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Bifurcations and new exact travelling wave solutions for the bidirectional wave equations

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Abstract. By using the method of dynamical system, the bidirectional wave equations are considered. Based on this method, all kinds of phase portraits of the reduced travelling wave system in the parametric space are given. All possible bounded travelling wave solutions such as dark soliton solutions, bright soliton solutions and periodic travelling wave solutions are obtained. With the aid of *Maple* software, numerical simulations are conducted for dark soliton solutions, bright soliton solutions and periodic travelling wave solutions to the bidirectional wave equations. The results presented in this paper improve the related previous studies.

Keywords. Bidirectional wave equations; dynamical system method; phase portrait; dark soliton solution; bright soliton solution; periodic travelling wave solution.

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1. Introduction

In this paper, we consider the travelling wave solutions of the bidirectional wave equations

$$\begin{cases} v_t + u_x + (uv)_x + a u_{xxx} - b v_{xxt} = 0, \\ u_t + v_x + u u_x + c v_{xxx} - d u_{xxt} = 0, \end{cases} \tag{1}$$

where a, b, c and d are real parameters. x represents the distance along the channel, t is the elapsed time, v is the dimensionless deviation of the water surface from its undisturbed position and u is the dimensionless horizontal velocity [1]. The bidirectional wave equations are a type of important mathematical physics equation which is used as a model equation for the propagation of long waves on the surface of water with a small amplitude and play a crucial role in nonlinear physics fields.

Recently, some important mathematical physics equations have been widely studied [2–5]. In particular, the bidirectional wave equations have been studied by some researchers. Some exact travelling wave solutions were obtained by Lee and Sakthivel [1] by using the modified tanh–coth function method. Chen [6] used the auxiliary ordinary equation method to obtain some exact solutions of eq. (1). However, we notice that the previous authors did not consider the dynamics of eq. (1) and did not find all possible travelling wave solutions. Therefore, it is essential to study the dynamics of eq. (1) and find some new travelling wave solutions of eq. (1). Here, we use the approach of dynamical system to solve eq. (1) and to give some new travelling wave solutions of eq. (1) [7–10]. The approach of dynamical system is concise, direct and effective which is based on the method of the bifurcation theory of planar dynamical system. Unlike other methods, the approach of dynamical system can not only obtain exact solutions but also study bifurcations of nonlinear travelling wave equations. Using this method, we can obtain some travelling wave solutions easily and enrich the diversity of solution structures of the bidirectional wave equations.

To find travelling wave solutions of (1), we assume that

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = kx - \omega t, \tag{2}$$

where k and ω are travelling wave parameters. Substituting (2) into (1), we have

$$\begin{cases} -\omega V' + kU' + kUV' + kU'V + ak^3U''' \\ \qquad + bk^2\omega V''' = 0, \\ -\omega U' + kV' + kUU' + ck^3V''' + dk^2\omega U''' = 0. \end{cases} \tag{3}$$

Then, we consider the following transformation:

$$U = mV, \tag{4}$$

where m is a constant to be determined later. Substituting (4) into (3), we have

$$\begin{cases} -\omega V' + mkV' + 2mkVV' + mak^3V''' \\ \qquad + bk^2\omega V''' = 0, \\ -m\omega V' + kV' + m^2kVV' + ck^3V''' \\ \qquad + mdk^2\omega V''' = 0. \end{cases} \tag{5}$$

Equating the two equations, we get the following conditions:

$$m = 2, \quad k = -\omega, \quad b - 2a = 2d - c. \tag{6}$$

In our work, we always assume that (1) satisfies (6). Under conditions (6), (5) is reduced to the following equation:

$$(2ak^2 - bk^2)V'' + 2V^2 + 3V + g = 0, \tag{7}$$

where g is an integration constant. Suppose $2ak^2 - bk^2 \neq 0$, then

$$\alpha = \frac{2}{2ak^2 - bk^2}, \quad \beta = \frac{3}{2ak^2 - bk^2}, \\ \gamma = \frac{g}{2ak^2 - bk^2}. \tag{8}$$

Finally, we have the following equation:

$$V'' + \alpha V^2 + \beta V + \gamma = 0 \tag{9}$$

which corresponds to the two-dimensional Hamiltonian system

$$\frac{dV}{d\xi} = y, \quad \frac{dy}{d\xi} = -\alpha V^2 - \beta V - \gamma \tag{10}$$

with the Hamiltonian

$$H(V,y) = \frac{1}{2}y^2 + \frac{1}{3}\alpha V^3 + \frac{1}{2}\beta V^2 + \gamma V. \tag{11}$$

