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Chaos in discrete fractional difference equations

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Abstract. Recently, the discrete fractional calculus (DFC) is receiving attention due to its potential applications in the mathematical modelling of real-world phenomena with memory effects. In the present paper, the chaotic behaviour of fractional difference equations for the tent map, Gauss map and $2x(\text{mod } 1)$ map are studied numerically. We analyse the chaotic behaviour of these fractional difference equations and compare them with their integer counterparts. It is observed that fractional difference equations for the Gauss and tent maps are more stable compared to their integer-order version.

Keywords. Fractional difference equation; chaos; Lyapunov exponent; Gauss map; tent map; discrete fractional calculus.

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1. Introduction

Fractional calculus (FC) has a history of more than 300 years, however it is being applied in several areas of science and engineering since the nineties of the last century [1–7]. FC enables us to interpolate the behaviour between integer-order systems. Fractional-order derivatives allow us to deal comfortably with memory effects in dynamical systems [2].

Discrete fractional calculus (DFC) has been gaining attention recently. Discrete fractional difference operator incorporates memory effect, and it can be used to model many physical phenomena, in particular, growth behaviour in biology [8]. For the history and basic theory of DFC, we refer our readers to [8–13].

The purpose of this paper is to understand the chaotic behaviour shown by fractional difference equations. Recently, such a study has been done for the discrete logistics map and discrete sine map [14,15]. In this paper, we analyse numerically the chaotic behaviour of three maps viz., discrete tent map, discrete $2x(\text{mod}1)$ map and discrete Gauss map. Study of these maps is important as they are standard one-dimensional maps, well known to show characteristic bifurcation and chaos for the integer-order, and have found applications in a number of fields ranging from biology to number theory [16].

The rest of this paper is organized as follows. Section 2 introduces some of the required preliminaries. Section 3 is dedicated to the study of the discrete tent map. Section 4 studies the discrete Gauss map, while §5 deals with the discrete $2x(\text{mod } 1)$ map. Section 6 sums up the conclusions.

2. Preliminaries

In the present section, we set up notations and recall some basic definitions from discrete fractional calculus [9,10,17].

Let $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ where $a \in \mathbb{R}$.

DEFINITION 1

If u is a real-valued function defined on $\mathbb{N}_a$ and $\alpha > 0$, then the discrete fractional sum of order α denoted as $\Delta_a^{-\alpha}$ is defined as

$$\Delta_a^{-\alpha} u(t) := \sum_{s=a}^{t-\alpha} \frac{(t-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)} u(s), \quad t \in \mathbb{N}_{a+\alpha}. \quad (1)$$

Here $\sigma(s) = s + 1$ is the forward shift operator and $t^{\underline{\alpha}}$ denotes the falling factorial function given as

$$t^{\underline{\alpha}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}. \quad (2)$$

Remark. Notice that the domain is taken as $\mathbb{N}_{a+\alpha}$ for the sum to be well-defined.

Regular forward difference operator is defined as $\Delta u(t) = u(t+1) - u(t)$.

DEFINITION 2

For $\alpha > 0$, such that $N-1 < \alpha < N$ for $N \in \mathbb{N}$, u: $\mathbb{N}_a \to \mathbb{R}$, the Caputo-type discrete fractional difference operator is defined as

$$\Delta_a^\alpha u(t) := \Delta_a^{-(N-\alpha)} \Delta^N u(t) \tag{3}$$

$$= \sum_{s=a}^{t-N+\alpha} \frac{(t-\sigma(s))^{\underline{N-\alpha-1}}}{\Gamma(N-\alpha)} \Delta^N u(s),$$
$$t \in \mathbb{N}_{a+N-\alpha}. \tag{4}$$

Remark. Point a in the above definitions is called the anchor point or starting point. Although integer-order forward difference operator Δ^N is independent of the choice of the starting point, fractional difference operators (and sum operators) Δ_a^α depend on the choice of the starting point.

