} \frac{1}{\eta!} \tilde{\Psi}^{(\eta)}(p,t) J_\eta(q), \quad (48)$$

using q-dependent coefficients

$$J_\eta(q) = \left(+i\frac{p_0}{\sqrt{2}}\right)^{\eta} H_\eta(\kappa_0 q) M(q). \quad (49)$$

5. Momentum, position and energy expectation values

In this section, (35) and (45) are applied to get explicit expressions for the momentum, position and energy expectation values. The results are then used to scrutinize in some detail the Ehrenfest theorem and the expression for the energy expectation value.

5.1 *Calculation of* $\langle\Psi(t)|\,\hat{p}^{\eta}\,|\Psi(t)\rangle$

By using the completeness relation for the position eigenkets $|q\rangle$ and recalling that $-i\hbar\partial/\partial q$ is the momentum operator $\hat{p}$ in position representation, the expectation value

$$\langle\Psi(t)|\,\hat{p}^{\eta}\,|\Psi(t)\rangle = (-i\hbar)^{\eta}\int_{-\infty}^{\infty}\Psi^{\star}(q,t)\Psi^{(\eta)}(q,t)\mathrm{d}q \tag{50}$$

gives the overlap between the wave function $\Psi(q,t)$ and its η-derivative, $\Psi^{(\eta)}(q,t)$. Similarly, using the completeness relations for the momentum eigenkets $|p\rangle$ and the momentum wave function $\tilde{\Psi}(p,t)$, given by (47), one also finds that

$$\begin{aligned}\langle\Psi(t)|\,\hat{p}^{\eta}\,|\Psi(t)\rangle &= \int_{-\infty}^{\infty} p^{\eta}|\tilde{\psi}(p,t)|^{2}\mathrm{d}p\\ &= p_0^{\eta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(+i)^{m-n}C(m,n,t)\\ &\quad\times G(m,n,\eta;Y).\end{aligned} \tag{51}$$

In the foregoing equation, one has coefficients ($m \geq 0$, $n \geq 0$ and $\tau = t - t_0$)

$$\begin{aligned}C(m,n,t) :=\ &\frac{\exp(i(m-n)\omega_0\tau)}{2^{(m+n)/2}\sqrt{m!n!}}\langle\Psi(t_0)|m\rangle\\ &\times\langle n|\Psi(t_0)\rangle = C^{\star}(n,m,t),\end{aligned} \tag{52}$$

and defines, for $n \geq 0$, $m \geq 0$ and $\eta \geq 0$, the auxiliary quantity

$$\begin{aligned}&G(m,n,\eta;Y) = G(n,m,\eta;Y)\\ &:= \int_{-\infty}^{\infty}\left(\frac{p}{p_0}\right)^{\eta}[\tilde{M}(p-\mathbb{P}(t))]^{2}H_m\left(\frac{p-\mathbb{P}(t)}{p_0}\right)\\ &\quad\times H_n\left(\frac{p-\mathbb{P}(t)}{p_0}\right)\mathrm{d}p\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_m(y)H_n(y)(y+Y)^{\eta}\exp(-y^2)\mathrm{d}y\\ &= \sum_{r=0}^{\eta}\binom{\eta}{r}Y^{\eta-r}\Lambda(m,n,r),\end{aligned} \tag{53}$$

where the change of variables $y := [p - \mathbb{P}(t)]/p_0$ and the abbreviation $Y(t) := \mathbb{P}(t)/p_0$ have been used. The integral involved in this equation is given by [15, (46) and (52)]

$$\begin{aligned}\Lambda(m,n,r) &:= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_m(y)H_n(y)\,y^{r}\exp(-y^2)\mathrm{d}y\\ &= \begin{cases}0 & \text{if } (r-n-m) \text{ is odd}\\ \lambda(m,n,r) & \text{otherwise,}\end{cases}\end{aligned} \tag{54}$$

where, with $s := \frac{1}{2}(r-n-m)$,

$$\begin{aligned}\lambda(m,n,r) &= \lambda(n,m,r)\\ &= r!\,2^{m+n-r}\sum_{\nu=\max(0,-s)}^{\min(m,n)}\binom{m}{\nu}\binom{n}{\nu}\frac{\nu!}{2^{\nu}(s+\nu)!}.\end{aligned} \tag{55}$$

If the initial state $|\Psi(t_0)\rangle = \sum_{\nu=0}^{\mathcal{N}}a_{\nu}\,|\nu\rangle$ is formed by a finite number $\mathcal{N}$ of eigenkets $|\nu\rangle$ of the harmonic oscillator, then the orthogonality relation $\langle n|\nu\rangle = \delta_{n,\nu}$ implies a finite number of nonvanishing coefficients $C(m,n,t)$ in (52) and summands (51).

5.2 *Calculation of* $\langle\Psi(t)|\,\hat{q}^{\eta}\,|\Psi(t)\rangle$

Similar to the procedure in §5.1, one writes

$$\langle\Psi(t)|\,\hat{q}^{\eta}|\Psi(t)\rangle = (i\hbar)^{\eta}\int_{-\infty}^{\infty}\tilde{\Psi}^{\star}(p,t)\tilde{\Psi}^{(\eta)}(p,t)\mathrm{d}p \tag{56}$$

and with $X(t) := \mathbb{Q}(t)/q_0$, one obtains the expression

$$\begin{aligned}\langle\Psi(t)|\,\hat{q}^{\eta}\,|\Psi(t)\rangle &= \int_{-\infty}^{\infty}q^{\eta}\,|\psi(q,t)|^{2}\,\mathrm{d}q\\ &= q_0^{\eta}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C(m,n,t)G(m,n,\eta;X).\end{aligned} \tag{57}$$

5.3 *Ehrenfest theorem*

For a quantum system with Hamiltonian $H(\hat{q},\hat{p},t) = \hat{p}^2/(2m_0) + V(\hat{q},t)$, the Ehrenfest theorem asserts that the mean position and momentum evolve according to equations which formally are reminiscent of their classical counterparts [16,17]:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Psi(t)|\,\hat{q}\,|\Psi(t)\rangle = \frac{1}{m_0}\,\langle\Psi(t)|\,\hat{p}|\Psi(t)\rangle,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Psi(t)|\,\hat{p}\,|\Psi(t)\rangle = -\left\langle\frac{\partial V(\hat{q},t)}{\partial\hat{q}}\right\rangle.\end{aligned} \tag{58}$$

One can perform a direct check of the Ehrenfest theorem by using (57) and (51), when $\eta = 1$. To do this, one verifies that $\{G(m,n,1;Y)|m,n=1,2,\ldots\}$ is a tridiagonal matrix such that, for a given m, the only nonzero elements of $G(m,n,1;Y) = \Lambda(m,n,0)Y + \Lambda(m,n,1)$ are those corresponding to the columns: (i) $n = m, m+1$, if $m = 0$ and (ii) $n = m-1, m, m+1$, if $m \geq 1$. Then, one writes

$$\langle\hat{q}\rangle(t) = q_0 \sum_{m=0}^{\infty} \mathcal{M}_m(X(t),t),$$

$$\langle\hat{p}\rangle(t) = p_0 \sum_{m=0}^{\infty} \tilde{\mathcal{M}}_m(Y(t),t), \tag{59}$$

with the auxiliary quantities

$$\mathcal{M}_m(X(t),t) := \sum_{n=m-1}^{m+1} C(m,n,t)G(m,n,1;X(t)) \tag{60}$$

and

$$\tilde{\mathcal{M}}_m(Y(t),t) := \sum_{n=m-1}^{m+1} (+i)^m(-i)^n C(m,n,t) \times G(m,n,1;Y(t)), \tag{61}$$

where, as before, $X(t) := \mathbb{Q}(t)/q_0$ and $Y(t) := \mathbb{P}(t)/p_0$.

5.4 *Expectation value of the energy*

To elucidate the structure of this quantity, one starts from the formula

$$\langle\hat{H}(t)\rangle = \frac{1}{2m_0}\langle\Psi(t)|\,\hat{p}^2\,|\Psi(t)\rangle + \frac{1}{2}m_0\omega_0^2\langle\Psi(t)|\,\hat{q}^2\,|\Psi(t)\rangle - F(t)\,\langle\Psi(t)|\,\hat{q}\,|\Psi(t)\rangle, \tag{62}$$

where the expectation values on the right-hand side of (62) are given by the first equation (59), and the relations (57) and (51) are evaluated when $\eta = 2$. For a given m, the only nonzero elements of $G(m,n,2;Y) = \Lambda(m,n,2) + 2\Lambda(m,n,1)Y + \Lambda(m,n,0)Y^2$ are those corresponding to the columns: (i) $n = m, m+1, m+2$, if $m = 0$, (ii) $n = m-1, m, m+1, m+2$, if $m = 1$ and (iii) $n = m-2, \ldots, m+2$, if $m \geq 2$. The first values of $G(m,n,2;Y)$ are shown in table 1.

5.5 *A particular case,* $|\Psi(t_0)\rangle = |m\rangle$

It is instructive to consider the νth eigenstate as initial state of the unperturbed harmonic oscillator, $|\Psi(t_0)\rangle = |\nu\rangle$. In this case, the coefficients $C(m,n,t)$ defined by (52) reduce to $C(m,\ n,\ t) = \delta_{m,\nu}\,\delta_{n,\nu}/(2^\nu \nu!)$, where $\delta_{m,\nu}$ is the Kronecker delta. Therefore, from (59)–(61) one obtains $\langle\Psi(t)|\,\hat{q}\,|\Psi(t)\rangle = \mathbb{Q}(t)$ and $\langle\Psi(t)|\,\hat{p}\,|\Psi(t)\rangle = \mathbb{P}(t)$. Similarly, from (53) it follows that $G(\nu,\nu,2;Y) = \Lambda(\nu,\nu,2) + \Lambda(\nu,\nu,0)Y^2$ (see also table 1), where

$$\frac{\Lambda(\nu,\nu,0)}{2^\nu\,\nu!} = 1, \quad \frac{\Lambda(\nu,\nu,2)}{2^\nu\,m!} = \frac{1}{2}(2\nu+1). \tag{63}$$

Then, (57) and (51) give

$$\langle\Psi(t)|\,\hat{q}^2\,|\Psi(t)\rangle = q_0^2\left[\frac{1}{2}(2\nu+1) + \left(\frac{\mathbb{Q}(t)}{q_0}\right)^2\right],$$

$$\langle\Psi(t)|\,\hat{p}^2|\Psi(t)\rangle = p_0^2\left[\frac{1}{2}(2\nu+1) + \left(\frac{\mathbb{P}(t)}{p_0}\right)^2\right]. \tag{64}$$

Having $\langle\Psi(t)|\,\hat{q}^\eta\,|\Psi(t)\rangle$ and $\langle\Psi(t)|\,\hat{p}^\eta\,|\Psi(t)\rangle$, for $\eta = 1$ and 2, one proceeds according to the rules of statistics

Table 1. Values of $G(m,n,2;Y) = \Lambda(m,n,2) + 2\Lambda(m,n,1)Y + \Lambda(m,n,0)Y^2$. On the mth row, the only nonzero contributions correspond to the columns: (i) $n = m, m+1, m+2$, if $m = 0$, (ii) $n = m-1, m, m+1, m+2$, if $m = 1$ and (iii) $n = m-2, \ldots, m+2$, if $m \geq 2$.

