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The effect of nonlinearity on unstable zones of Mathieu equation

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Abstract. Mathieu equation is a well-known ordinary differential equation in which the excitation term appears as the non-constant coefficient. The mathematical modelling of many dynamic systems leads to Mathieu equation. The determination of the locus of unstable zone is important for the control of dynamic systems. In this paper, the stable and unstable regions of Mathieu equation are determined for three cases of linear and nonlinear equations using the homotopy perturbation method. The effect of nonlinearity is examined in the unstable zone. The results show that the transition curves of linear Mathieu equation depend on the frequency of the excitation term. However, for nonlinear equations, the curves depend also on initial conditions. In addition, increasing the amplitude of response leads to an increase in the unstable zone.

Keywords. Mathieu equation; unstable zone; parametric excitation; nonlinear.

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1. Introduction

The parametric excitation arises in many phenomena in physics and engineering, such as dynamic stability of elastic columns, dynamics of meshing gears, stability in ion trap spectrometer and so on. Mathematically, theses problems led to the differential equation with the time-varying coefficients [1]. In simple cases, the governing equation (called Hill's equation) of the system is a homogeneous, linear, second-order differential equation with the periodic coefficient, called [2]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \bar{f}(t)y = 0; \qquad \bar{f}(t+T) = \bar{f}(t), \tag{1}$$

where T is the periodic time of $\bar{f}(t)$. If $\bar{f}(t) = \delta + 2\epsilon\cos(2t)$, eq. (1) reduces to the classical Mathieu equation [1]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (\delta + 2\epsilon\cos(2t))y = 0. \tag{2}$$

For example, in vibration systems, δ is the frequency of oscillation and $2\epsilon\cos(2t)$ represents the parametric excitation of strength ϵ. The stability of the solution depends on the amount of δ and ϵ. The curves, called the transition curves, plotted on ϵ–δ plane, separate the stable and unstable regions. Note that along the transition curves, the solutions are periodic.

Nayfeh [1] determines the transition curves of eq. (2) using the method of strained parameters and the method of multiple scales. Broer and Simo [3] studied the transition curves, named resonance tongue, using the geometric approach for the linear Mathieu and Hill's equations. Zhou *et al* [4] derived the equation of motion for the ions within the practical quadrupole ion trap and characterized it by the nonlinear Mathieu equation. They used the Poincare–Lighthill–Kuo (PLK) method to determine the stability regions of the Mathieu equation. Jazar *et al* [5] derived the governing equations of the lateral vibration of a microcantilever resonator in a linear domain as the forced Mathieu equation and used the energy-rate method to determine the stability chart. In this method, the transition curves are determined numerically based on a computer program.

In this article, the effect of nonlinearity on transition curves of Mathieu equation is examined. The transition curves are determined by the perturbation homotopy method which is an approximate analytical method. The method proposes a general formula to construct an infinite number of linear ordinary differential equations from the original equation. Therefore, for any form of the excitation term, the transition curves can be determined easier and faster than the perturbation methods.

First, the homotopy perturbation method is described briefly and the transition curves are then determined for three cases of linear and nonlinear Mathieu equation. The area of unstable zones are determined to examine the effect of nonlinear term.

2. Homotopy perturbation method

In 1992, Liao [6] proposed an analytical method, namely the homotopy method, with which the series solutions of nonlinear ordinary differential equation can be obtained. Six years later, Jihuan He [7] proposed the homotopy perturbation method (HPM) by which a large class of nonlinear differential problems could be solved easily. The method has been worked out and extended over a number of years by numerous researchers.

In this paper, to determine the transition curves of the following equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \delta y + f(t, y, \dot{y}, \ddot{y}, ...) = 0 \tag{3}$$

the homotopy equation is constructed as follows:

$$\begin{aligned} H(Y, t; p) &= \ddot{Y}(t; p) + \Delta(p)Y \\ &\quad + pf(t, Y, \dot{Y}, \ddot{Y}, ...) = 0, \end{aligned} \tag{4}$$

where $f(t, Y, \dot{Y}, \ddot{Y}, ...)$ is a periodic function and $Y(t; p)$ and $\Delta(p)$ are defined as

$$Y(t; p) = y_0 + \sum_{j=1}^{\infty} \left(y_j \frac{p^j}{j!} \right), \tag{5}$$

$$\Delta(p) = \delta_0 + \sum_{j=1}^{\infty} \left(\delta_j \frac{p^j}{j!} \right), \tag{6}$$

where $p \in [0 \quad 1]$ is the embedding parameter which monotonically increases from zero to unity. As $p = 0$, the homotopy equation (4) reduces to

$$H(Y, t; 0) = \ddot{Y}(t; 0) + \omega_0^2 Y = 0, \tag{7}$$

where $\omega_0^2 = \delta_0 = \Delta(0)$ and $Y(t; 0) = y_0$ represents the initial approximation of the solution. When $p = 1$, eq. (4) deforms to the original differential equation:

$$H(Y, t; 1) = \ddot{y}(t) + \delta y + f(t, y, ...) = 0, \tag{8}$$

where

$$y(t) = Y(t; 1) = y_0 + \sum_{j=1}^{\infty} (y_j / j!), \tag{9}$$

$$\delta = \delta_0 + \sum_{j=1}^{\infty} (\delta_j / j!). \tag{10}$$

In order to find y_j and δ_j, eqs (5) and (6) are substituted into (4) and the coefficients of each power of p are then set to zero. Equation (7) will be obtained if the coefficients of p^0 is set to zero. The coefficients of p^n should be determined from the following equation:

$$\left.\frac{\partial^n H}{\partial p^n}\right|_{p=0} = \frac{\partial^n Y}{\partial p^n} + \sum_{i=0}^{n} \binom{n}{i} \frac{\mathrm{d}^{n-i} \Delta}{\mathrm{d}p^{n-i}} \frac{\partial^i Y}{\partial p^i} + n \frac{\partial^{n-1} f}{\partial p^{n-1}} = 0, \tag{11}$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \tag{12}$$

and

$$\left.\frac{\partial^n Y}{\partial p^n}\right|_{p=0} = y_n, \qquad \left.\frac{\mathrm{d}^n \Delta}{\mathrm{d}p^n}\right|_{p=0} = \delta_n. \tag{13}$$

In this way, the original differential equation is transformed to an infinite number of sub-problems. Eliminating the secular terms, the relations between δ_j and other parameters can be obtained.