In addition, when the integration constant g is 0 (that is $\gamma = g/(2ak^2 - bk^2) = 0$), (1) is reduced to the following equation:

$$V'' + \alpha V^2 + \beta V = 0, \tag{12}$$

which corresponds to the two-dimensional Hamiltonian system

$$\frac{dV}{d\xi} = y, \quad \frac{dy}{d\xi} = -\alpha V^2 - \beta V \tag{13}$$

with the Hamiltonian

$$H(V,y) = \frac{1}{2}y^2 + \frac{1}{3}\alpha V^3 + \frac{1}{2}\beta V^2. \tag{14}$$

According to the Hamiltonian, we can get all kinds of phase portraits in the parametric space. Because the phase orbits defined by the vector fields of system (10) determine all their travelling wave solutions of eq. (1), we can investigate the bifurcations of phase portraits of system (10) to seek the travelling wave solutions of eq. (1). The rest of the paper is organized as follows: In §2, we give all phase portraits of system (10) and discuss the bifurcations of phase portraits of system (10). In §3, according to the dynamics of the phase orbits of system (10) given by §2, we give all possible exact solutions of eq. (1) for $\gamma \neq 0$ and $\gamma = 0$. Finally, a conclusion is given in §4.

2. Bifurcations of phase portraits of system (10)

2.1 *The case of* $\gamma \neq 0$

We first consider the bifurcations of phase orbits of system (10) when $\gamma \neq 0$. Let the right-hand terms of system (10) be zeros, i.e. $y = 0$ and $-\alpha V^2 - \beta V - \gamma = 0$. Obviously, the abscissas of equilibrium points of system (10) are the real roots of $f(u) = \alpha v^2 + \beta v + \gamma$. Then, we find that the system (10) has two equilibrium points at $S_1((-\beta + \sqrt{\Delta})/2\alpha, 0)$ and $S_2((-\beta - \sqrt{\Delta})/2\alpha, 0)$ if $\Delta > 0$, where $\Delta = \beta^2 - 4\alpha\gamma$. If $\Delta = 0$, system (10) has a unique equilibrium at $O(-\beta/2\alpha, 0)$. If $\Delta < 0$, system (10) has no equilibrium. For the Hamiltonian $H(V,y) = \frac{1}{2}y^2 - \frac{1}{3}\alpha V^3 - \frac{1}{2}\beta V^2 - \gamma V = h$, we write $h_1 = H((-\beta + \sqrt{\Delta})/2\alpha, 0) = (\beta^3 - 6\alpha\beta\gamma - \sqrt{\Delta^3})/12\alpha^2$, $h_2 = H((-\beta - \sqrt{\Delta})/2\alpha, 0) = (\beta^3 - 6\alpha\beta\gamma + \sqrt{\Delta^3})/12\alpha^2$. With the change of the parameter group of α, β and γ, the phase portraits for (10) when $\gamma \neq 0$ are shown in figures 1 and 2.

From figures 1 and 2, we summarize crucial conclusions as follows:

(1) When $\Delta > 0$, system (10) has bounded orbits; when $\Delta \leq 0$, system (10) has no bounded orbits.
(2) When $\Delta > 0$, system (10) has a unique homoclinic orbit Γ which is asymptotic to the saddle and enclosing the centre.
(3) When $\Delta > 0$, there is a family of periodic orbits which are enclosing the centre and filling up the interior of the homoclinic orbit Γ.