$m \backslash n$	0	1	2	3	4	5	6	7
0	$\frac{1}{2}+Y^2$	$2Y$	2	0	0	0	0	0
1	$2Y$	$3+2Y^2$	$8Y$	12	0	0	0	0
2	2	$8Y$	$20+8Y^2$	$48Y$	96	0	0	0
3	0	12	$48Y$	$168+48Y^2$	$384Y$	960	0	0
4	0	0	96	$384Y$	$1728+384Y^2$	$3840Y$	11520	0
…	0	0	0	…	…	…	…	$\ddots$

and calculate the mean-square deviations or fluctuations in $\hat{q}$ and $\hat{p}$, which are given by

$$
\begin{aligned}
(\Delta q)^2 &= \langle\Psi(t)|\,\hat{q}^2\,|\Psi(t)\rangle - (\langle\Psi(t)|\hat{q}|\Psi(t)\rangle)^2 \\
&= \left(\nu + \frac{1}{2}\right) q_0^2, \\
(\Delta p)^2 &= \langle\Psi(t)|\,\hat{p}^2\,|\Psi(t)\rangle - (\langle\Psi(t)|\hat{p}|\Psi(t)\rangle)^2 \\
&= \left(\nu + \frac{1}{2}\right) p_0^2,
\end{aligned} \tag{65}
$$

for $\nu = 0, 1, 2, \ldots$. After recalling that $q_0 p_0 = \hbar$, one gets the Heisenberg uncertainty relation $(\Delta q)(\Delta p) = (\nu + \frac{1}{2})\hbar \geq \hbar/2$. For the particular case $|\Psi(t_0)\rangle = |\nu\rangle$, one obtains that the expectation value of energy is given by

$$
\begin{aligned}
\langle\hat{H}(t)\rangle = {} & \left(\nu + \frac{1}{2}\right)\hbar\omega_0 + \left[\frac{\mathbb{P}^2(t)}{2m_0} + \frac{1}{2}m_0\omega_0^2\mathbb{Q}^2(t)\right] \\
& -F(t)\mathbb{Q}(t).
\end{aligned} \tag{66}
$$

Therefore, the expectation value of energy is the sum of the quantized energy $\hbar\omega_0(\nu + \frac{1}{2})$ plus the classical energy of the driven harmonic oscillator.

6. Cross-Wigner functions

As already commented, the two-point function $\rho(\theta|\, q_a, p_a, q_b, p_b, t)$ given by eq. (5) provides a representation of the Liouville equation in the phase space. In this section, in order to maintain the treatment as simple as possible, considerations are restricted to the case of a pure state described by the density operator $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$, so that (5) becomes $\rho(\theta|\, q_a,\ p_a,\ q_b,\ p_b,\ t) = [\Psi(\theta|\, q_a, p_a, t)]^\star \Psi(\theta|q_b, p_b, t)$, and (7) reduces to

$$
\begin{aligned}
&\rho(\theta, q', p'|q, p, t) \\
&\quad = \left[\Psi\left(\theta|\, q' - \frac{1}{2}q,\, p' - \frac{1}{2}p,\, t\right)\right]^\star \\
&\qquad \times \Psi\left(\theta|\, q' + \frac{1}{2}q,\, p' + \frac{1}{2}p,\, t\right).
\end{aligned} \tag{67}
$$

Here, the components of (q', p') are treated as parameters, whereas (q, p) are the independent variables. In addition, one has two representations for the wave function $\Psi(\theta|\cdots)$: (i) pq, $\theta = 0$, $\Psi_-(q, p, t)$ as given by (48) and (ii) qp, $\theta = 1$, $\Psi_+(q, p, t)$ as given by (41).

In this section, I wish to illustrate that the phase-space formulation in terms of wave functions $\Psi_\pm(q, p, t)$ and the Wigner approach to quantum mechanics are directly linked. To do this, the starting point is given by the phase-space functions [1, (72)]

$$
\begin{aligned}
W(\theta|\, q', p', t) = {} & \frac{1}{2\pi\hbar}\int w(q', p) w^\star(q, p') \\
& \times \rho(\theta, q', p'|q, p, t)\mathrm{d}q\mathrm{d}p.
\end{aligned} \tag{68}
$$

6.1 *Phase-space function $W(\theta|\, q', p', t)$ in the pq representation, $\theta = 0$*

One begins by recalling the expression [1, (74)]

$$
\begin{aligned}
W(0|\, q', p', t) = {} & \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\rho_{\mu\eta}(q',\ p',\ t) \\
& \times\tilde{\mathcal{J}}_{\mu\eta}(q',\ p'),
\end{aligned} \tag{69}
$$

where the cross-Wigner functions [1, (75)]

$$
\begin{aligned}
&\rho_{\mu\eta}(q',\ p',\ t) \\
&:= \int w(q',\ p)\left[\tilde{\Psi}^{(\mu)}\left(p' - \frac{1}{2}p, t\right)\right]^\star \\
&\quad \times\tilde{\Psi}^{(\eta)}\left(p' + \frac{1}{2}p, t\right)\mathrm{d}p \\
&= \left(+\frac{i}{\hbar}\right)^\mu\left(-\frac{i}{\hbar}\right)^\eta \\
&\quad \times\int w^\star(q, p')\left(q' - \frac{1}{2}q\right)^\mu\left(q' + \frac{1}{2}q\right)^\eta \\
&\quad \times\left\langle q' + \frac{1}{2}q\right|\,\hat{\rho}(t)\,\left|q' - \frac{1}{2}q\right\rangle\mathrm{d}q.
\end{aligned} \tag{70}
$$

are escorted by the referential functions $\tilde{\mathcal{J}}_{\mu\eta}(q',\ p')$ [1, (76)]. Here, $\rho_{\mu\eta}(q', p', t) = \rho^\star_{\eta\mu}(q', p', t)$ can be calculated using either the expressions (45)–(46) for the ηth derivative $\tilde{\Psi}^{(\eta)}(p, t)$ of the momentum wave function or the expression (40) for the position wave function $\Psi(q, t)$. Henceforth, the indices (m, μ, ℓ, M) and (n, η, k, N) will be associated with $\Psi^\star(\cdots)$ and $\Psi(\cdots)$, respectively.

6.1.1 *Cross-Wigner functions $\rho_{\mu\eta}(q', p', t)$ based on the eq.* (70).

After inserting (45) into eq. (70), it is desirable to define the quantity

$$
\begin{aligned}
&\mathbb{Y}_{m,\mu,n,\eta}(q' - \mathbb{Q}(t), p', t) \\
&:= \int w(q' - \mathbb{Q}(t), p)\exp\left(-\frac{1}{4}\left[\frac{p}{p_0}\right]^2\right) \\
&\quad \times\mathbb{Y}^\star_{m,\,\mu}\left(p' - \frac{1}{2}p, t\right)\mathbb{Y}_{n,\,\eta}\left(p' + \frac{1}{2}p, t\right)\mathrm{d}p.
\end{aligned} \tag{71}
$$

The expression (46) and the change of integration variable $y = \frac{1}{2}\chi_0 p$ allow us to write

$$\mathbb{Y}_{m,\mu,n,\eta}(q' - \mathbb{Q}(t), p', t)$$
$$= 2\,(+i)^{\mu-\eta}\chi_0^{\mu+\eta-1}\sum_{\ell=0}^{\mu}\sum_{k=0}^{\eta}(+i)^{\ell-k}\binom{\mu}{\ell}\binom{\eta}{k}$$
$$\times [2\kappa_0\mathbb{Q}(t)]^{\mu+\eta-\ell-k}\int \exp(i\chi y)$$
$$\times \exp(-ay^2)\Pi_{m,\ell}(-y-b)\Pi_{n,k}(y+c)\mathrm{d}y, \qquad (72)$$

where $\Pi_{n,k}(x)$ are the polynomials defined in (37), and $\{\chi, a, b, c\}$ are the parameters

$$\chi = 4\kappa_0(q' - \mathbb{Q}(t)), \quad a = 2,$$
$$b = -c = -\chi_0(p' - \mathbb{P}(t)). \qquad (73)$$

Now, using (B.3) one calculates the integral in (72) and finds that

$$\mathbb{Y}_{m,\mu,n,\eta}(q' - \mathbb{Q}(t), p', t)$$
$$= 2\,(+i)^{\mu-\eta}\chi_0^{\mu+\eta-1}\sum_{\ell=0}^{\mu}\sum_{k=0}^{\eta}(+i)^{\ell-k}\binom{\mu}{\ell}\binom{\eta}{k}$$
$$\times [2\kappa_0\mathbb{Q}(t)]^{\mu+\eta-\ell-k}\sum_{M=0}^{m+\ell}\sum_{N=0}^{n+k}$$
$$\times \theta_M(m,\ell)\,\theta_N(n,k)(-1)^M F^{\star}_{MN}(\mathcal{X}, a, b, c). \qquad (74)$$

The procedure for calculating $\theta_M(m,\ell)$ and $\theta_N(n,k)$ is explained in Appendix B, and the integral $F_{MN}(\mathcal{X}, a, b, c)$ is defined and evaluated in Appendix C. In (74), please note the presence of the complex conjugate of $F_{MN}(\chi, a, b, c)$.