3. Transition curves

The transition curves separate the stable and unstable solutions on δ–ϵ plane and along the curves, the solutions are periodic. The periodic solutions of Mathieu equation can be obtained for certain values of δ and ϵ [1]. If the frequency of $f(t, y, ...)$ is equal to $2m$ ($m = 1, 2, ...$) the transition curves for linear Mathieu equation start at $(\delta, \epsilon) = ((m)^2 k^2, 0)$ for all $k \in N$ [3,8]. In the next sections, for three cases of function $f(t, y, ...)$ the transition curves will be determined.

3.1 *Case* 1: $f(t, y) = 2\epsilon \cos(mt) y$

Consider eq. (3) as follows:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \delta y + 2\epsilon \cos(mt) y = 0. \tag{14}$$

Using eq. (11), eq. (14) is transformed to a set of ordinary differential equations with constant coefficients as follows:

$$\ddot{y}_0 + \omega_0^2 y_0 = 0, \tag{15}$$

$$\ddot{y}_1 + \omega_0^2 y_1 = -2\epsilon \cos(mt) y_0 - \delta_1 y_0 \tag{16}$$

and

$$\ddot{y}_n + \omega_0^2 y_n = -n\epsilon \cos(mt) y_{n-1} - \sum_{i=0}^{n-1} \tbinom{n}{i} \delta_{n-i} y_i \tag{17}$$

etc.

From eq. (15), y_0 is calculated as

$$y_0 = a\cos(\omega_0 t) + b\sin(\omega_0 t). \tag{18}$$

Substituting eq. (18) into (16), we obtain

$$\begin{aligned}\ddot{y}_1 + \omega_0^2 y_1 = &-\epsilon a\cos(m+\omega_0)t - \epsilon a\cos(m-\omega_0)t\\ &-\epsilon b\sin(m+\omega_0)t + \epsilon b\sin(m-\omega_0)t\\ &-\delta_1 a\cos(\omega_0 t) - \delta_1 a\sin(\omega_0 t).\end{aligned} \tag{19}$$

Neglecting the secular terms,

$$\text{if}\quad \omega_0 \neq m/2 \Rightarrow \delta_1 = 0 \tag{20}$$

$$\text{if}\quad \omega_0 = m/2 \Rightarrow \begin{cases} a = 0 \text{ and } \delta_1 = \epsilon & \text{(a)}\\ b = 0 \text{ and } \delta_1 = -\epsilon & \text{(b)}\end{cases}, \tag{21}$$

from eqs (20) or (21) and eq. (19), y_1 can be determined, respectively as follows:

$$\begin{aligned}\omega_0 \neq m/2 \Rightarrow y_1 = &\frac{\epsilon}{m(m+2\omega_0)}[a\cos(m+\omega_0)\\ &+ b\sin(m+\omega_0)]\\ &+\frac{\epsilon}{m(m-2\omega_0)}[a\cos(m-\omega_0)\\ &+ b\sin(m-\omega_0)],\end{aligned} \tag{22}$$

$$\omega_0 = m/2 \Rightarrow \begin{cases} y_1 = \dfrac{\epsilon b}{2m^2}\sin(3m/2t)\\ y_1 = \dfrac{\epsilon a}{2m^2}\cos(3m/2t)\end{cases}. \tag{23}$$

Consider $n = 2$ in eq. (17), using eqs (18) and (22) or (23) and avoiding the secular term, the relations of δ_2 can be determined. The results are summarized as follows:

$$\begin{aligned}\delta &= \left(\frac{m}{2}\right)^2 + \epsilon - \frac{\epsilon^2}{2m^2} + O(\epsilon^3),\\ \delta &= \left(\frac{m}{2}\right)^2 - \epsilon - \frac{\epsilon^2}{2m^2} + O(\epsilon^3),\\ \delta &= m^2 - \frac{\epsilon^2}{3m^2} + O(\epsilon^3),\\ \delta &= m^2 + \frac{5\epsilon^2}{3m^2} + O(\epsilon^3).\end{aligned} \tag{24}$$

Equations (24), for $m = 2$, agree with ref. [1] in which the multiple scale method is used.

3.2 *Case* 2: $f(t, y) = 2\epsilon(\cos(mt) + \cos(rmt))y$

In this section, eq. (3) is considered as follows:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \delta y + 2\epsilon(\cos(mt) + \cos(rmt))y = 0,$$
$$r = 2, 3, 4, \ldots. \tag{25}$$

Same as in the previous section, using eq. (11) the following equations are obtained:

$$\begin{aligned}\ddot{y}_0 + \omega_0^2 y_0 &= 0,\\ \ddot{y}_1 + \omega_0^2 y_1 &= -\delta_1 y_0 - 2\epsilon(\cos(mt) + \cos(rmt))y_0,\\ \ddot{y}_2 + \omega_0^2 y_2 &= -\delta_2 y_0 - 2\delta_1 y_1 - 4\epsilon(\cos(mt)\\ &\quad + \cos(rmt))y_1.\end{aligned} \tag{26}$$

The solution of the first equation is represented by eq. (18). Substituting y_0 into the second equation and avoiding the secular terms, δ_1 is calculated. y_1 will be calculated as follows:

For $\omega_0 = m/2$:

$$\begin{aligned}\delta_1 = \epsilon \Rightarrow y_1 = &\left(\frac{\epsilon b}{2m^2}\right)\sin\left(\frac{3mt}{2}\right)\\ &+\left(\frac{\epsilon b}{r(r+1)m^2}\right)\sin\left(\frac{(2r+1)mt}{2}\right)\\ &-\left(\frac{\epsilon b}{r(r-1)m^2}\right)\sin\left(\frac{(2r-1)mt}{2}\right),\\ \delta_1 = -\epsilon \Rightarrow y_1 = &\left(\frac{\epsilon a}{2m^2}\right)\cos\left(\frac{3mt}{2}\right)\\ &+\left(\frac{\epsilon a}{r(r+1)m^2}\right)\cos\left(\frac{(2r+1)mt}{2}\right)\\ &+\left(\frac{\epsilon a}{r(r-1)m^2}\right)\cos\left(\frac{(2r-1)mt}{2}\right).\end{aligned} \tag{27}$$