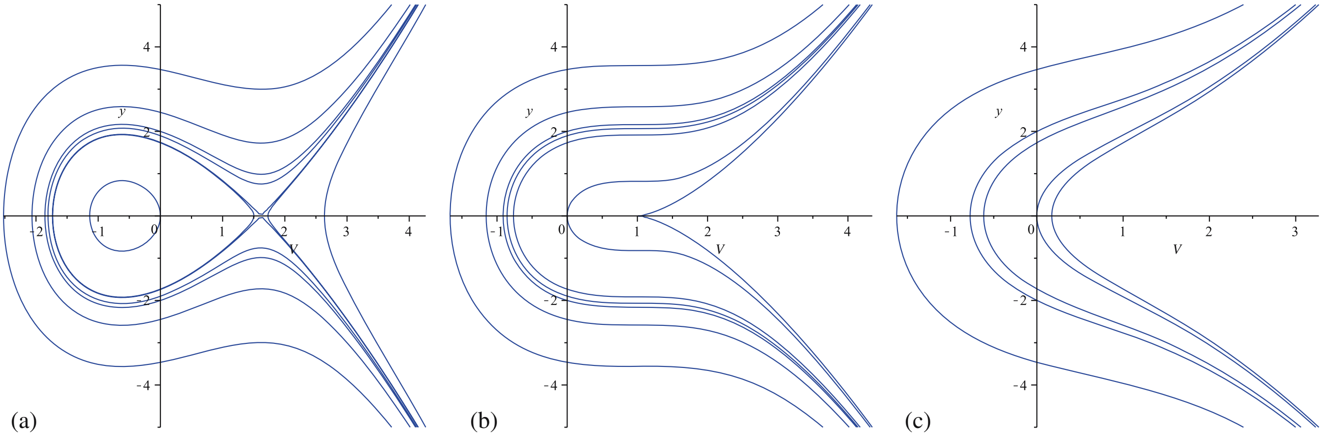

Figure 1. The bifurcations of phase portraits of (10) when $\alpha < 0$ and (**a**) $\triangle > 0$, (**b**) $\triangle = 0$ and (**c**) $\triangle < 0$.

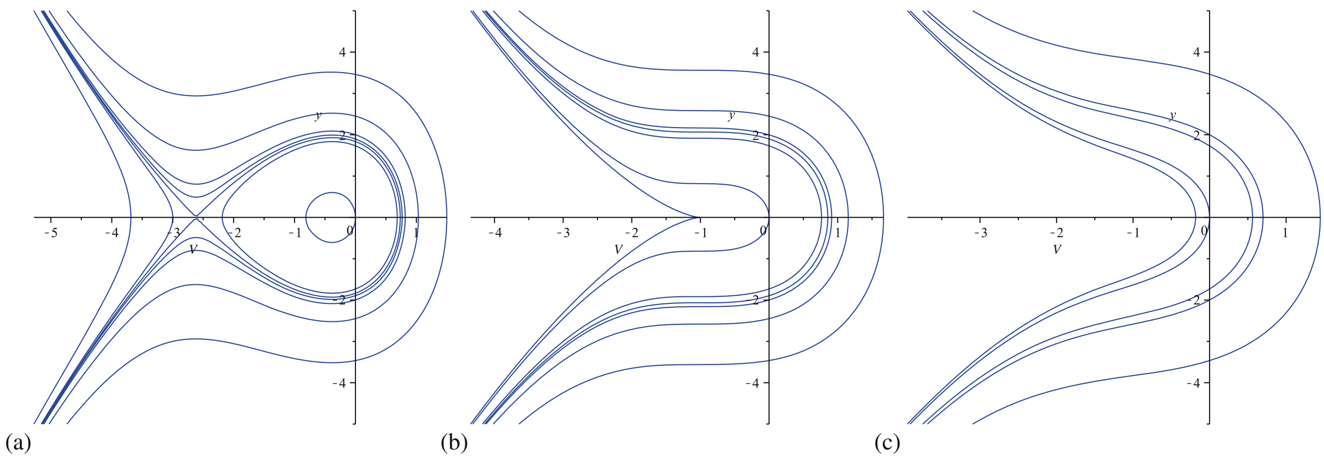

Figure 2. The bifurcations of phase portraits of (10) when $\alpha > 0$ and (**a**) $\triangle > 0$, (**b**) $\triangle = 0$ and (**c**) $\triangle < 0$.

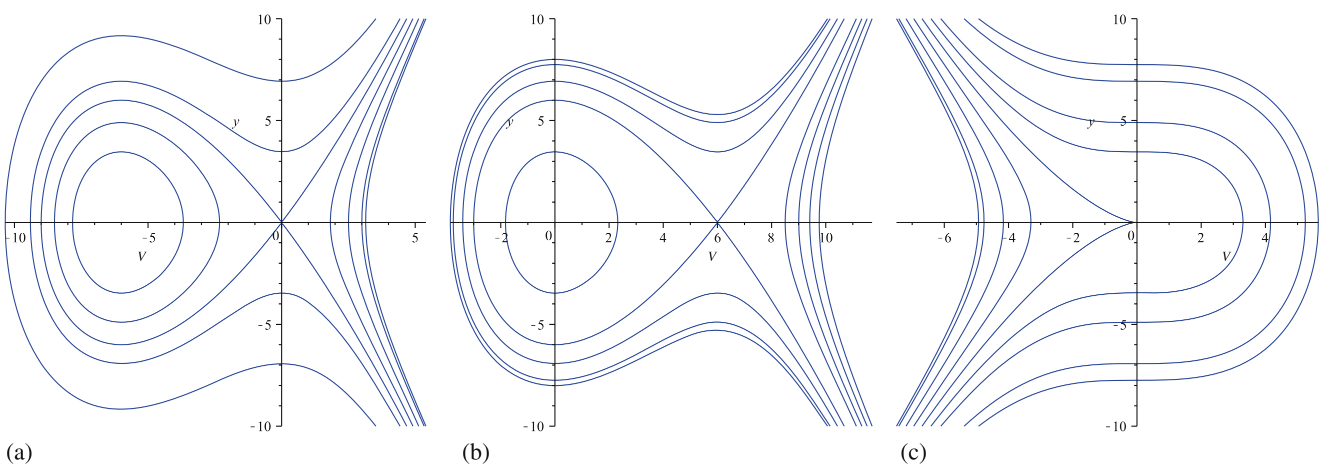

Figure 3. The bifurcations of phase portraits of (13) when $\alpha < 0$ and (**a**) $\beta < 0$, (**b**) $\beta > 0$ and (**c**) $\beta = 0$.