Finally, for the cross-Wigner functions one obtains the expression

$$\rho_{\mu\eta}(q',\ p',\ t) = [\tilde{M}(p' - \mathbb{P}(t))]^2$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(+i)^{m-n}\,a_m^{\star}(\tau)a_n(\tau)}{2^{m+n}\sqrt{m!n!}}$$
$$\times \mathbb{Y}_{m,\mu,n,\eta}(q' - \mathbb{Q}(t), p', t). \qquad (75)$$

6.1.2 *Cross-Wigner functions* $\rho_{\mu\eta}(q', p', t)$ *based on eq.* (70). After inserting (40) into the eq. (70) and using the transform $x = q/(2q_0)$, the cross-Wigner function $\rho_{\mu\eta}(q', p', t)$ can be written as

$$\rho_{\mu\eta}(q', p', t) = 2q_0\frac{(+i)^{\mu-\eta}}{p_0^{\mu+\eta}}[M(q' - \mathbb{Q})]^2$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m^{\star}(\tau)a_n(\tau)}{2^{(m+n)/2}\sqrt{m!n!}}(-1)^{m+\mu}$$
$$\times \mathcal{F}(\mu, m, \eta, n|\mathcal{X}, a, b, c, \Lambda, \lambda), \qquad (76)$$

where the parameters $\{\chi, a, b, c\}$ take the values

$$\mathcal{X} = 2\frac{p' - \mathbb{P}}{p_0}, \quad a = 1, \quad \Lambda = -\lambda = -\frac{q'}{q_0},$$
$$b = -c = -\frac{q' - \mathbb{Q}(t)}{q_0}. \qquad (77)$$

The integral $\mathcal{F}(\mu, M, \eta, N|\mathcal{X}, a, b, c, \Lambda, \lambda)$ is defined and evaluated in Appendix D.

6.1.3 *Properties of* $\rho_{\mu\eta}(q', p', t)$. The functions $\rho_{\mu\eta}(q', p', t)$ have the following properties [1, (77) and (78)]:

$$\frac{1}{2\pi\hbar}\int \rho_{\mu\eta}(q', p', t)\mathrm{d}q' = [\tilde{\Psi}^{(\mu)}(p', t)]^{\star}$$
$$\times \tilde{\Psi}^{(\eta)}(p', t) \qquad (78)$$

and

$$\frac{1}{2\pi\hbar}\int \rho_{\mu\eta}(q', p', t)\mathrm{d}p' = (+i)^{\mu-\eta}\left(\frac{q'}{\hbar}\right)^{\mu+\eta}$$
$$\times \langle q'|\,\hat{\rho}(t)|q'\rangle. \qquad (79)$$

In particular, for $\mu = \eta = 0$ and for any time $t \geq t_0$, the right-hand sides of (78) and (79) give the quantum probability densities for momentum and position, respectively, i.e., $|\tilde{\Psi}(p', t)|^2$ and $|\Psi(q', t)|^2$.

6.2 *Phase-space function* $W(\theta|\,q', p', t)$ *in the qp representation,* $\theta = 1$

In this case, instead of (69) one has the phase-space function [1, (81)]

$$W(1|\,q', p', t) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty}\frac{1}{\mu!\,\eta!}\tilde{\rho}_{\mu\eta}(q', p', t)$$
$$\times \mathcal{J}_{\mu\eta}(q',\ p'), \qquad (80)$$

with the cross-Wigner functions [1, (82)]

$$\begin{aligned}
&\tilde{\rho}_{\mu\eta}(q',p',t)\\
&\quad := \int w^{\star}(q,p')\left[\Psi^{(\mu)}\left(q'-\frac{1}{2}q,t\right)\right]^{\star}\\
&\qquad \times\Psi^{(\eta)}\left(q'+\frac{1}{2}q,t\right)\mathrm{d}q\\
&\quad = \left(-\frac{i}{\hbar}\right)^{\mu}\left(+\frac{i}{\hbar}\right)^{\eta}\\
&\qquad \times\int w(q',p)\left(p'-\frac{1}{2}p\right)^{\mu}\left(p'+\frac{1}{2}p\right)^{\eta}\\
&\qquad \times\left\langle p'+\frac{1}{2}\right|\hat{\rho}(t)\left|p'-\frac{1}{2}p\right\rangle\mathrm{d}p,
\end{aligned}\tag{81}$$

and the referential functions $\mathcal{J}_{\mu\eta}(q',p')$ [1, (83)]. From a comparison between (81) and (70), one concludes that momentum and position wave functions interchange their roles in the definitions of the phase-space functions $\tilde{\rho}_{\mu\eta}(q',p',t)$ and $\rho_{\mu\eta}(q',p',t)$. Therefore, one may proceed by following a similar treatment to the one described in §6.1. The results are shown in the following.

6.2.1 *Cross-Wigner functions based on the first equality of eq.* (81). In this case, one uses the expression (35) for $\Psi^{(\eta)}(q,t)$, and obtains

$$\begin{aligned}
\tilde{\rho}_{\mu\eta}(q',p',t) &= [M(q'-\mathbb{Q}(t))]^2\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m^{\star}(\tau)a_n(\tau)}{2^{m+n}\sqrt{m!\,n!}}\\
&\quad\times\mathbb{X}_{m,\mu,n,\eta}(q',p'-\mathbb{P}(t),t),
\end{aligned}\tag{82}$$

where

$$\begin{aligned}
&\mathbb{X}_{m,\mu,n,\eta}(q',p'-\mathbb{P}(t),t)\\
&\quad := \int w^{\star}(q,p'-\mathbb{P}(t))\exp\left(-\left[\frac{q}{2q_0}\right]^2\right)\\
&\qquad\times\mathbb{X}^{\star}_{m,\mu}\left(q'-\frac{1}{2}q,t\right)\mathbb{X}_{n,\eta}\left(q'+\frac{1}{2}q,t\right)\mathrm{d}q
\end{aligned}\tag{83}$$

is an auxiliary quantity that can be expressed in the form

$$\begin{aligned}
&\mathbb{X}_{m,\mu,n,\eta}(q',p'-\mathbb{P}(t),t) := 2(-i)^{\mu-\eta}\kappa_0^{\mu+\eta-1}\\
&\quad\times\sum_{\ell=0}^{\mu}\sum_{k=0}^{\eta}(-i)^{\ell-k}\binom{\mu}{\ell}\binom{\eta}{k}[2\chi_0\mathbb{P}(t)]^{\mu+\eta-\ell-k}\\
&\quad\times\sum_{M=0}^{m+\ell}\sum_{N=0}^{n+k}\theta_M(m,\ell)\,\theta_N(n,k)(-1)^M\\
&\quad\times F_{MN}(\mathcal{X},a,b,c).
\end{aligned}\tag{84}$$

Here, $F_{MN}(\mathcal{X},a,b,c)$ is given in Appendix C, and the set $\{\chi,a,b,c\}$ takes the values

$$\begin{aligned}
&\chi = 4\chi_0[p'-\mathbb{P}(t)],\quad a=2,\\
&b=-c=\kappa_0[q'-\mathbb{Q}(t)].
\end{aligned}\tag{85}$$

6.2.2 *Cross-Wigner functions based on the second equality of eq.* (81). Similarly, one writes the result as

$$\begin{aligned}
\tilde{\rho}_{\mu\eta}(q',p',t) &= 2p_0\frac{(-i)^{\mu-\eta}}{q_0^{\mu+\eta}}[\tilde{M}(p'-\mathbb{P}(t))]^2\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(+i)^{m-n}a_m^{\star}(\tau)a_n(\tau)}{2^{(m+n)/2}\sqrt{m!\,n!}}(-1)^{m+\mu}\\
&\quad\times\mathcal{F}^{\star}(\mu,m,\eta,n|\mathcal{X},a,b,c,\Lambda,\lambda),
\end{aligned}\tag{86}$$

with

$$\begin{aligned}
&\mathcal{X} = 2\frac{q'-\mathbb{Q}}{q_0},\quad a=1,\quad b=-c=-\frac{p'-\mathbb{P}(t)}{p_0},\\
&\Lambda=-\lambda=-\frac{p'}{p_0}.
\end{aligned}\tag{87}$$

In (86), one notes the presence of the complex conjugate of $\mathcal{F}(\cdots)$, which is a consequence of the definition of $\mathcal{F}(\cdots)$ in terms of the exponential function $\exp(-i\mathcal{X}y)$ (see eq. (D.1)).

6.2.3 *Properties of* $\tilde{\rho}_{\mu\eta}(q',p',t)$. The cross-Wigner function $\tilde{\rho}_{\mu\eta}(q',p',t)$ implies the expressions [1, (84) and (85)]:

$$\begin{aligned}
\frac{1}{2\pi\hbar}\int\tilde{\rho}_{\mu\eta}(q',p',t)\mathrm{d}p' &= [\Psi^{(\mu)}(q',t)]^{\star}\\
&\quad\times\Psi^{(\eta)}(q',t)
\end{aligned}\tag{88}$$

and

$$\begin{aligned}
\frac{1}{2\pi\hbar}\int\tilde{\rho}_{\mu\eta}(q',p',t)\mathrm{d}q' &= (-i)^{\mu-\eta}\left(\frac{p'}{\hbar}\right)^{\mu+\eta}\\
&\quad\times\langle p'|\hat{\rho}(t)|p'\rangle.
\end{aligned}\tag{89}$$

Similar to (78) and (79), when $\mu=\eta=0$, integrating $\tilde{\rho}_{\mu\eta}(q',p',t)$ over momentum produces the space probability distribution $|\Psi(q',t)|^2$, and integrating over position gives the momentum probability distribution $|\tilde{\Psi}(p',t)|^2$.

6.3 *Diagonal cross-Wigner functions*

Consider the functions $W(\theta|q',p',t)$ given by (69) and (80), with $\theta=0$ and 1. These functions act as bridges connecting $\rho(\theta, q', p'|q, p, t)$ with the cross-Wigner functions $\rho_{\mu\eta}(q', p', t)$ and $\tilde{\rho}_{\mu\eta}(q', p', t)$. When the diagonal elements $\mu = \eta$ are considered in (70) and (81), one finds that the cross-Wigner functions $\rho_{\mu\mu}(q', p', t)$ and $\tilde{\rho}_{\mu\mu}(q', p', t)$ are real-valued quantities, so that the usual definition of the Wigner function is obtained for $\mu = 0$. Thus, the method formulated in [1], that has been also used in this paper, is consistent with theoretical methods and numerical approaches described in the literature, in which real-valued Wigner functions defined in phase space are used.