For $\omega_0 = rm/2$:

$$\begin{aligned}\delta_1 = \epsilon \Rightarrow y_1 = &\left(\frac{\epsilon b}{(r+1)m^2}\right)\sin\left(\frac{(r+2)mt}{2}\right)\\ &-\left(\frac{\epsilon b}{(1-r)m^2}\right)\sin\left(\frac{(r-2)mt}{2}\right)\\ &+\left(\frac{\epsilon b}{2r^2m^2}\right)\sin\left(\frac{3rmt}{2}\right),\\ \delta_1 = -\epsilon \Rightarrow y_1 = &\left(\frac{\epsilon a}{(r+1)m^2}\right)\cos\left(\frac{(r+2)mt}{2}\right)\\ &+\left(\frac{\epsilon a}{(1-r)m^2}\right)\cos\left(\frac{(r-2)mt}{2}\right)\\ &+\left(\frac{\epsilon a}{2r^2m^2}\right)\cos\left(\frac{3rmt}{2}\right).\end{aligned} \tag{28}$$

Substituting y_0 and y_1 into the third equation of (26) and eliminating the secular terms, δ_2 is then calculated. So the transition curves are obtained as follows:

$$\delta = \left(\frac{m}{2}\right)^2 + \epsilon - \frac{\epsilon^2}{2m^2}\frac{r^2+3}{r^2-1} + O(\epsilon^3), \quad r \neq 2$$

$$\delta = \left(\frac{m}{2}\right)^2 - \epsilon - \frac{\epsilon^2}{2m^2}\frac{r^2+3}{r^2-1} + O(\epsilon^3), \quad r \neq 2$$

$$\delta=\left(\frac{m}{2}\right)^2+\epsilon-\frac{\epsilon^2}{6m^2}+O(\epsilon^3),\quad r=2$$

$$\delta=\left(\frac{m}{2}\right)^2-\epsilon-\frac{13\epsilon^2}{6m^2}+O(\epsilon^3),\quad r=2$$

$$\delta=\left(\frac{rm}{2}\right)^2+\epsilon-\frac{\epsilon^2}{2m^2}\frac{1-5r^2}{r^2(r^2-1)}+O(\epsilon^3),\quad r\neq 2$$

$$\delta=\left(\frac{rm}{2}\right)^2-\epsilon+\frac{\epsilon^2}{2m^2}\frac{3r^2+1}{r^2(r^2-1)}+O(\epsilon^3),\quad r\neq 2$$

$$\delta=m^2+\epsilon-\frac{5\epsilon^2}{24m^2}+O(\epsilon^3),\quad r=2$$

$$\delta=m^2-\epsilon+\frac{37\epsilon^2}{24m^2}+O(\epsilon^3),\quad r=2. \tag{29}$$

3.3 *Case* 3: $f(t,y)=2\epsilon\cos(mt)(y+y^3)$

In this section, the nonlinear Mathieu equation is considered as

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\delta y+2\epsilon\cos(mt)(y+y^3)=0. \tag{30}$$

Using eq. (11), the following ordinary differential equations are obtained:

$$\ddot{y}_0+\omega_0^2y_0=0,$$

$$\ddot{y}_1+\omega_0^2y_1=-\delta_1y_0-2\epsilon\cos(mt)(y_0+y_0^3),$$

$$\ddot{y}_2+\omega_0^2y_2=-\delta_2y_0-2\delta_1y_1 - 4\epsilon\cos(mt)(y_1+3y_0^2y_1). \tag{31}$$

The solution of the first equation is represented by eq. (18). Substituting y_0 into the second equation and avoiding the secular terms, δ_1 is calculated. y_1 will be then calculated as follows:

For $\omega_0=m/2$:

$$\delta_1=\epsilon(1+b^2)\Rightarrow y_1=\left(\frac{\epsilon b}{2m^2}\right)\left(1+\frac{3b^2}{4}\right)\sin\left(\frac{3mt}{2}\right) - \left(\frac{\epsilon b^3}{24m^2}\right)\sin\left(\frac{5mt}{2}\right),$$

$$\delta_1=-\epsilon(1+a^2)\Rightarrow y_1=\left(\frac{\epsilon a}{2m^2}\right)\left(1+\frac{3a^2}{4}\right)\cos\left(\frac{3mt}{2}\right) + \left(\frac{\epsilon a^3}{24m^2}\right)\cos\left(\frac{5mt}{2}\right). \tag{32}$$

For $\omega_0=m/4$:

$$\delta_1=-\frac{\epsilon b^2}{4},$$

$$a=0\Rightarrow y_1=\left(\frac{-2\epsilon b}{m^2}\right)\left(1+\frac{3b^2}{4}\right)\sin\left(\frac{3mt}{4}\right) + \left(\frac{2\epsilon b}{3m^2}\right)\left(1+\frac{3b^2}{4}\right)\sin\left(\frac{5mt}{4}\right) - \frac{\epsilon b^3}{12m^2}\sin\left(\frac{7mt}{4}\right),$$