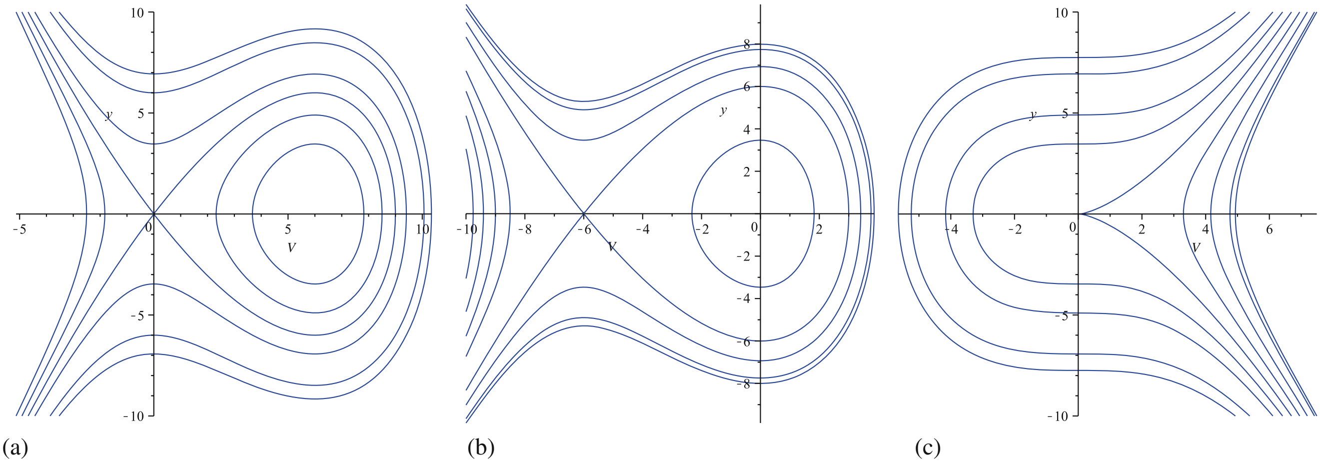

Figure 4. The bifurcations of phase portraits of (13) when $\alpha > 0$ and (**a**) $\beta < 0$, (**b**) $\beta > 0$ and (**c**) $\beta = 0$.

2.2 *The case of* $\gamma = 0$

We consider the bifurcations of phase portraits of (10) when $\gamma = 0$. We consider the phase portraits of (13). Let the right-hand terms of system (13) be zeros, i.e. $y = 0$ and $-\alpha V^2 - \beta V = 0$. We find that system (13) has two equilibrium points $S(-(\beta/\alpha), 0)$ and $O(0, 0)$. For the Hamiltonian $H(V, y) = \frac{1}{2}y^2 + \frac{1}{3}\alpha V^3 + \frac{1}{2}\beta V^2 = h$, we write $h_0 = H(0, 0) = 0$, $h_3 = H(-(\beta/\alpha), 0) = (\beta^3/6\alpha^2)$. With the change of the parameter group of α and β, the system has different phase portraits for (13) which are shown in figures 3 and 4.

For $\alpha = 2/(2ak^2 - bk^2)$ and $\beta = 3/(2ak^2 - bk^2)$, α and β are of the same sign. From the first image of figure 3 and the second image of figure 4, we summarize crucial conclusions as follows:

(1) System (13) has a unique homoclinic orbit Γ which is asymptotic to the saddle and enclosing the centre.

(2) There is a family of periodic orbits which are enclosing the centre and filling up the interior of the homoclinic orbit Γ.