However, the method presented in this paper leads to additional possibilities, e.g., for $\mu \neq \eta$, the function $\rho_{\mu\eta}(q', p', t)$, given in (70), is a weighted measure of the interference pattern generated at the point (q', p') and time t, by the overlaps between the components $\tilde{\Psi}^{(\mu)}_{[k]}\left(p' - \frac{1}{2}p, t\right)$ and $\tilde{\Psi}^{(\eta)}_{[k]}\left(p' + \frac{1}{2}p, t\right)$ of the states in the mixture $\hat{\rho}(t)$. In other words, $\rho_{\mu\eta}(q', p', t)$ describes the cumulative contributions, at the orders μ and η, originating from $\mathcal{N}$ pure states in $\hat{\rho}(t)$.

7. Harmonic oscillator in an oscillating electric field

As an example, consider a one-dimensional harmonic oscillator of charge e_0 exposed to a time-dependent electric field $\mathcal{E}(t) = \mathcal{E}_0 \cos(\Omega t + \vartheta)$, for $t_0 < t \leq t_f$, and $\mathcal{E}(t) = 0$, otherwise, which is polarized in the direction of motion of the oscillator. The constants $\mathcal{E}_0 > 0$, $\Omega > 0$ and ϑ are the amplitude, the angular frequency and the phase of the electric field, respectively. The external driven force is given by $F(t) := F_0 \cos(\Omega t + \vartheta)$, where $F_0 = e_0\mathcal{E}_0$ and $T_0 = 2\pi/\Omega$ is the period. In detail, the Hamiltonian of the system under consideration is given by $\hat{H}(t) = \hat{K} + \hat{V}(t)$, with $\hat{K} = \hat{p}^2/(2m_0) + \frac{1}{2}m_0\omega_0^2\hat{q}^2$ and $\hat{V}(\hat{q}, t) = -F_0 \cos(\Omega t + \vartheta)\hat{q}$.

7.1 *Functions* $\mathbb{Q}(t, t_0)$, $\mathbb{P}(t, t_0)$ *and* $\gamma(t, t_0)$

For the calculation of functions $\mathbb{Q}(t, t_0)$ and $\mathbb{P}(t, t_0)$ given by (15), one requires the functions $Q(t, t_0)$ and $P(t, t_0)$. When $F(t)$ is substituted into (10)–(12), one gets

$$Q(t, t_0) = Q_0\left[2\cos(\Omega t_0 + \vartheta) - \frac{\Omega_+}{\omega_0}\cos(\Omega_- t + \omega_0 t_0 + \vartheta) + \frac{\Omega_-}{\omega_0}\cos(\Omega_+ t - \omega_0 t_0 + \vartheta)\right] \tag{90}$$

and

$$P(t, t_0) = P_0\left[-2\sin(\Omega t_0 + \vartheta) + \frac{\Omega_+}{\Omega}\sin(\Omega_- t + \omega_0 t_0 + \vartheta) + \frac{\Omega_-}{\Omega}\sin(\Omega_+ t - \omega_0 t_0 + \vartheta)\right]. \tag{91}$$

Similarly, from (12) one obtains the phase factor

$$\gamma(t, t_0) = \frac{\Omega E_q}{\Omega_+\Omega_-}\Bigg[-\Omega(t - t_0) - \cos(\Omega(t + t_0) + 2\vartheta) \times \sin(\Omega(t - t_0)) + \frac{\Omega\Omega_+}{2\omega_0\Omega_-}\sin(\Omega_-(t - t_0)) - \frac{\Omega\Omega_-}{2\omega_0\Omega_+}\sin(\Omega_+(t - t_0)) + \frac{\Omega}{\omega_0}\cos(\Omega(t + t_0) + 2\vartheta)\sin(\omega_0(t - t_0))\Bigg].$$

In the foregoing equations, the following auxiliary quantities have been introduced:

$$\Omega_\pm := \Omega \pm \omega_0, \qquad Q_0 = \frac{F_0}{2m_0\Omega_+\Omega_-},$$
$$P_0 = \frac{F_0\Omega}{2\Omega_+\Omega_-}, \qquad E_q = \frac{F_0^2}{4m_0\Omega^2}, \tag{92}$$

where E_q is the so-called ponderomotive (or quiver) energy.

Hereinafter, an electron is considered, atomic units (a.u.) are used, and data from ref. [18] are chosen, namely, mass $m_0 = 1$, charge $e_0 = -e = -1$, natural frequency $\omega_0 = 0.01$, field frequency $\Omega = 1$, phase

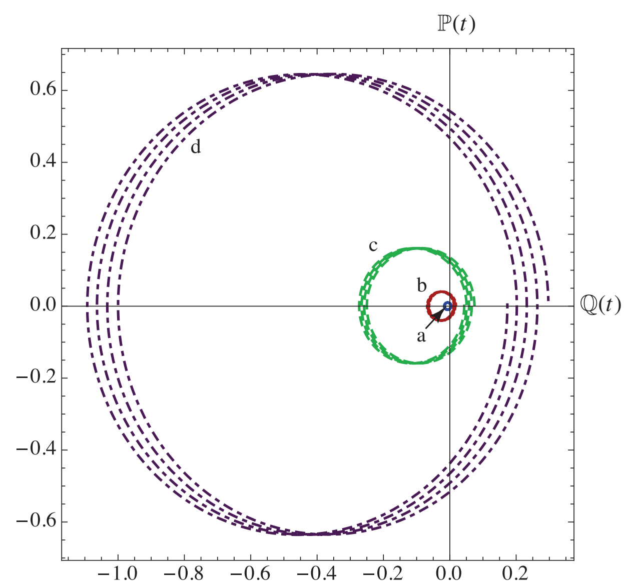

Figure 1. Parametric plot of $\mathbb{Q}(t)$ and $\mathbb{P}(t)$ as a function of time t ($24\pi \leq t \leq 32\pi$), for the peak field strength $\mathcal{E}_0$, in a.u.: (**a**) 0.01, (**b**) 0.04, (**c**) 0.16 and (**d**) 0.64.

$\vartheta = 0$, and the peak field strength $\mathcal{E}_0 = 0.64$ a.u. The initial state is chosen as a linear combination of the ground state and the first excited state of the harmonic oscillator: $|\Psi(t_0)\rangle = a_0(t_0)\,|0\rangle + a_1(t_0)\,|1\rangle$, with coefficients such that $|a_0(t_0)|^2 + |a_1(t_0)|^2 = 1$, e.g. $a_0(t_0) = 3i/5$ and $a_1(t_0) = 4/5$. In general, a time interval of 16 field cycles is considered, $0 \le t \le 32\pi$ a.u.

The function $\gamma(t, t_0)$ given in (92) can be divided into a linear term $-\gamma_0(t - t_0)$, where $\gamma_0 = E_q\,\Omega^2/(\Omega_+\Omega_-)$, plus periodic nonlinear contributions $\gamma(t, t_0) + \gamma_0(t - t_0)$ on time $t - t_0$. From (26), (40) and (47), one notes that, with increasing time, the linear term $-\gamma_0(t - t_0)$ induces large oscillations in the phases of the wave functions $\Psi_\pm(q, p, t)$, $\Psi(q, t)$ and $\tilde{\Psi}(p, t)$. Notwithstanding the fact that the presence of $\gamma(t, t_0)$ is essential to solve the Schrödinger equation, this phase factor does not play any role in the calculation of expected values or in the calculation of cross-Wigner functions.

Figure 1 shows, over the time interval $24\pi \le t \le 32\pi$, parametric curves of the functions $\mathbb{Q}(t)$ and $\mathbb{P}(t)$ – defined by eqs (15), (90) and (91) – for several values of peak field strength $\mathcal{E}_0$: (a) 0.01, (b) 0.04, (c) 0.16 and (d) 0.64. As expected, the biggest tours of $(\mathbb{Q}(t), \mathbb{P}(t))$ in the phase space are held when $\mathcal{E}_0 = 0.64$.

7.2 *Husimi and phase-space wave function* $\Psi_+(q, p, t)$

For $\mathcal{E}_0 = 0.64$, and for the phase-space points $(q, p) = (0.4, 0.6)$ and $(0.4, 0.8)$, the Husimi function and the phase-space wave function $\Psi_+(q, p, t)$ given by (28) and (26), respectively, are shown in figure 2. In general, at each point (q, p) of the phase space, the behaviour of the Husimi function $\rho_{\mathrm{H}}(q, p, t)$ and the wave functions $\Psi_\pm(q, p, t)$ stem from the time dependence of the quantum Hamiltonian (non-conservation