$$\delta_1=-\frac{\epsilon a^2}{4},$$

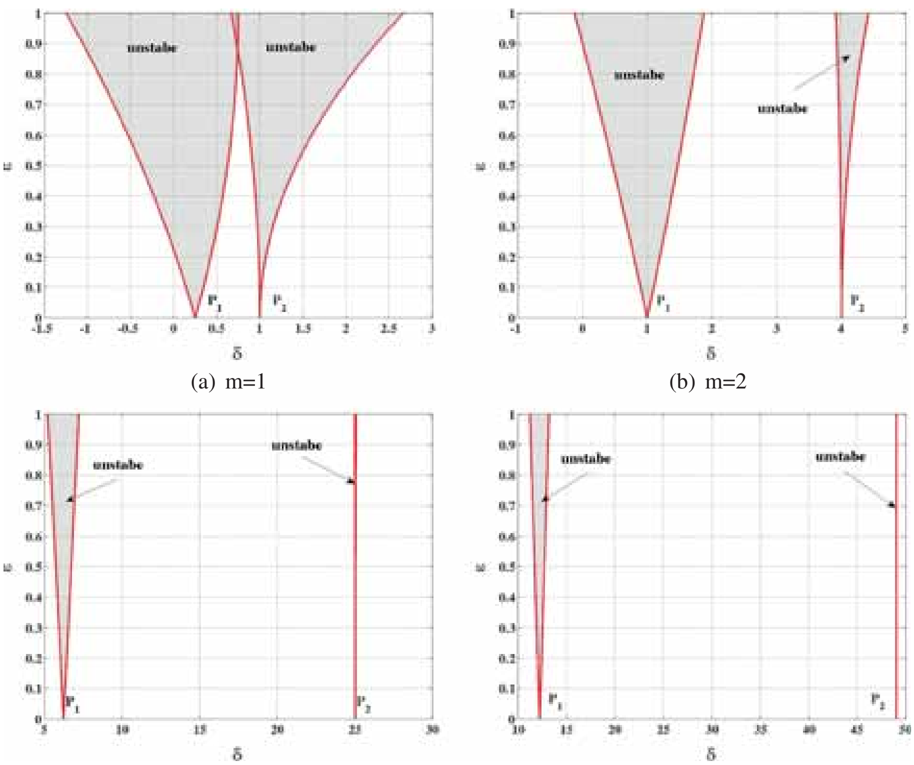
$$b=0\Rightarrow y_1=\left(\frac{2\epsilon a}{m^2}\right)\left(1+\frac{3a^2}{4}\right)\cos\left(\frac{3mt}{4}\right) + \left(\frac{2\epsilon a}{3m^2}\right)\left(1+\frac{3a^2}{4}\right)\cos\left(\frac{5mt}{4}\right) + \frac{\epsilon a^3}{12m^2}\cos\left(\frac{7mt}{4}\right),$$

$$\delta_1=\frac{\epsilon a^2}{4},$$

$$a=b\Rightarrow y_1=\left(\frac{2\epsilon a}{m^2}\right)\left(1+\frac{3a^2}{4}\right)\left(\cos\left(\frac{3mt}{4}\right) - \sin\left(\frac{3mt}{4}\right)\right)+\left(\frac{2\epsilon a}{3m^2}\right)\left(1+\frac{3a^2}{4}\right) \times\left(\cos\left(\frac{5mt}{4}\right)+\sin\left(\frac{5mt}{4}\right)\right) + \frac{\epsilon a^3}{6}\left(\sin\left(\frac{7mt}{4}\right)-\cos\left(\frac{7mt}{4}\right)\right),$$

$$\delta_1=\frac{\epsilon a^2}{4},$$

$$a=-b\Rightarrow y_1=\left(\frac{2\epsilon a}{m^2}\right)\left(1+\frac{3a^2}{4}\right) \times\left(\cos\left(\frac{3mt}{4}\right)+\sin\left(\frac{3mt}{4}\right)\right) + \left(\frac{2\epsilon a}{3m^2}\right)\left(1+\frac{3a^2}{4}\right) \times\left(\cos\left(\frac{5mt}{4}\right)-\sin\left(\frac{5mt}{4}\right)\right) + \frac{\epsilon a^3}{6}\left(\cos\left(\frac{7mt}{4}\right)-\sin\left(\frac{7mt}{4}\right)\right). \tag{33}$$



(a) m=1 (b) m=2

(c) m=5 (d) m=7

Figure 1. Transition curves of $(\mathrm{d}^2y/\mathrm{d}t^2) + \delta y + 2\epsilon\cos(mt)y = 0$.

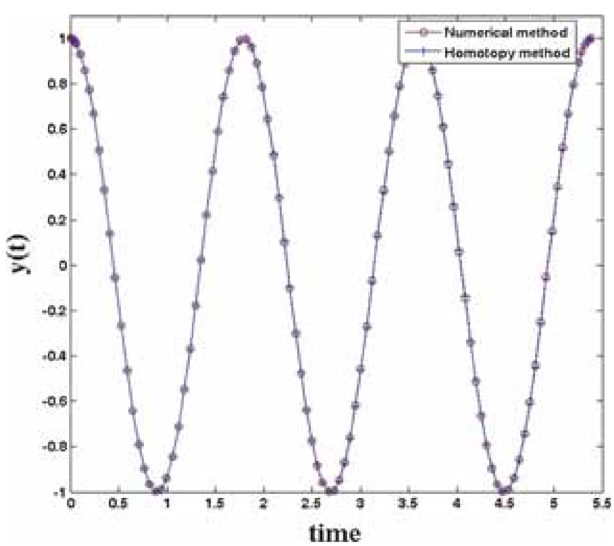


Figure 2. Periodic solution of eq. (14), $m = 7$, $\epsilon = 0.01$.

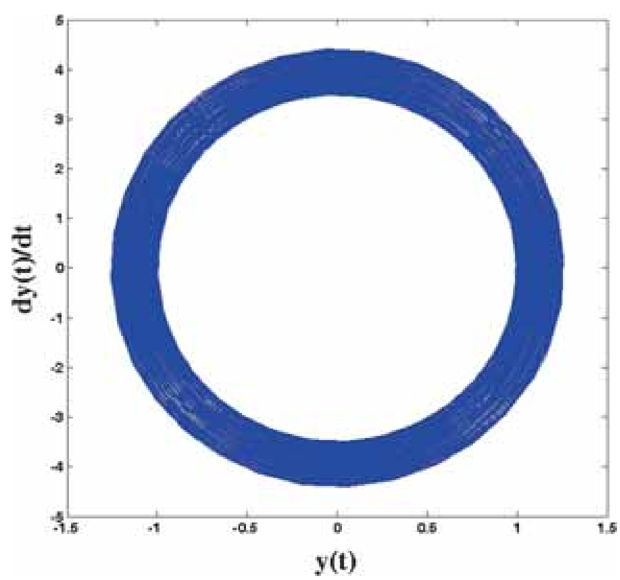


Figure 3. Unstable solution of eq. (14), $m = 7$, $\epsilon = 0.1$, $\delta = 12.25$.