Theorem 1 (see [18]). *For the fractional difference equation where $N-1 < \alpha < N$, $N \in \mathbb{N}$,*

$$\Delta_a^\alpha u(t) = f(t+\alpha-1, u(t+\alpha-1))$$
$$\Delta^k u(a) = u_k, \quad u_k \in \mathbb{R}, \ k = 0, 1, 2, \ldots, N-1, \tag{5}$$

the equivalent integral equation is given as

$$u(t) = \sum_{k=0}^{N-1} \frac{(t-a)^{\underline{k}}}{k!} u_k + \sum_{s=a+N-\alpha}^{t-\alpha} \frac{(t-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(s+\alpha-1,$$
$$u(s+\alpha-1)), \quad t \in \mathbb{N}_{a+N}. \tag{6}$$

Hereafter, we always assume anchor point $a = 0$ and restrict ourselves to $0 < \alpha < 1$. In view of this, eq. (6) becomes

$$u(t) = u_0 + \sum_{s=1-\alpha}^{t-\alpha} \frac{(t-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(s+\alpha-1,$$
$$u(s+\alpha-1)), \quad t \in \mathbb{N}_1, \tag{7}$$

where $u(0) = u_0$. As $s+\alpha \in \mathbb{N}$, let $s+\alpha = j$ and using the fact that

$$(t-\sigma(s))^{\underline{\alpha-1}} = \frac{\Gamma(t-s)}{\Gamma(t-s-\alpha+1)},$$

we have

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} f(j-1, u(j-1)),$$
$$t \in \mathbb{N}_1. \tag{8}$$

3. Chaos in the discrete tent map

A tent map is defined as

$$f(x) := \mu \min\{x, 1-x\}, \quad x \in [0, 1], \tag{9}$$

where μ is a parameter. The name comes from the shape of the graph which looks like a tent with a sharp corner at $x = 1/2$. We take $0 \le \mu \le 2$ to ensure that f maps [0, 1] into [0, 1] and all orbits remain bounded. Integer-order difference equation for the tent map is given as

$$u(t+1) = \mu \min\{u(t), 1-u(t)\}, \quad u(0) = c,$$
$$u(t+1) - u(t) = \mu \min\{u(t), 1-u(t)\} - u(t),$$
$$u(0) = c,$$
$$\Delta u(t) = \min\{(\mu-1)u(t), \mu-(\mu+1)u(t)\},$$
$$u(0) = c. \tag{10}$$

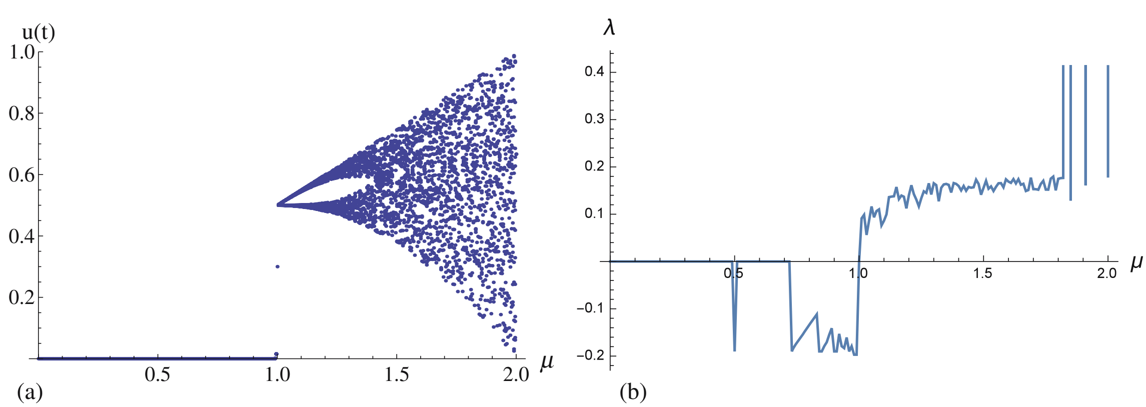


Figure 1. The bifurcation diagram (**a**) and Lyapunov exponent distribution (**b**) for integer-order difference equation of the tent map with $u(0) = 0.3$ and $\alpha = 1$.