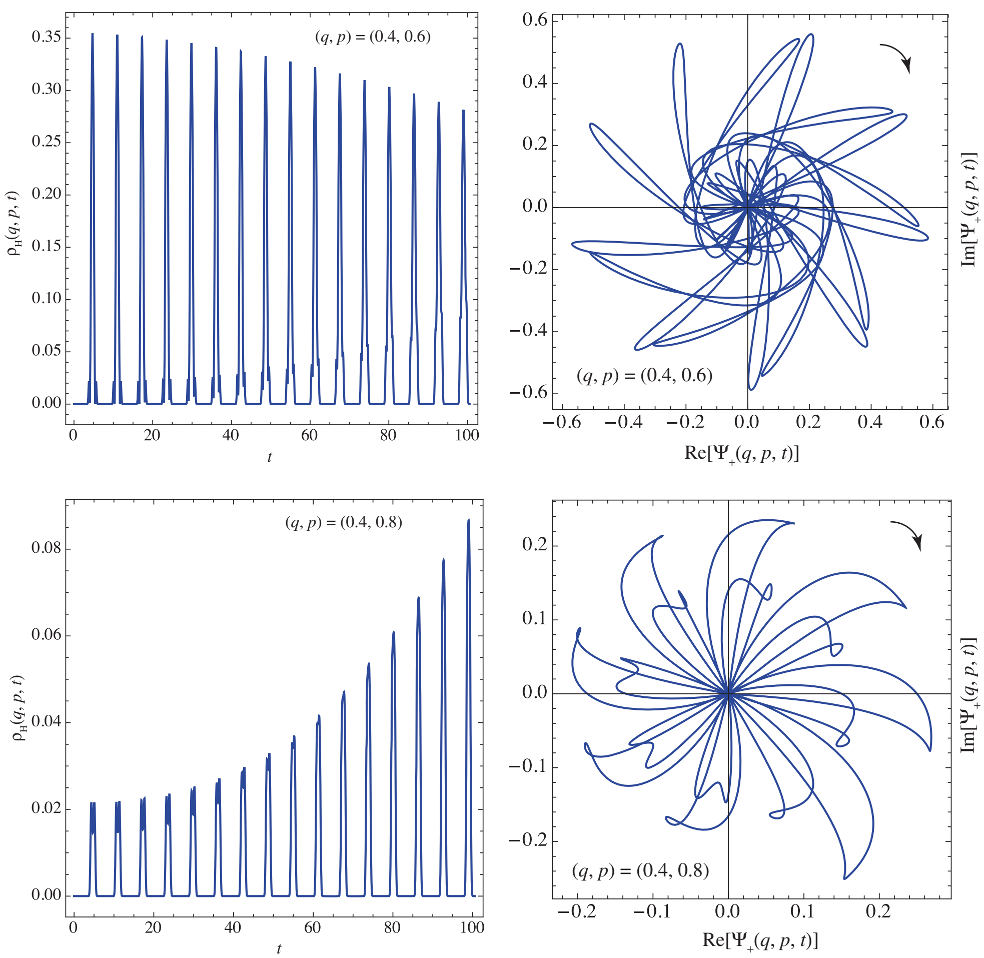

Figure 2. Husimi function (left panels) and phase-space wave function $\Psi_+(q, p, t)$ (right panels) evaluated at the points $(q, p) = (0.4, 0.6)$ and $(q, p) = (0.4, 0.8)$ as a function of time t, for a peak field strength $\mathcal{E}_0 = 0.64$ a.u. The parametric plots are generated with the real and imaginary parts of $\Psi_+(q, p, t)$, and the arrows indicate the direction in which the patterns are generated.

of energy) and from the phase-factors $A_\pm(q,p,t) = \exp(\pm i\,\zeta_\pm(q,p,t)/\hbar)$, whose phases $\zeta_\pm(q,p,t)$ are given by (25). In the right panels of figure 2, the parametric representation of the phase-space wave function $\Psi_+(q,p,t)$ displays in the plane $\mathrm{Re}[\Psi_+(q,p,t)]$–$\mathrm{Im}[\Psi_+(q,p,t)]$ a structure of leaves that gradually appears over time and, for the values of (q,p) used in the examples, the global patterns are generated in clockwise direction.

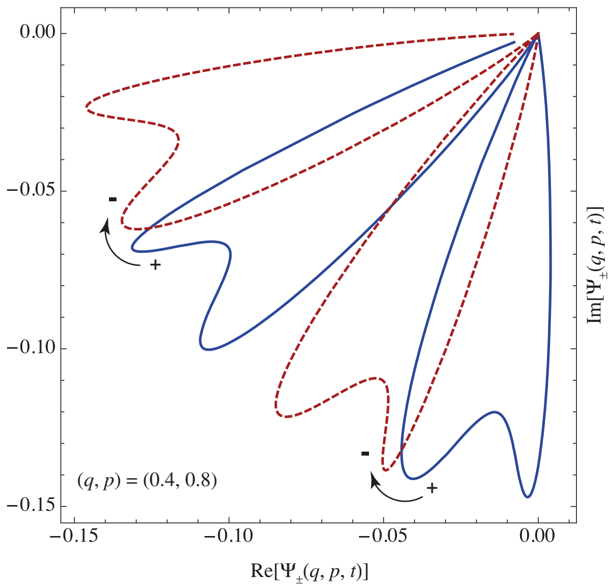

Figure 3. Graphical representation of transform (93) for a time interval $0 \le t \le 12$ a.u. In the course of time, a clockwise rotation over an angle $qp/\hbar$ moves each point $(\mathrm{Re}[\Psi_+(q,p,t)], \mathrm{Im}[\Psi_+(q,p,t)])$ to the position $(\mathrm{Re}[\Psi_-(q,p,t)], \mathrm{Im}[\Psi_-(q,p,t)])$.

By virtue of the relation $\Psi_-(q,p,t) = w^\star(q,p)\,\Psi_+(q,p,t)$, one can write

$$\begin{bmatrix}\mathrm{Re}[\Psi_-(q,p,t)]\\ \mathrm{Im}[\Psi_-(q,p,t)]\end{bmatrix} = \begin{bmatrix}\cos(qp/\hbar) & \sin(qp/\hbar)\\ -\sin(qp/\hbar) & \cos(qp/\hbar)\end{bmatrix} \times \begin{bmatrix}\mathrm{Re}[\Psi_+(q,p,t)]\\ \mathrm{Im}[\Psi_+(q,p,t)]\end{bmatrix} \tag{93}$$

and, therefore, for time t and phase-space point (q,p), the parametric representation of $\Psi_-(q,p,t)$ can be obtained from the one of $\Psi_+(q,p,t)$ by a clockwise rotation around the z-axis over an angle $qp/\hbar$ while keeping the phase-space axes fixed (active rotation). In figure 3, this property is illustrated by considering a time interval $0 \le t \le 12$ a.u. and the phase-space point $(q,p) = (0.4, 0.8)$ a.u.

7.3 *Position and momentum marginal probability distributions*

The position and momentum wave functions are given by expressions (40) and (47), respectively: $\Psi(q,t)$ and $\tilde{\Psi}(p,t)$. One verifies that these functions are normalized to unity, and plots the position and momentum probability density functions for any value of time $t \ge t_0$, as shown in figure 4 at four time points. For the superposition state consisting of the ground state and the first excited state of the harmonic oscillator, one has

$$|\Psi(q,t)|^2 = [M(q-\mathbb{Q}(t))]^2 \times \left[|a_0(0)|^2 + 2\,|a_1(0)|^2\left(\frac{q-\mathbb{Q}(t)}{q_0}\right)^2 + \mathcal{I}_+(t-t_0)\frac{q-\mathbb{Q}(t)}{q_0}\right] \tag{94}$$

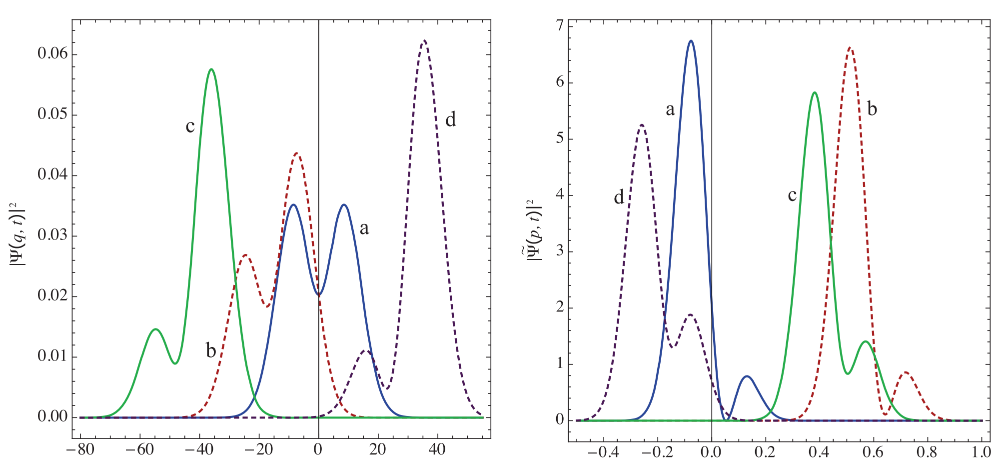

Figure 4. Position and momentum probability density functions, $|\Psi(q,t)|^2$ and $|\tilde{\Psi}(p,t)|^2$, at four time points: (a) $t = 0$, (b) $t = 27$, (c) $t = 77$ and (d) $t = 100$ a.u.

and

$$|\tilde{\Psi}(p,t)|^2 = [\tilde{M}(p-\mathbb{P}(t))]^2 \times\left[\,|a_0(0)|^2 + 2\,|a_1(0)|^2\left(\frac{p-\mathbb{P}(t)}{p_0}\right)^2 - i\,\tilde{\mathcal{I}}_-(t-t_0)\left(\frac{p-\mathbb{P}(t)}{p_0}\right)\right], \tag{95}$$

with the interference terms

$$\mathcal{I}_\pm(\tau) := \sqrt{2}\,a_0^\star(0)a_1(0)\exp\left(-\frac{i}{\hbar}\omega_0\tau\right) \pm\sqrt{2}\,a_0(0)a_1^\star(0)\exp\left(+\frac{i}{\hbar}\omega_0\tau\right). \tag{96}$$

It is apparent from eqs (94) and (95) that the position and momentum probability distributions have the form $f(x) := \exp(-x^2)[A_0+2A_1x^2+A_2x]$, where x can be identified either with $(q-\mathbb{Q}(t))/q_0$ or $(p-\mathbb{P}(t))/p_0$, and the coefficients A_0 and A_1 are time-independent whereas $A_2(\tau)$ is time-dependent. Because the relation $\mathrm{d}f(x)/\mathrm{d}x = 0$ implies the cubic equation $-4A_1x^3 - 2A_2x^2 + (4A_1 - 2A_0)x + A_2 = 0$, in each case, interior to the domains of the functions $|\Psi(q,t)|^2$ and $|\tilde{\Psi}(p,t)|^2$, there are three relative extrema (figure 4), i.e., two maxima and one minimum.

As time passes, the relative extrema of the probability density function $|\Psi(q,t)|^2$ change their positions in the coordinate q according to a triplet of functions, $q_1(t) < q_2(t) < q_3(t)$, whose behaviours are illustrated in the left panel of figure 5. The q-separation between the relative extrema of $|\Psi(q,t)|^2$ over a long time exhibits a periodic behaviour, as shown in the right panel of figure 5, in which the distances $q_3(t) - q_2(t)$, $q_2(t) - q_1(t)$ and $q_3(t) - q_1(t)$ are plotted. The inset in the right panel shows the overall pattern formed by the functions $q_1(t)$, $q_2(t)$ and $q_3(t)$ during the time interval $0 \le t \le 650.0$ a.u. At this point, it is to be noted that $Q(t,t_0)$ and $\mathbb{Q}(t,t_0)$ are both functions of time, and that the local and overall patterns are determined by the frequencies ω_0, Ω and $\Omega_\pm$.