3. Exact explicit travelling wave solutions of eq. (1)

In this section, we consider the exact solutions of eq. (1). Because only bounded travelling waves are meaningful to a physical model, we just pay attention to the bounded solutions of eq. (1). By using the first equation of (10) and the Jacobian elliptic functions [11], we have the following results:

3.1 *The case of* $\gamma \neq 0$

(1) When $\alpha < 0$ and $h = h_2$, there exists a dark soliton solution which corresponds to a smooth homoclinic orbit Γ of (10) defined by $H(\psi, y) = h_2$, and we have the parametric representation:

$$V(\xi) = \frac{\beta + \sqrt{\Delta} - 3\sqrt{\Delta}\,\mathrm{sech}^2((\sqrt{\Delta}^{4}/2)\xi)}{2|\alpha|}, \tag{15}$$

where $\Delta = \beta^2 - 4\alpha\gamma$.

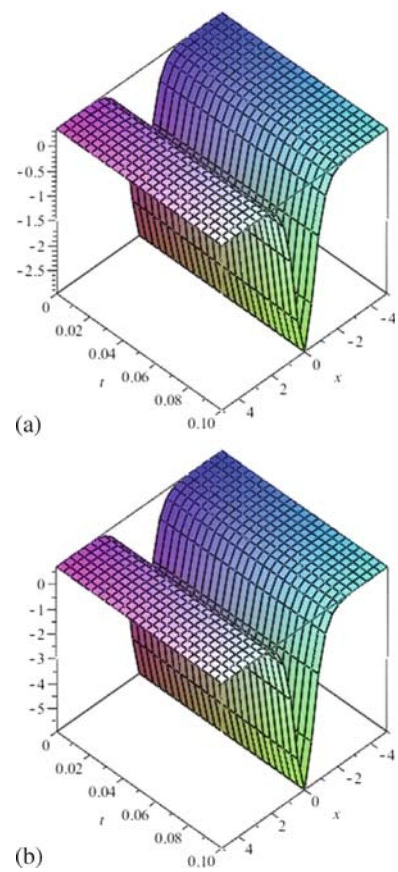

Figure 5. The 3D graphics of (19). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

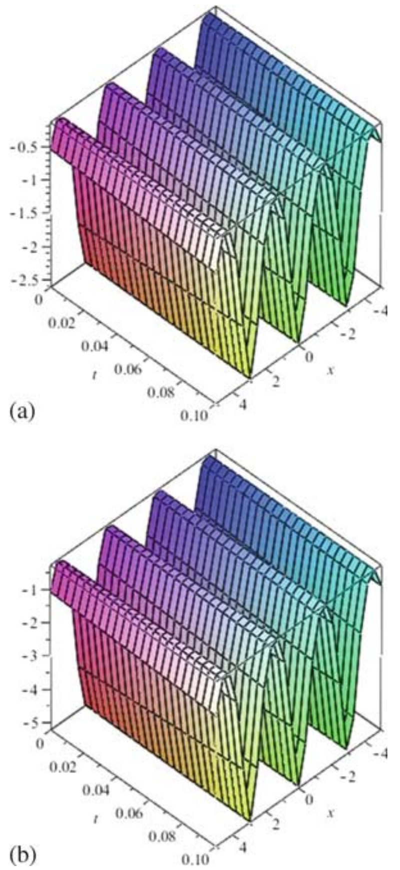

Figure 6. The 3D graphics of (20). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

(2) When $\alpha < 0$ and $h \in (h_1, h_2)$, there exists a family of periodic solutions which correspond to the family of periodic orbits Γ^h of (10) defined by $H(\phi, y) = h, h \in (h_1, h_2)$, and we have the parametric representation:

$$V(\xi) = z_3 + (z_2 - z_3)\mathrm{sn}^2 \times \left(\frac{\sqrt{6B(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right), \quad (16)$$

where $z_1 > z_2 > z_3$ and the parameters z_1, z_2, z_3 are defined by $y^2 = 2h - \gamma V - \beta V^2 - \frac{2}{3}\alpha V^3 = -\frac{2}{3}\alpha(z_1 - V)(z_2 - V)(V - z_3)$.

(3) When $\alpha > 0$ and $h = h_2$, there exists a bright soliton solution which corresponds to a smooth homoclinic orbit Γ of (10) defined by $H(\psi, y) = h_2$, and we have the parametric representation:

$$V(\xi) = \frac{-\beta - \sqrt{\Delta} + 3\sqrt{\Delta}\,\mathrm{sech}^2((\sqrt{\Delta}^{4}/2)\xi)}{2|\alpha|}, \quad (17)$$

where $\Delta = \beta^2 - 4\alpha\gamma$.