Substituting y_0 and y_1 into the third equation of (31) and eliminating the secular terms, δ_2 is then calculated. The relations of δ are then calculated as

$$\delta = \left(\frac{m}{2}\right)^2 + \epsilon(1+a^2) - \frac{\epsilon^2}{2m^2}(1+3a^2+7a^4/4) + O(\epsilon^3),$$

$$\delta = \left(\frac{m}{2}\right)^2 - \epsilon(1+a^2) - \frac{\epsilon^2}{2m^2}(1+3a^2+7a^4/4) + O(\epsilon^3),$$

$$\delta = \left(\frac{m}{4}\right)^2 - \epsilon\frac{a^2}{4} - \frac{\epsilon^2}{2m^2}\left(\frac{16}{3} + 16a^2 + \frac{73a^4}{8}\right) + O(\epsilon^3),$$

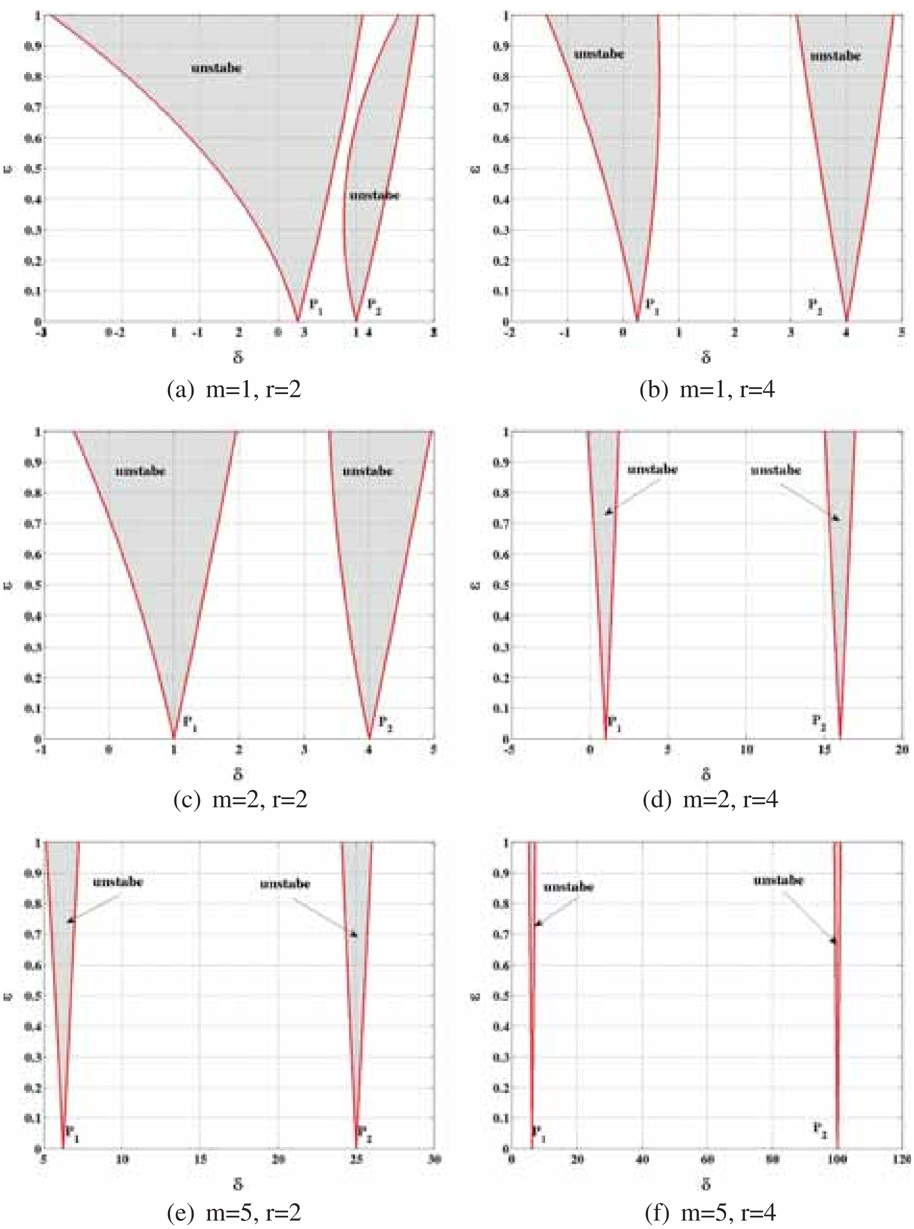
$$\delta = \left(\frac{m}{4}\right)^2 + \epsilon\frac{a^2}{2} - \frac{\epsilon^2}{2m^2}\left(\frac{16}{3} + 32a^2 + \frac{73a^4}{2}\right) + O(\epsilon^3). \quad (34)$$

For nonlinear Mathieu equation, the transition curves depend on the amplitude of response, a.

4. Results

The locus of transition curves for three cases of $f(t, y)$ are plotted in figures 1, 4 and 8. The curves separate the δ–ϵ plane into regions of stability and instability. In all the figures, unstable zones are marked by gray colour. In these zones, the solutions are unbounded and in the stable zones the solutions are bounded and aperiodic. However, along the transition curves the solutions are periodic. In this paper the area of unstable zones is calculated to examine the effect of nonlinear terms on the locus of transition curves.

The transition curves, for Case 1, are shown in figure 1 for $m = 1$, 2, 5 and 7. Along the curves, the periodicity of solutions is equal to $4\pi/m$ or $2\pi/m$ (see eqs (22)–(24)). In figures 1a–1d, the left curve passing through $P_1 : ((m/2)^2, 0)$ represents the cosine solutions and the right curve passing through $P_2 : (m^2, 0)$ represents the sine periodic solutions and so on.