At this point, with the help of eq. (95) a similar procedure leads to the patterns generated by the probability density function $|\tilde{\Psi}(p,t)|^2$, which are shown in figure 6.

7.4 *Cross-Wigner functions*

Starting with the initial state $|\Psi(0)\rangle = a_0(0)\,|0\rangle + a_1(0)\,|1\rangle$, with $a_0(0) = 3i/5$ and $a_1(0) = 4/5$, the real-valued cross-Wigner functions $\rho_{00}(q,p,t)$ and $\rho_{11}(q,p,t)$ are calculated by applying eqs (76) and (77), and the results are shown in figures 7 for the time $t \approx 15.71$ a.u. In figure 8, a parametric representation of the complex-valued function $\rho_{01}(q,p,t) = \rho_{10}^\star(q,p,t)$ is shown for the time interval $0 \le t \le 100.5$ a.u. and the phase-space points $(q,p) = (0.4, 0.6)$ and $(q,p) = (0.4, 0.8)$.

Functions $\rho_{\mu\eta}(q,p,t)$ can take either positive and negative values, and they satisfy the relations (78) and (79). Thus, except by proportionality factors involved in these equations, (i) integrating the diagonal cross-Wigner function $\rho_{\mu\mu}(q,p,t)$ over q leads to the marginal quantity $|\tilde{\Psi}^{(\mu)}(p,t)|^2$ and (ii) integrating the non-diagonal $(\mu \neq \eta)$ cross-Wigner function

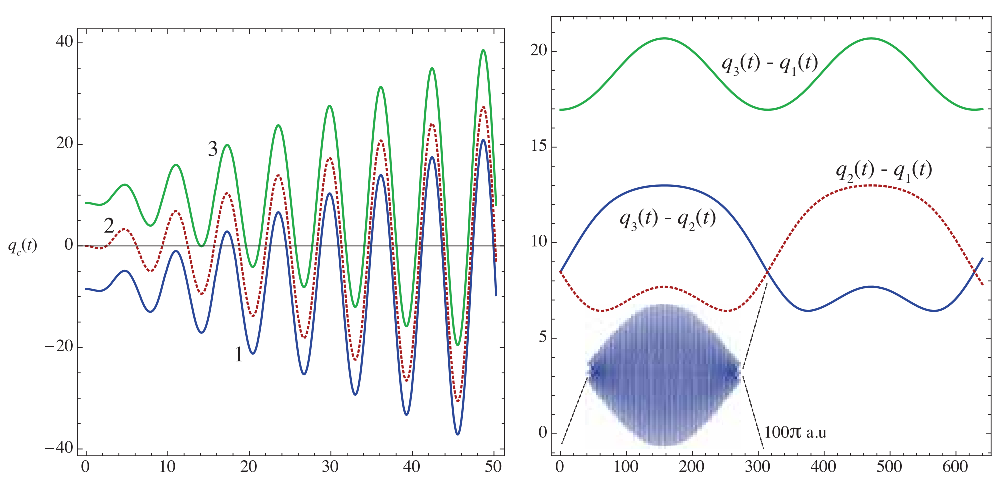

Figure 5. At each time t, the probability density function $|\Psi(q,t)|^2$ has three relative extrema (left panel), whose positions in coordinate q are given by the functions $q_c(t)$: $q_1(t) < q_2(t) < q_3(t)$. In the right panel, the distances $q_3(t) - q_2(t)$, $q_2(t) - q_1(t)$ and $q_3(t) - q_1(t)$ are plotted over a long time interval, $t_0 \le t \le 640.0$ a.u. The inset is the overall pattern formed by $q_c(t)$, when $0 \le t \le 100\pi$ a.u.

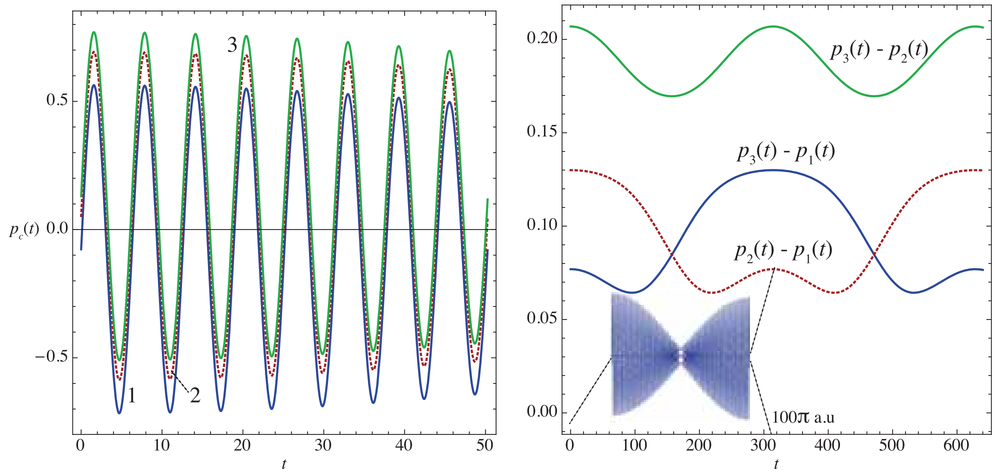

Figure 6. At each time t, the probability density function $|\tilde{\Psi}(p,t)|^2$ has three relative extrema (left panel), whose positions in coordinate p are given by the functions $p_c(t)$: $p_1(t) < p_2(t) < p_3(t)$. In the right panel, the distances $p_3(t) - p_2(t)$, $p_2(t) - p_1(t)$ and $p_3(t) - p_1(t)$ are plotted over a long time interval, $t_0 \leq t \leq 640.0$ a.u. The inset is the overall pattern formed by $p_c(t)$, when $0 \leq t \leq 100\pi$ a.u.

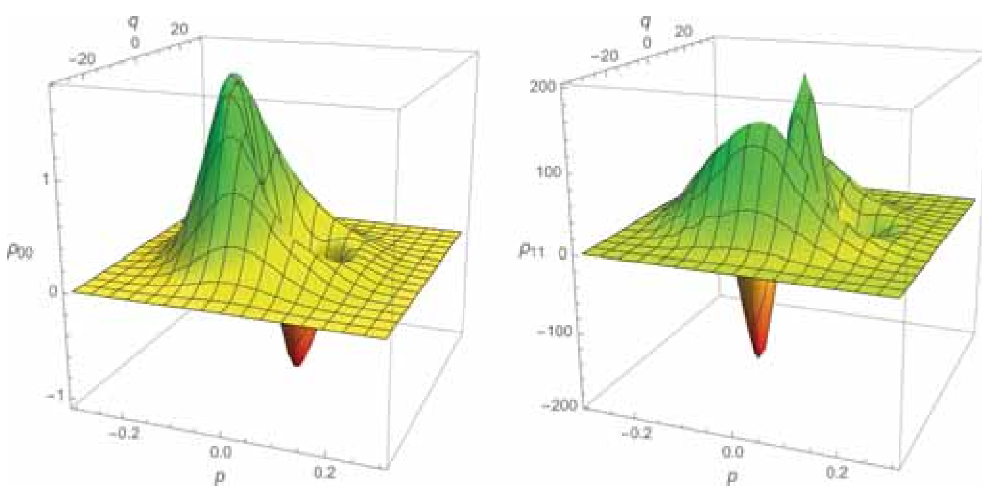

Figure 7. Plot of cross-Wigner functions $\rho_{00}(q, p, t)$ and $\rho_{11}(q, p, t)$ for $t \approx 15.71$ a.u., when the system starts from an initial state $|\Psi(0)\rangle = a_0(0)\,|0\rangle + a_1(0)\,|1\rangle$, with $a_0(0) = 3i/5$ and $a_1(0) = 4/5$. Note the negative values.

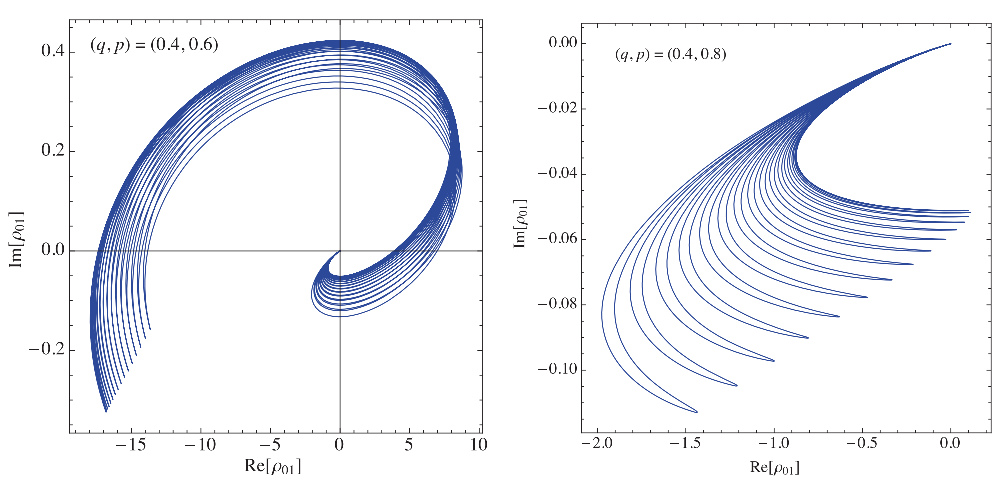

Figure 8. Parametric plot of the cross-Wigner function $\rho_{01}(q, p, t)$ for $0 \leq t \leq 100.5$ a.u., when the system starts from an initial state $|\Psi(0)\rangle = a_0(0)\,|0\rangle + a_1(0)\,|1\rangle$, with $a_0(0) = 3i/5$ and $a_1(0) = 4/5$.

$\rho_{\mu\eta}(q, p, t)$ over q gives the product $[\tilde{\Psi}^{(\mu)}(p, t)]^{\star}\,\tilde{\Psi}^{(\eta)}(p, t)$, whereas if one integrates $\rho_{\mu\eta}(q, p, t)$ over p, one gets $(q)^{\mu+\eta}|\Psi(q, t)|^2$.