(4) When $\alpha > 0$ and $h \in (h_1, h_2)$, there exists a family of periodic solutions which correspond to the family of periodic orbits Γ^h of (10) defined by $H(\phi, y) = h, h \in (h_1, h_2)$, and we have the parametric representation:

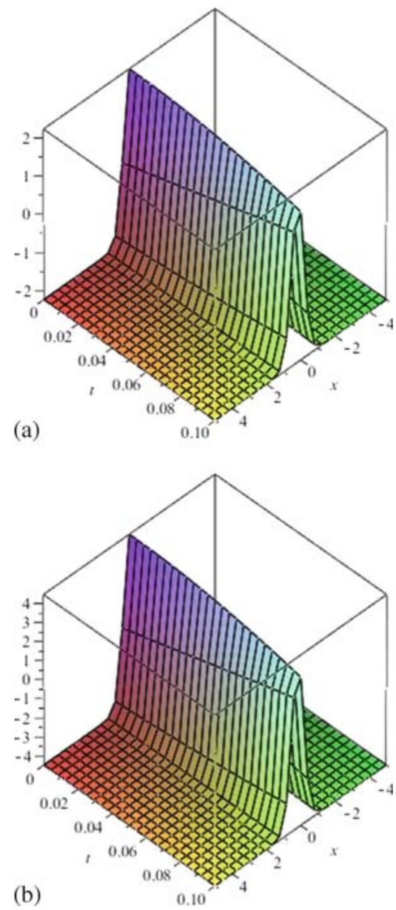

Figure 7. The 3D graphics of (21). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

$$V(\xi) = z_1 - (z_1 - z_2)\mathrm{sn}^2 \times \left(\frac{\sqrt{6\alpha(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_1 - z_2}{z_1 - z_3}}\right), \quad (18)$$

where $z_1 > z_2 > z_3$ and the parameters z_1, z_2, z_3 are defined by $y^2 = 2h - \gamma V - \beta V^2 - \frac{2}{3}\alpha V^3 = \frac{2}{3}\alpha(z_1 - V)(V - z_2)(V - z_3)$.

By using these results and considering condition (6), we obtain exact explicit travelling wave solutions of eq. (1) as follows:

(1) When $\alpha < 0$ and $h = h_2$

$$\begin{cases} u(x,t) = \dfrac{\beta + \sqrt{\Delta} - 3\sqrt{\Delta}\mathrm{sech}^2((\sqrt{\Delta}^{4}/2)(kx - \omega t))}{|\alpha|}, \\ v(x,t) = \dfrac{\beta + \sqrt{\Delta} - 3\sqrt{\Delta}\mathrm{sech}^2((\sqrt{\Delta}^{4}/2)(kx - \omega t))}{2|\alpha|}. \end{cases} \quad (19)$$

(2) When $\alpha < 0$ and $h \in (h_1, h_2)$

$$\begin{cases} u(x,t) = 2\left(z_3 + (z_2 - z_3)\mathrm{sn}^2\left(\dfrac{\sqrt{6B(z_1 - z_3)}}{6}(kx - \omega t), \sqrt{\dfrac{z_2 - z_3}{z_1 - z_3}}\right)\right), \\ v(x,t) = z_3 + (z_2 - z_3)\mathrm{sn}^2\left(\dfrac{\sqrt{6B(z_1 - z_3)}}{6}(kx - \omega t), \sqrt{\dfrac{z_2 - z_3}{z_1 - z_3}}\right). \end{cases} \tag{20}$$

(3) When $\alpha > 0$ and $h = h_2$

$$\begin{cases} u(x,t) = \dfrac{-\beta - \sqrt{\Delta} + 3\sqrt{\Delta}\mathrm{sech}^2((\sqrt{\Delta}^4/2)(kx - \omega t))}{\alpha}, \\ v(x,t) = \dfrac{-\beta - \sqrt{\Delta} + 3\sqrt{\Delta}\mathrm{sech}^2((\sqrt{\Delta}^4/2)(kx - \omega t))}{2\alpha}. \end{cases} \tag{21}$$

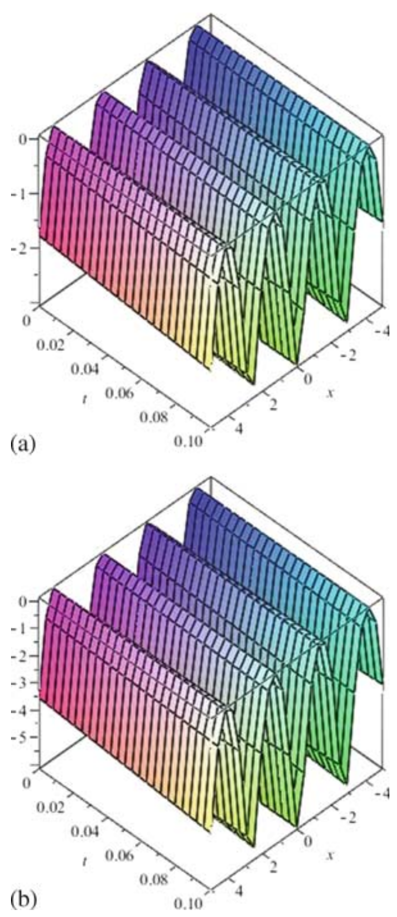

Figure 8. The 3D graphics of (22). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