(a) m=1, r=2
(b) m=1, r=4
(c) m=2, r=2
(d) m=2, r=4
(e) m=5, r=2
(f) m=5, r=4

Figure 4. Transition curves of $(\mathrm{d}^2y/\mathrm{d}t^2) + \delta y + 2\epsilon(\cos(mt) + \cos(rmt))y = 0$.

The area of unstable zone between the curves passing through P_1 is equal to unity as in the following equation:

$$A_1(p_1) = \int_0^1 |\delta_1(\epsilon) - \delta_2(\epsilon)|\, \mathrm{d}\epsilon = \int_0^1 2\epsilon\, \mathrm{d}\epsilon = 1, \tag{35}$$

where δ_1 and δ_2 represent the transition curves passing through point P_1 (the first two equations of (24)). The area of unstable zone between the curves passing through P_2 is equal to $2/3m^2$:

$$A_1(p_2) = \int_0^1 |\delta_1(\epsilon) - \delta_2(\epsilon)|\, \mathrm{d}\epsilon = \int_0^1 \frac{2\epsilon^2}{m^2}\, \mathrm{d}\epsilon = \frac{2}{3m^2}, \tag{36}$$

where δ_1 and δ_2 represent the transition curves passing through point P_2 (the third and fourth equations of (24)). So the curves for sine and cosine are substantially coincident, as m increases (compare figures 1a and 1d).

For a special case of $m = 7$, $\epsilon = 0.01$ and $\delta = 12.24$ (calculated by second equation of (24)), the periodic solution of eq. (14) is calculated using eqs (18) and (23) ($y = y_0 + y_1$). The results are compared with numerical method in figure 2, and a good agreement is seen between the two methods.

Figure 3 shows the trajectory in the phase plane for eq. (14) considering $m = 7$, $\epsilon = 0.1$ and $\delta = 12.25$. The selected parameters are located in the unstable zone in figure 1d and the solutions are calculated numerically.

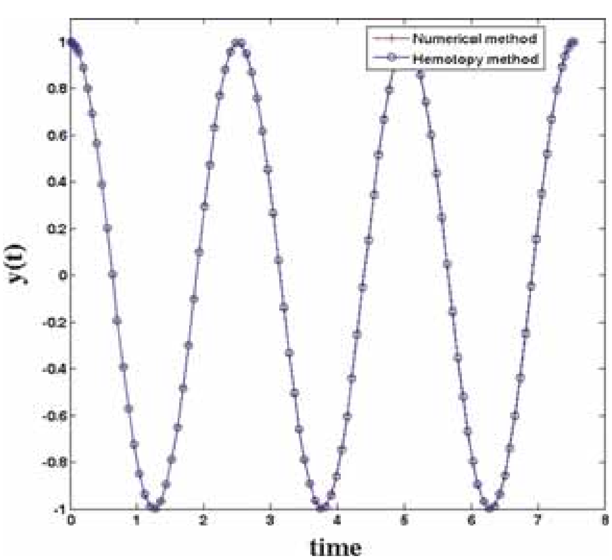


Figure 5. Periodic solution of eq. (25), $m = 5$, $r = 2$, $\epsilon = 0.01$.

Figure 4 depicts the transition curves of Case 2 (eqs (29)) for different values of m and r. With respect to the first four equations of (29) and eqs (18) and (27), along the curves passing through $P_1 : ((m/2)^2, 0)$ the periodicity of solution is $4\pi/m$ and along the curves passing through $P_2 : ((rm/2)^2, 0)$ the periodicity of solution is $4\pi/rm$ (see the second four equations of (29) and eqs (28) and (18)). The left curves passing through P_1 or P_2 in figures 4a–4f represent the cosine periodic solutions and the right curves represent the sine solution.

Figure 5 shows the periodic solution of eq. (25) for $m = 5$, $r = 2$, $\epsilon = 0.01$ and $\delta = 6.2400$ (calculated by the fourth equation of (29)). The solutions are determined by numerical and homotopy methods which have a good agreement with each other.

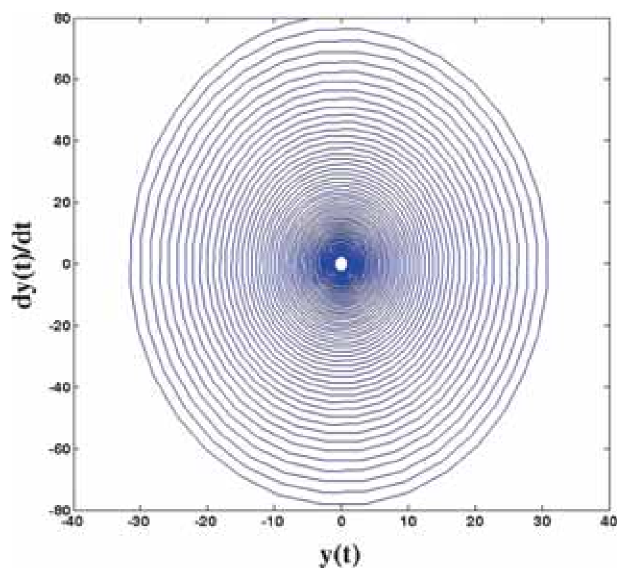


Figure 6. Unstable solution of eq. (25), $m = 5$, $r = 2$, $\epsilon = 0.1$, $\delta = 6.25$.