8. Summary and conclusion

As a general conclusion, the phase-space representations of quantum mechanics using Glauber coherent states is an appropriate and efficient method for dealing with physical systems, in particular for getting an exact treatment of the driven harmonic oscillator. This approach is an alternative and a complement to other methods described in the literature, e.g. [3–10].

Notwithstanding the mandatory consequences of the uncertainty relation for position and momentum, $\Delta q \Delta p \geq \hbar/2$, a pure state of the driven harmonic oscillator can be described by the complex-valued wave functions $\Psi(\theta|q, p, t)$ in the phase space, in terms of the pq ($\theta = 0$) and qp ($\theta = 1$) representations. Similarly, in the general case of a mixed state, the weighted mixture of pure states represented in eq. (7) by the function $\rho(\theta, q', p'|q, p, t)$ gives account of the state of the system at time t. Incidentally, eqs (6) define a two-dimensional cell in phase space, which is centred at the point (q', p') and has edge lengths q and p. In the case of a pure state, $\rho(\theta, q', p'|q, p, t)$ becomes the product of probability amplitudes $[\Psi(\theta|q' - \frac{1}{2}q, p' - \frac{1}{2}p, t)]^{\star}\Psi(\theta|q' + \frac{1}{2}q, p' + \frac{1}{2}p, t)$.

The usual Wigner function, the cross-Wigner functions and the Husimi distribution arise in a very natural way, and (§6) analytical expressions has been deduced for them. The results obtained in this paper reinforce and encourage the treatment of quantum systems in phase space using either wave functions or cross-Wigner functions.

A final comment is relevant. In classical mechanics (cm), if one establishes that at time t_0 the solution curve or trajectory of the driven classical harmonic oscillator passes through the phase-space point $(\mathbb{Q}(t_0), \mathbb{P}(t_0))$, then the solution of the equations of motion (17) is given by

$$\begin{bmatrix} \mathbb{Q}(t) \\ \mathbb{P}(t) \end{bmatrix}_{\mathrm{cm}} = \begin{bmatrix} \cos(\omega_0\tau) & \sin(\omega_0\tau)/(m_0\omega_0) \\ -m_0\omega_0\sin(\omega_0\tau) & \cos(\omega_0\tau) \end{bmatrix} \times \begin{bmatrix} \mathbb{Q}(t_0) + Q(t, t_0) \\ \mathbb{P}(t_0) + P(t, t_0) \end{bmatrix}, \tag{97}$$

where $Q(t, t_0)$ and $P(t, t_0)$ are defined by (10) and (11). Thus, comparing (97) and (15), one concludes that the classical trajectory with initial condition $(\mathbb{Q}(t_0), \mathbb{P}(t_0)) = (0, 0)$ is the only one relevant for the quantum treatment described in this work.

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Appendix A. Some formulae used in this work

The Hermite polynomials satisfy the following relations:

1. [19, Section 5.6.4],

$$H_\eta(y + \sigma) = \sum_{k=0}^{\eta} \binom{\eta}{k} H_k(y)(2\sigma)^{\eta-k} = \sum_{k=0}^{\eta} \binom{\eta}{\eta - k} H_{\eta-k}(\sigma)(2y)^k. \tag{A.1}$$

2. [14(b)],

$$H_m(y)H_k(y) = \sum_{r=0}^{\min(m,k)} 2^r r! \binom{m}{r}\binom{k}{r} \times H_{m+k-2r}(y). \tag{A.2}$$

3. [20, 8.958.2],

$$\sum_{m=0}^{n} \binom{n}{m} H_{n-m}(x)H_m(y) = 2^{n/2} H_n\left(\frac{x+y}{\sqrt{2}}\right). \tag{A.3}$$

4. The Fourier transformation of the Hermite polynomials is given by

$$\int_{-\infty}^{\infty} \exp(ixy)\exp\left(-\frac{1}{2}y^2\right) H_m(y)\mathrm{d}y = (+i)^m\sqrt{2\pi}\exp\left(-\frac{1}{2}x^2\right) H_m(x). \tag{A.4}$$

5. Then, given a Hermite polynomial $H_n(y + c)$, one can write the identity

$$(y+b)^m H_n(y+c) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!\, 2^{n-2k}}{k!\,(n-2k)!} \times (y+b)^m (y+c)^{n-2k} = \sum_{k=0}^{[n/2]} \sum_{\ell=0}^{m+n-2k} \frac{(-1)^k\, n!\, 2^{n-2k}}{k!\,(n-2k)!} \times A_\ell(m, n-2k; b, c)y^\ell, \tag{A.5}$$

where the coefficients $A_\ell(m, n-2k; b, c)$ are given by expression (D.2).

Appendix B. On the polynomials $\Pi_{n,k}(x)$

From the definition of $\Pi_{n,k}(x)$ given by (37), and formula (A.2), it follows that

$$\Pi_{n,k}(x) := \sum_{\lambda=0}^{n} \sum_{r=0}^{\min(\lambda,k)} \binom{n}{\lambda}(-2)^r r! \binom{\lambda}{r}\binom{k}{r} H_{n-\lambda}(x) \times H_{\lambda+k-2r}(x)$$
$$= \sum_{\lambda=0}^{n} \sum_{r=0}^{\min(\lambda,k)} \sum_{u=0}^{\min(n-\lambda,\lambda+k-2r)} \Theta(n,k;\lambda;r,u) \times H_{n+k-2r-2u}(x), \tag{B.1}$$

with coefficients

$$\Theta(n,k;\lambda;r,u) := (-1)^r 2^{r+u} r! u! \binom{n}{\lambda}\binom{\lambda}{r}\binom{k}{r} \times \binom{n-\lambda}{u}\binom{\lambda+k-2r}{u}. \tag{B.2}$$

Equation (B.1) can be written in a more elegant way, namely

$$\Pi_{n,k}(x) = \sum_{N=0}^{n+k} \theta_N(n,k) H_N(x),$$
$$\Pi_{n,k}(-x) = (-1)^{n+k}\, \Pi_{n,k}(x), \tag{B.3}$$

where the values $\theta_N(n,k)$ are determined by equating the coefficients of x^N in both sides of the equality

$$\sum_{N=0}^{n+k} \theta_N(n,k) x^N = \sum_{\lambda=0}^{n} \sum_{r=0}^{\min(\lambda,k)} \sum_{u=0}^{\min(n-\lambda,\lambda+k-2r)} \times \Theta(n,k;\lambda;r,u) x^{n+k-2r-2u}. \tag{B.4}$$

After exploring various examples, one finds that $\theta_N(n,k) = 0$, if $(n+k)$ is even and N is odd, or if $(n+k)$ is odd and N is even. Using this fact and the reflection formula $H_N(-x) = (-1)^N H_N(x)$ for the Hermite polynomials, one confirms that the polynomial $\Pi_{n,k}(x)$ is even or odd depending on the value of $n+k$.

At this point, in (B.3) and (B.4) consider the particular situation in which $k = 0$ and, from (38), recall that $\Pi_{n,0}(x) = 2^{n/2} H_n(x\sqrt{2})$. Then, a formula linking the Hermite polynomials $H_n(x\sqrt{2})$ with $H_N(x)$ [14(c)] allows one to write

$$\Pi_{n,0}(x) = 2^{n/2} H_n(x\sqrt{2}) = \sum_{j=0}^{[n/2]} \frac{n!}{j!(n-2j)!} 2^{n-j} \times H_{n-2j}(x). \tag{B.5}$$

Hence, by comparing this relation with (B.3), one finds the coefficients

$$\theta_N(n,0) = \theta_j(n) := \frac{n!}{j!(n-2j)!} 2^{n-j}, \tag{B.6}$$

if $N = n-2j$ and $j = 0, \ldots, [n/2]$, and $\theta_N(n,0) = 0$, otherwise.

Appendix C. Evaluation of the integral $F_{MN}(\mathcal{X}, a, b, c)$

For $a > 0$, consider the integral

$$F_{MN}(\mathcal{X}, a, b, c) := \int_{-\infty}^{\infty} \mathrm{d}y \exp(-i\mathcal{X}y) \exp(-ay^2) \times H_M(y+b) H_N(y+c). \tag{C.1}$$

By using (A.1) and (A.2), and the integral

$$\int_{-\infty}^{\infty} \exp(-ixy)\, y^k \exp(-ay^2)\mathrm{d}y = (-i)^k \sqrt{\frac{\pi}{a}} \left(\frac{1}{2\sqrt{a}}\right)^k \exp\left(-\frac{x^2}{4a}\right) H_k\left(\frac{x}{2\sqrt{a}}\right), \tag{C.2}$$

one finds that

$$F_{MN}(\mathcal{X}, a, b, c) = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\mathcal{X}^2}{4a}\right) \times \sum_{k=0}^{M}\sum_{\ell=0}^{N} \binom{M}{M-k}\binom{N}{N-\ell}\left(-\frac{i}{\sqrt{a}}\right)^{k+\ell} \times H_{k+\ell}\left(\frac{\mathcal{X}}{2\sqrt{a}}\right) H_{M-k}(b) H_{N-\ell}(c). \tag{C.3}$$

Note that $F_{MN}(-\mathcal{X}, a, b, c) = F^{\star}_{MN}(\mathcal{X}, a, b, c)$, where $\star$ denotes the complex conjugate.