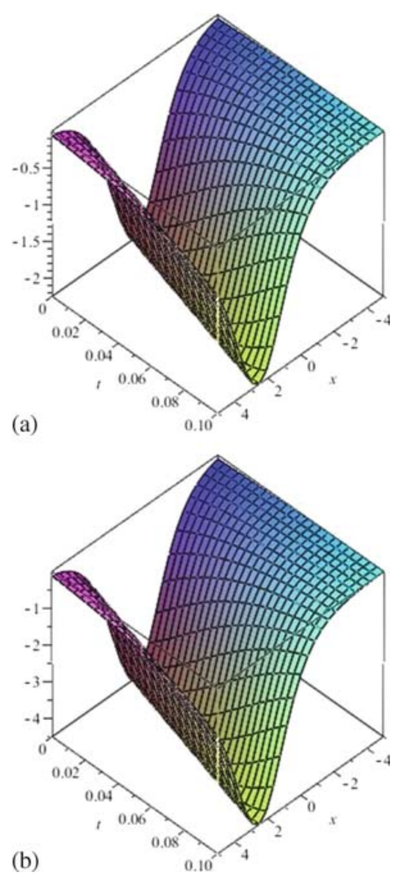

Figure 9. The 3D graphics of (25). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

(4) When $\alpha > 0$ and $h \in (h_1, h_2)$

$$\begin{cases} u(x,t) = 2\left(z_1 - (z_1 - z_2)\mathrm{sn}^2\left(\dfrac{\sqrt{6\alpha(z_1 - z_3)}}{6}(kx - \omega t), \sqrt{\dfrac{z_1 - z_2}{z_1 - z_3}}\right)\right), \\ v(x,t) = z_1 - (z_1 - z_2)\mathrm{sn}^2\left(\dfrac{\sqrt{6\alpha(z_1 - z_3)}}{6}(kx - \omega t), \sqrt{\dfrac{z_1 - z_2}{z_1 - z_3}}\right). \end{cases} \tag{22}$$

3.2 *The case of* $\gamma = 0$

(1) When $\beta < 0$, there exists a smooth dark soliton solution which corresponds to a smooth homoclinic orbit Γ of (13) defined by $H(\phi, y) = 0$, and we have the parametric representation:

$$V(\xi) = \frac{3\beta - 3\beta\tanh^2((\sqrt{-\beta}/2)\xi)}{-2\alpha}. \tag{23}$$

(2) When $\beta > 0$, there exists a smooth dark soliton solution which corresponds to a smooth homoclinic orbit Γ of (13) defined by $H(\phi, y) = h_3$, and we have the parametric representation:

$$V(\xi) = -\frac{\beta}{\alpha}\left(1 - \frac{3}{2}\mathrm{sech}^2\left(\frac{\sqrt{\beta}}{2}\xi\right)\right). \tag{24}$$

By using these results and considering condition (6), we obtain exact explicit travelling wave solutions of eq. (1) as follows:

(1) When $\beta < 0$ and $h = 0$

$$\begin{cases} u(x,t) = -\dfrac{3\beta - 3\beta\tanh^2((\sqrt{-\beta}/2)(kx - \omega t))}{\alpha}, \\ v(x,t) = -\dfrac{3\beta - 3\beta\tanh^2((\sqrt{-\beta}/2)(kx - \omega t))}{2\alpha}. \end{cases} \tag{25}$$

(2) When $\beta > 0$ and $h = h_3$

$$\begin{cases} u(x,t) = -2\dfrac{\beta}{\alpha}\left(1 - \dfrac{3}{2}\mathrm{sech}^2\left(\dfrac{\sqrt{\beta}}{2}(kx - \omega t)\right)\right), \\ v(x,t) = -\dfrac{\beta}{\alpha}\left(1 - \dfrac{3}{2}\mathrm{sech}^2\left(\dfrac{\sqrt{\beta}}{2}(kx - \omega t)\right)\right). \end{cases} \tag{26}$$

Based on the above discussions, by using the numerical simulation method, we simulate all the exact bounded travelling wave solutions of eq. (1) with the aid of *Maple* software.