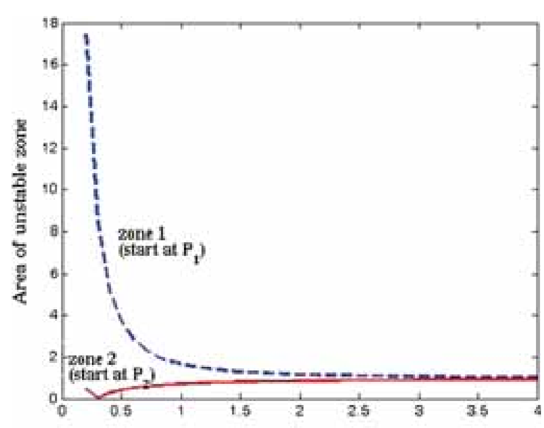
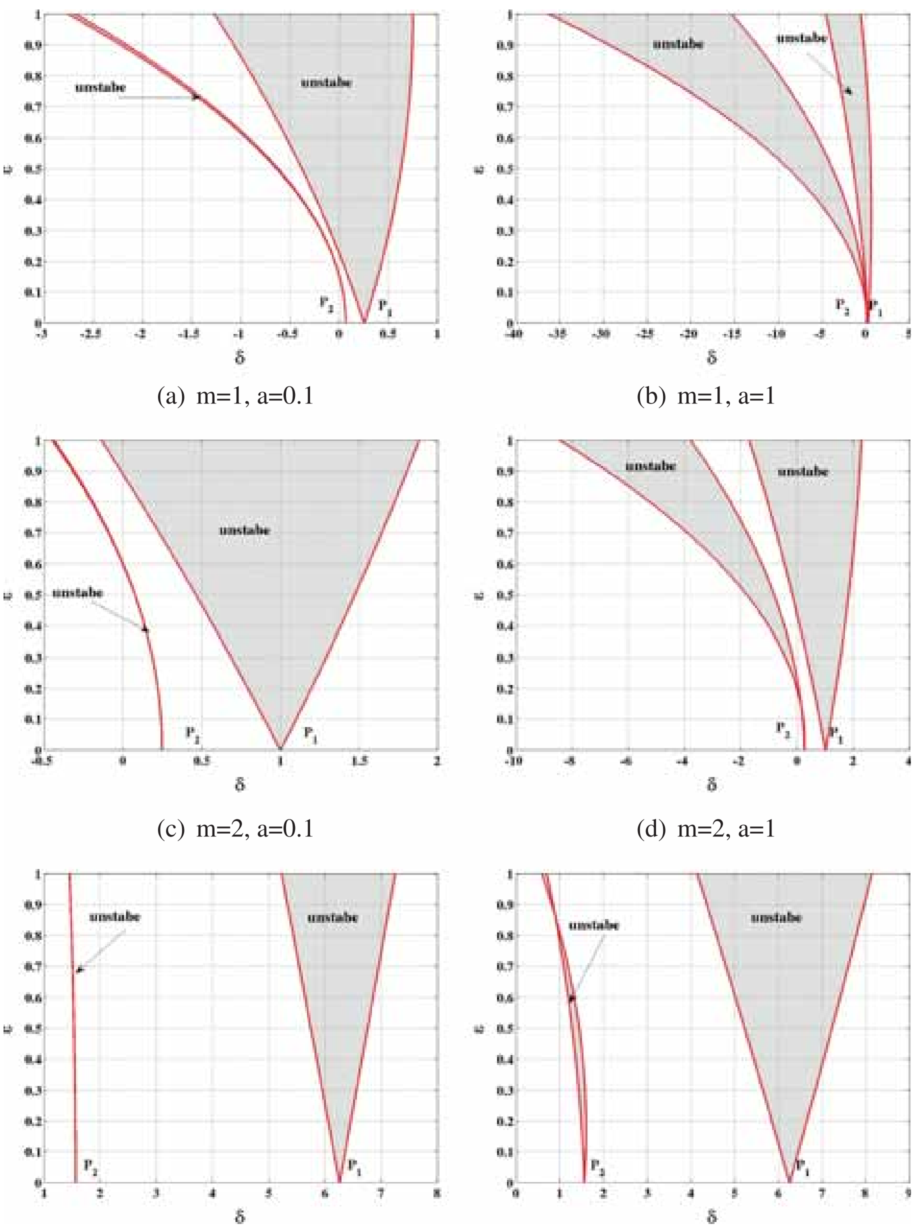


Figure 7. The area of unstable zone, $r = 2$.

The unstable solution of eq. (25) is calculated numerically for $m = 5$, $\epsilon = 0.1$ and $\delta = 6.25$. Figure 6 shows the trajectory in the phase plane.

Similar to eq. (35) and using eq. (29), for $r = 2$ (figures 4a, 4c and 4e) the area of unstable zone between the curves passing through P_1 (zone 1) is equal to $1+(2/3m^2)$ and for the one between the curves passing through P_2 (zone 2) is equal to $1 - (7/12m^2)$. Figure 7 shows the area of unstable zone for $r = 2$. The area of zone (1) decreases rapidly, for $m < 1$ and then approaches unity as m increases. The area of zone (2) will be equal to zero at $m = \sqrt{7/12}$. For $r \neq 2$ (figures 4b, 4d and 4f) the area of unstable zone between the curves passing through P_1 is unity and for the one between the curves passing through P_2 is equal to $1 + (1/3m^2r^2)$. So the area of unstable zone approaches unity as m and r increase.

The transition curves of Case 3 (eqs (34)), is shown in figure 8 for different values of m and a. For nonlinear Mathieu equation, the transition curves not only depend on m but also a, the amplitude of response which is determined by initial conditions. With respect to eqs (32)–(34), along the transition curves, the periodicity of solutions is $2\pi/m$ or $8\pi/m$. The left curves



(a) m=1, a=0.1

(b) m=1, a=1

(c) m=2, a=0.1

(d) m=2, a=1

(e) m=5, a=0.1

(f) m=5, a=1

Figure 8. Transition curves of $(\mathrm{d}^2y/\mathrm{d}t^2) + \delta y + 2\epsilon\cos(mt)(y + y^3) = 0$.

passing through P_1 in figures 8a–8d, represent the cosine periodic solutions and the right curves represent the sine solution. However, along the curves passing through P_2, the solutions are combinations of sine and cosine.

For special parameters located on transition curves, for example $m = 5$, $\epsilon = 0.01$ and $\delta = 6.2399$ (calculated by the second equation of (34)), the periodic solution is determined by numerical and homotopy methods in which $y = y_0 + y_1$. Figure 9 shows the periodic solutions.

Figure 10 shows the trajectory in phase plane for eq. (30) when $m = 5$, $\epsilon = 0.1$ and $\delta = 6.25$. The selected parameters are located on the unstable zone in figure 8e and the solutions are calculated numerically.