Appendix D. Evaluation of the integral $\mathcal{F}(m, M, n, N|\mathcal{X}, a, b, c, \Lambda, \lambda)$

$$\mathcal{F}(\mu, M, \eta, N|\mathcal{X}, a, b, c, \Lambda, \lambda) := \int_{-\infty}^{\infty} \exp(-i\mathcal{X}y)\exp(-ay^2)(y+\Lambda)^{\mu} \times H_M(y+b)(y+\lambda)^{\eta} H_N(y+c)\mathrm{d}y. \tag{D.1}$$

On the one hand, for $\mu = \eta = 0$, one has $\mathcal{F}(0, M, 0, N|\mathcal{X}, a, b, c, \Lambda, \lambda) = F_{MN}(\mathcal{X}, a, b, c)$. On the other hand, using the binomial theorem followed by a change of indices $\ell := \mu + \nu$, for μ and η integers, one gets the relation

$$(y+\Lambda)^{\mu}(y+\lambda)^{\eta} = \sum_{r=0}^{\mu+\eta} A_r(\mu, \eta; \Lambda, \lambda) y^r, \tag{D.2}$$

where, for $\Lambda \neq 0$ and $\lambda \neq 0$, the coefficients $A_r(\mu, \eta; \Lambda, \lambda)$ are given by

$$A_r(\mu, \eta; \Lambda, \lambda) = \sum_{j=0}^{r} \binom{\mu}{j} \binom{\eta}{r-j} \Lambda^{\mu-j} \lambda^{\eta-r+j}. \quad \text{(D.3)}$$

In addition, the following special cases should be considered: (i) for $\Lambda = 0$ and $\lambda \neq 0$,

$$A_r(\mu, \eta; 0, \lambda) = \binom{\eta}{r-\mu} \lambda^{\eta-r+\mu}, \quad \text{if } \mu \leq r \leq \mu + \eta, \quad \text{(D.4)}$$

(ii) for $\Lambda \neq 0$ and $\lambda = 0$,

$$A_r(\mu, \eta; \Lambda, 0) = \binom{\mu}{r-\eta} \Lambda^{\eta-r+\mu}, \quad \text{if } \eta \leq r \leq \mu + \eta, \quad \text{(D.5)}$$

(iii) for $\Lambda = 0$ and $\lambda = 0$,

$$A_r(\mu, \eta; 0, 0) = 1 \quad \text{if } r = \mu + \eta, \quad \text{(D.6)}$$

where $A_r(\mu, \eta; \Lambda, \lambda) = 0$, otherwise.

Then, the identity $(i\partial/\partial\mathcal{X})^r \exp(-i\mathcal{X}y) = y^r \times \exp(-i\mathcal{X}y)$ implies that

$$\mathcal{F}(\mu, M, \eta, N|\mathcal{X}, a, b, c, \Lambda, \lambda) = \sum_{r=0}^{\mu+\eta} A_r(\mu, \eta; \Lambda, \lambda) \left(i\frac{\partial}{\partial\mathcal{X}}\right)^r F_{MN}(\mathcal{X}, a, b, c). \quad \text{(D.7)}$$

Thus, because $F_{MN}(\mathcal{X}, a, b, c)$ is given by expression (C.3), one can define an auxiliary function $\mathbb{G}_{k+\ell,r}(\chi, a)$ by the relation

$$\begin{aligned}\mathbb{G}_{k+\ell,r}(\chi, a) &:= \exp\left(+\frac{\chi^2}{4a}\right)\left(i\frac{\partial}{\partial\mathcal{X}}\right)^r F(\mathcal{X}) G_{k+\ell}(\mathcal{X}) \\ &= \left(\frac{i}{2\sqrt{a}}\right)^r \sum_{s=0}^{r} \binom{r}{s} \theta(k+\ell-s) \\ &\quad \times \frac{2^s (k+\ell)!}{(k+\ell-s)!} (-1)^{r-s} H_{r-s}\left(\frac{\chi}{2\sqrt{a}}\right) \\ &\quad \times H_{k+\ell-s}\left(\frac{\chi}{2\sqrt{a}}\right),\end{aligned} \quad \text{(D.8)}$$

where $F(\mathcal{X}) = \exp(-\chi^2/(4a))$ and $G_{k+\ell}(\mathcal{X}) = H_{k+\ell}(\chi/(2\sqrt{a}))$. For getting (D.8), one uses Leibnitz's rule for the rth derivative of a product of functions, Rodrigues's formula for the Hermite polynomials, and the expression

$$\left(\frac{\partial}{\partial x}\right)^s H_{k+\ell}(x) = \theta(k+\ell-s) \frac{2^s (k+\ell)!}{(k+\ell-s)!} H_{k+\ell-s}(x) \quad \text{(D.9)}$$

for the sth derivative of the Hermite polynomial $H_{k+\ell}(x)$. Here, $\theta(y)$ is the unit step function: it is equal to 0 for $y < 0$ and 1 for $y \geq 0$.

To summarize, the result of the calculation can be cast into the form

$$\begin{aligned}\mathcal{F}(\mu, M, \eta, N|\mathcal{X}, a, b, c, \Lambda, \lambda) &= \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\chi^2}{4a}\right) \\ &\times \sum_{k=0}^{M} \sum_{\ell=0}^{N} \binom{M}{M-k} \binom{N}{N-\ell} \left(-\frac{i}{\sqrt{a}}\right)^{k+\ell} \\ &\times \left[\sum_{r=0}^{\mu+\eta} A_r(\mu, \eta; \Lambda, \lambda) \mathbb{G}_{k+\ell,r}(\chi, a)\right] \\ &\times H_{M-k}(b) H_{N-\ell}(c).\end{aligned} \quad \text{(D.10)}$$

From (C.3) and (D.10) it is immediately verificable that $\mathcal{F}(0, M, 0, N|\mathcal{X}, a, b, c, \Lambda, \lambda) = F_{MN}(\mathcal{X}, a, b, c)$, because (D.8) implies that $\mathbb{G}_{k+\ell,0}(\chi, a) = H_{k+\ell} \times (\mathcal{X}/(2\sqrt{a}))$.

Appendix E. A system with Hamiltonian $\hat{H}(t) = \hat{K} + V(t)$

Instead of the driven harmonic oscillator, consider now a system with a Hamiltonian $\hat{H}$ that splits into a time-independent unperturbed part $\hat{K}$ and a perturbation $\hat{V}(t)$, that is, $\hat{H} = \hat{K} + \hat{V}(t)$. Instead of eq. (9), one write the Schrödinger evolution operator in the form, with $\tau = t - t_0$,

$$\hat{U}(t, t_0) = \exp\left(-\frac{i}{\hbar}\tau\hat{K}\right) \hat{U}_{\mathrm{I}}(t, t_0), \quad \mathcal{U}_{\mathrm{I}}(\hat{q}, \hat{p}; t_0, t_0) = \hat{1}, \quad \text{(E.1)}$$

where $\hat{U}_{\mathrm{I}}(t, t_0)$ is the evolution operator in the interaction picture and $\hat{U}_0(t, t_0) := \exp(-i\tau\hat{K}/\hbar)$ is the time-evolution operator associated with the Hamiltonian $\hat{K}$. By inserting the identity $\hat{1} = \exp(+\frac{i}{\hbar}\tau\hat{K}) \times \exp\left(-\frac{i}{\hbar}\tau\hat{K}\right)$ to the right-hand side of $\hat{U}_{\mathrm{I}}(t, t_0)$, the generalization of (14) is given by

$$\hat{U}(t, t_0) = [\hat{U}_0(t, t_0)\hat{U}_{\mathrm{I}}(t, t_0)\hat{U}_0^{+}(t, t_0)] \times \exp\left(-\frac{i}{\hbar}\tau\hat{K}\right). \quad \text{(E.2)}$$

Formally, the solution of the equation of motion governing the time evolution of $\hat{U}_{\mathrm{I}}(t, t_0)$ can be expressed as $\hat{U}_{\mathrm{I}}(t, t_0) := \mathcal{U}(\hat{q}, \hat{p}; t, t_0)$, where $\mathcal{U}(\hat{q}, \hat{p}; t, t_0)$ is a function of the position and momentum operators $\hat{q}$

and $\hat{p}$, beside the time dependence due to $\hat{V}(t)$. Now, according to the Weyl prescription, one can write

$$\mathcal{U}(\hat{q}, \hat{p}; t, t_0) = \frac{1}{(2\pi\hbar)^f} \int \mathcal{U}_\mathrm{W}(q', p'; t, t_0) \times \hat{D}(q', p') \mathrm{d}q' \mathrm{d}p', \tag{E.3}$$

where the scalar function $\mathcal{U}_\mathrm{W}(q', p'; t, t_0) = \mathrm{Tr}(\mathcal{U}(\hat{q}, \hat{p}; t, t_0)\hat{D}(-q', -p'))$ is the Weyl symbol associated with $\mathcal{U}(\hat{q}, \hat{p}; t, t_0)$ and $\hat{D}(q, p)$ is the Weyl operator defined in (1).

Consequently, the evolution operator in the Schrödinger picture can be written as

$$\hat{U}(t, t_0) = \left[\frac{1}{(2\pi\hbar)^f} \int \mathcal{U}_\mathrm{W}(q', p'; t, t_0) \times \hat{D}_\bullet(q', p', \tau) \mathrm{d}q' \mathrm{d}p'\right] \exp\left(-\frac{i}{\hbar}\tau \hat{K}\right), \tag{E.4}$$

where $\hat{D}_\bullet(q', p', \tau)$ is the time-dependent Weyl operator defined by the relation

$$\hat{D}_\bullet(q', p', \tau) := \exp\left(\frac{i}{\hbar}[p'\hat{Q}(\tau) - q'\hat{P}(\tau)]\right), \tag{E.5}$$

and

$$\hat{Q}(\tau) := \hat{U}_0(t, t_0)\, \hat{q}\, \hat{U}_0^+(t, t_0) = Q(\hat{q}, \hat{p}; \tau),$$
$$\hat{P}(\tau) := \hat{U}_0(t, t_0)\, \hat{p}\, \hat{U}_0^+(t, t_0) = P(\hat{q}, \hat{p}; \tau) \tag{E.6}$$

are time-dependent operators, which satisfy the commutation relation $[\hat{Q}(\tau), \hat{P}(\tau)] = i\hbar$. Thus, for a system with Hamiltonian $\hat{H} = \hat{K} + \hat{V}(t)$, the state of the system at time t takes the form

$$|\Psi(t)\rangle = \left[\frac{1}{(2\pi\hbar)^f} \int \mathcal{U}_\mathrm{W}(q', p'; t, t_0)\hat{D}_\bullet(q', p', \tau) \times \mathrm{d}q' \mathrm{d}p'\right] \exp\left(-\frac{i}{\hbar}\tau \hat{K}\right)|\Psi(t_0)\rangle. \tag{E.7}$$

This generalization of eq. (16) is of course formal as long as explicit expressions of the operators $\hat{Q}(\tau)$ and $\hat{P}(\tau)$, the function $\mathcal{U}(\hat{q}, \hat{p}; t, t_0)$ and the corresponding Weyl symbol $\mathcal{U}_\mathrm{W}(q', p'; t, t_0)$ are in general not known, and they can be difficult to obtain for a given specific system beyond the system discussed in this paper.

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