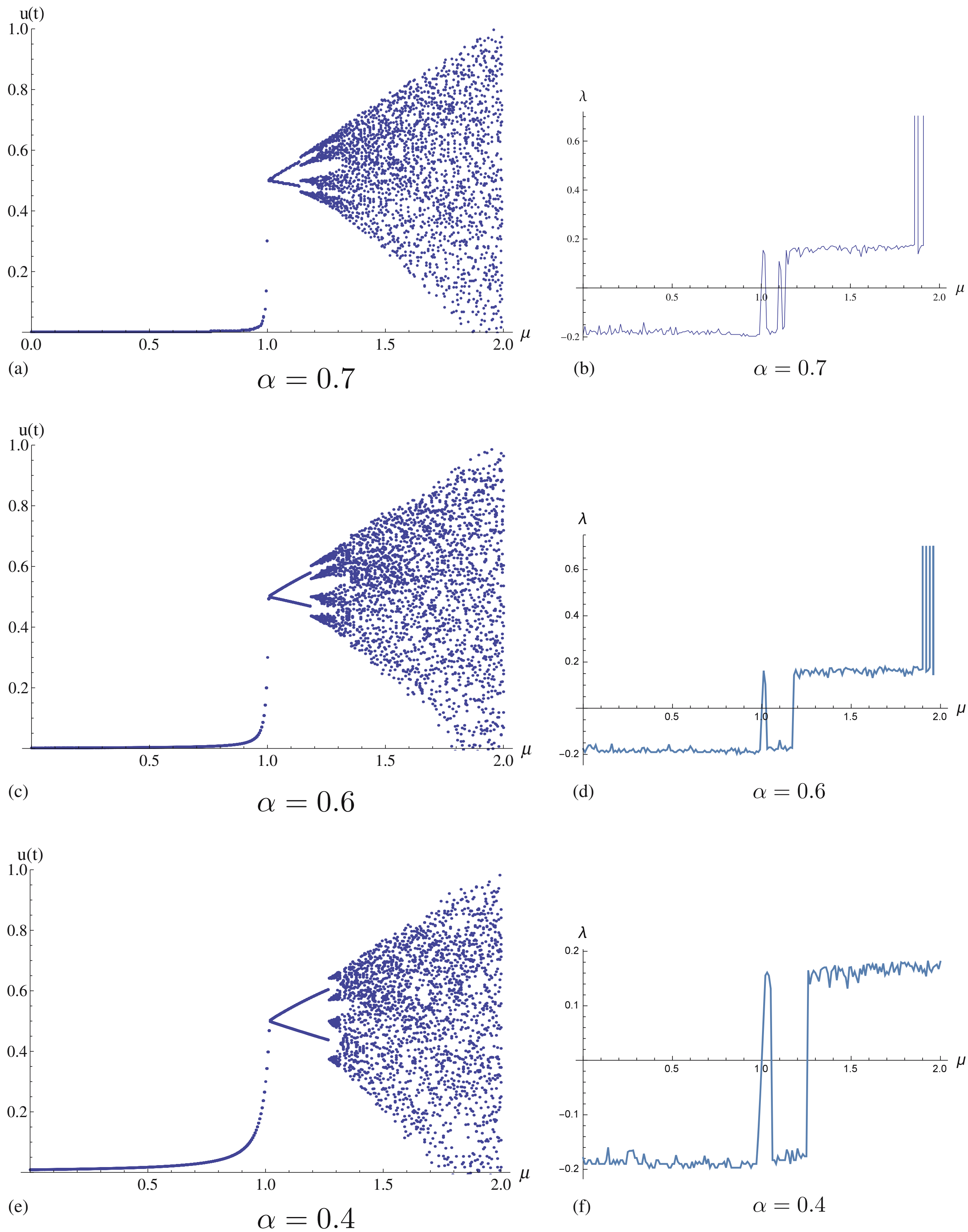


Figure 2. The bifurcation diagram and Lyapunov exponent distribution for fractional-order difference equation of the tent map with $u(0) = 0.3$ and $\alpha = 0.7$, 0.6 and 0.4.