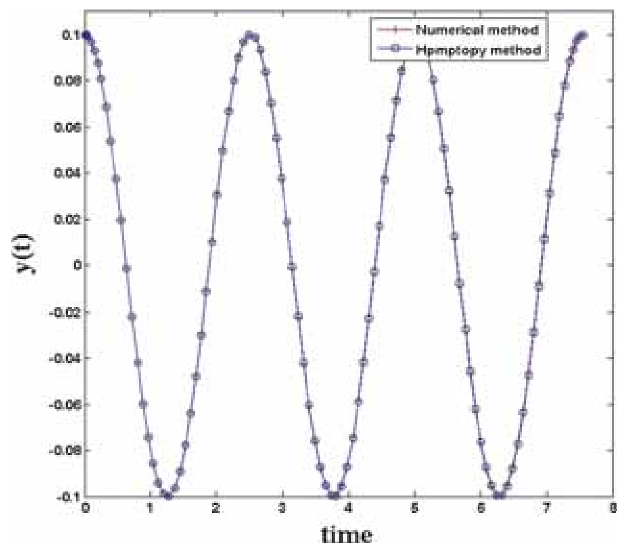


Figure 9. Periodic solution of eq. (30), $m = 5$, $a = 0.1$, $\epsilon = 0.01$.

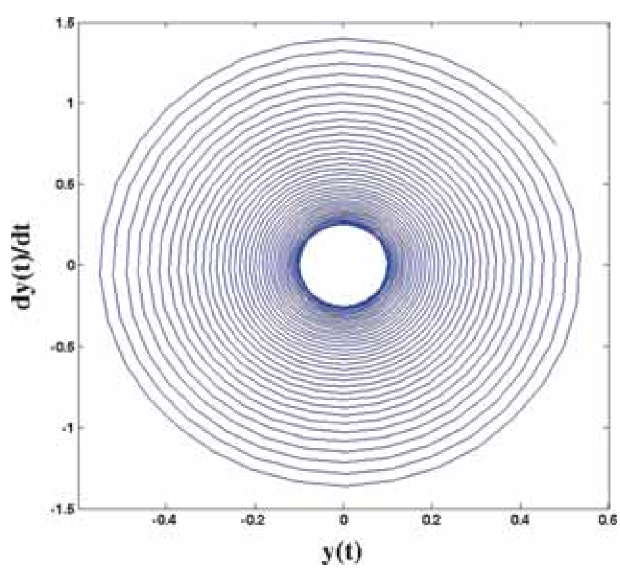


Figure 10. Unstable solution of eq. (30), $m = 5$, $a = 0.1$, $\epsilon = 0.1$, $\delta = 6.25$.

Using the first two equations of (34), the area of unstable zone between the curves passing through $P_1 : ((m/2)^2, 0)$ will be equal to

$$A_3(p_1) = \int_0^1 |\delta_1(\epsilon) - \delta_2(\epsilon)|\,\mathrm{d}\epsilon = \int_0^1 2\epsilon(1 + a^2)\,\mathrm{d}\epsilon = 1 + a^2. \tag{37}$$

So increasing a leads to increasing unstable zone area. Similarly, using the first two equations of (34), the area of unstable zone between the curves passing through $P_2 : ((m/4)^2, 0)$ can be calculated as

$$A_3(p_2) = \int_0^1 |\delta_1(\epsilon) - \delta_2(\epsilon)|\,\mathrm{d}\epsilon = \int_0^1 \frac{3a^2\epsilon}{4} - \left(16a^2 + \frac{219a^4}{8}\right)\frac{\epsilon^2}{2m^2}\,\mathrm{d}\epsilon = \frac{3a^2}{8} - \frac{1}{m^2}\left(\frac{8a^2}{3} + \frac{73a^4}{16}\right). \tag{38}$$

The area will be equal to zero at $m = \sqrt{((73a^2/6)+1)}$. Figure 11 shows the area vs. m for different values of a.

If the frequency of excitation term is equal to m, the transition curves of the linear equation (Case 1) start at $(\delta, \epsilon) = ((m/2)^2k^2, 0)$. For the linear equation with two excitation terms (Case 2), they start at $((m/2)^2k^2, 0)$ and $((rm/2)^2k^2, 0)$ and for nonlinear equation (Case 3), they start at $((m/4)^2k^2, 0)$. For linear equations, the area of unstable zones approaches unity as excitation term increases. However, for nonlinear equation it depends on a and increases with amplitude.

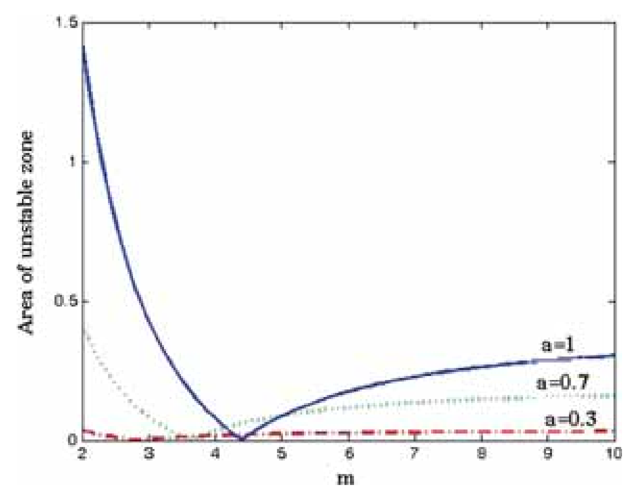


Figure 11. The area of unstable zone (start at P_2) for different values of a (nonlinear Mathieu equation).

5. Conclusion

In this paper, the stable and unstable zones of Mathieu equation are determined using the homotopy perturbation method. The transition curves are determined for three cases of linear and nonlinear equations. For linear equation, the locus of curves depends on the frequency of exciting term. However, for nonlinear equation, it depends also on the initial condition or amplitude of response. The area of unstable zone increases with increasing a. For linear equations, the transition curve starts at $m/2$ (half frequency of excitation term). However, for nonlinear equation, it starts at $m/4$.

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