In figure 5, we take $m = 2, k = 1, c = -1, a = \frac{1}{6}, b = 1, c = \frac{1}{3}, d = \frac{1}{2}, -5 \leq x \leq 5, 0 \leq t \leq 0.1$. In figure 6, we take $m = 2, k = 1, c = -1, a = \frac{1}{6}, b = 1, c = \frac{1}{3}, d = \frac{1}{2}, h = -\frac{1}{6}, -5 \leq x \leq 5, 0 \leq t \leq 0.1$. In figure 7, we take $m = 2, k = 1, c = -1, a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{8}{3}, d = 1, -5 \leq x \leq 5, 0 \leq t \leq 0.1$. In figure 8, we take $m = 2, k = 1, c = -1, a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{8}{3}, d = 1, h = -\frac{1}{6}, -5 \leq x \leq 5, 0 \leq t \leq 0.1$. In figure 9, we take $m = 2, k = 1, c = -1, a = \frac{1}{6}, b = 1, c = \frac{1}{3}, d = \frac{1}{2}, -5 \leq x \leq 5, 0 \leq t \leq 0.1$. In figure 10, we take $m = 2, k = 1, c = -1, a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{8}{3}, d = 1, -5 \leq x \leq 5, 0 \leq t \leq 0.1$.

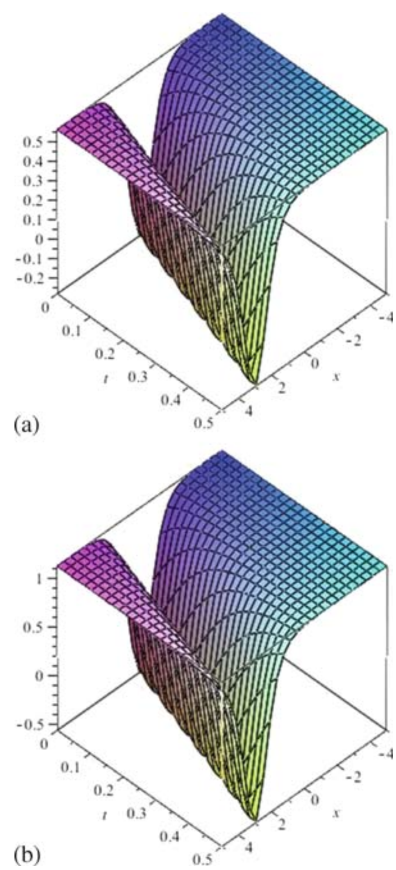

Figure 10. The 3D graphics of (26). (**a**) The 3D graphics of v and (**b**) the 3D graphics of u.

4. Conclusion

To summarize, by using the dynamical system method, the new exact travelling wave solutions (solitary wave solutions and periodic wave solutions) have been obtained for the bidirectional wave equations. Among them, eqs (19), (21), (25) and (26) are solitary wave solutions which are expressed by the hyperbolic functions. Equations (20) and (22) are periodic travelling wave solution which are expressed by Jacobian elliptic functions. The hyperbolic function solutions and the Jacobian elliptic function solutions in this paper are different from the solutions presented by other methods before. These results enrich the diversity of solution structures of the bidirectional wave equations.

From the above discussions, it is clear that the dynamical system method is a very powerful method to seek exact travelling wave solutions for nonlinear travelling wave equations. This method reduces large amount of calculations and allows us to solve complicated nonlinear evolution equations in mathematical physics. Moreover, this method can also be applied to other nonlinear travelling wave equations which can be reduced to integrable system.

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References

[1] J Lee and R Sakthivel, *Pramana – J. Phys.* **76**, 819 (2011)
[2] B Anitha, S Rathakrishnan, P Sagayaraj and A J A Pragasam, *Chin. J. Phys.* **52**, 939 (2014)
[3] C Chun and R Sakthivel, *Comput. Phys. Commun.* **181**, 1021 (2010)
[4] R Sakthivel, C Chun and J Lee, *Z. Naturforsch. A* **65**, 633 (2010)
[5] J Lee and R Sakthivel, *Pramana – J. Phys.* **81(6)**, 893 (2013)
[6] M Chen, *Appl. Math. Lett.* **11**, 45 (1998)
[7] Jibin Li, *Singular nonlinear travelling wave equations: Bifurcations and exact solutions* (Science Press, Beijing, 2013)
[8] Jibin Li and Huihui Dai, *On the study of singular nonlinear travelling equations: Dynamical system approach* (Science Press, Beijing, 2007)
[9] Shaolong Xie, Lin Wang and Yuzhong Zhang, *Commun. Nonlinear Sci. Numer. Simulat.* **17**, 1130 (2012)
[10] Heng Wang and Shuhua Zheng, *Chaos, Solitons and Fractals* **82**, 83 (2016)
[11] P M Byrd and M D Friedmann, *Handbook for elliptic integrals for engineers and scientists* (Springer, Berlin, 1971)