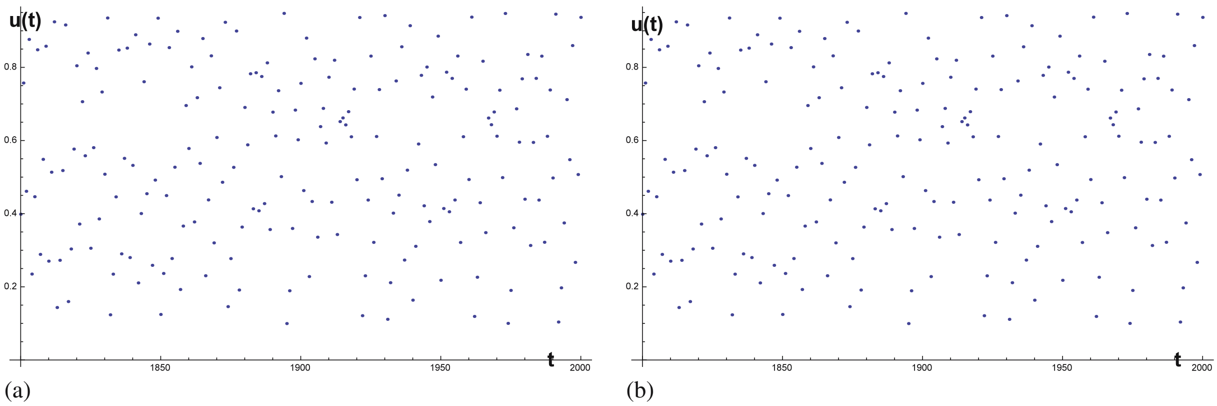


Figure 3. $\mu = 1.90$ orbit of the tent map for (**a**) $\alpha = 1$ and (**b**) $\alpha = 0.6$, with $u(0) = 0.3$ showing chaos.

Fractional difference equation equivalent to eq. (10) is given as

$$\begin{aligned}\Delta_0^\alpha u(t) &= \min\{(\mu-1)u(t+\alpha-1),\\ &\quad \mu-(\mu+1)u(t+\alpha-1)\},\\ u(0) &= c,\ t\in\mathbb{N}_{1-\alpha},\quad 0<\alpha\le 1.\end{aligned} \tag{11}$$

In view of eq. (8), the integral form of (11) is

$$\begin{aligned}u(t) &= c+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{t}\frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)}\min\{(\mu-1)u(j-1),\\ &\quad \mu-(\mu+1)u(j-1)\},\quad t\in\mathbb{N}_1.\end{aligned} \tag{12}$$

It should be noted that the present state $u(t)$ of the integer order map depends only on the previous state $u(t-1)$ explicitly, whereas in the case of fractional order map, state $u(t)$ depends on the values $u(0)$, $u(1)$, $u(2)$, ..., $u(t-1)$ explicitly. This phenomenon is called the memory effect of the discrete fractional map.

In figure 1, the bifurcation diagram along with the Lyapunov exponent distribution has been plotted for the integer-order difference equation, while the same has been plotted for fractional-order difference equations with $\alpha = 0.7$, 0.6 and 0.4 in figure 2 (initial condition $u(0) = 0.3$).

From the figures, we put forth the following observations:

(1) Tent map is known for the sudden onset of chaos after $\mu = 1$ as seen from figure 1. However, in the fractional version we observe the existence of a ‘stability window’ between the bifurcation at $\mu = 1$ and the onset of chaos (see figure 2).
(2) The width of this ‘stability window’ increases as the order of derivative α decreases.
(3) We observe that as α decreases, the number of steps required for orbits to converge to the fixed

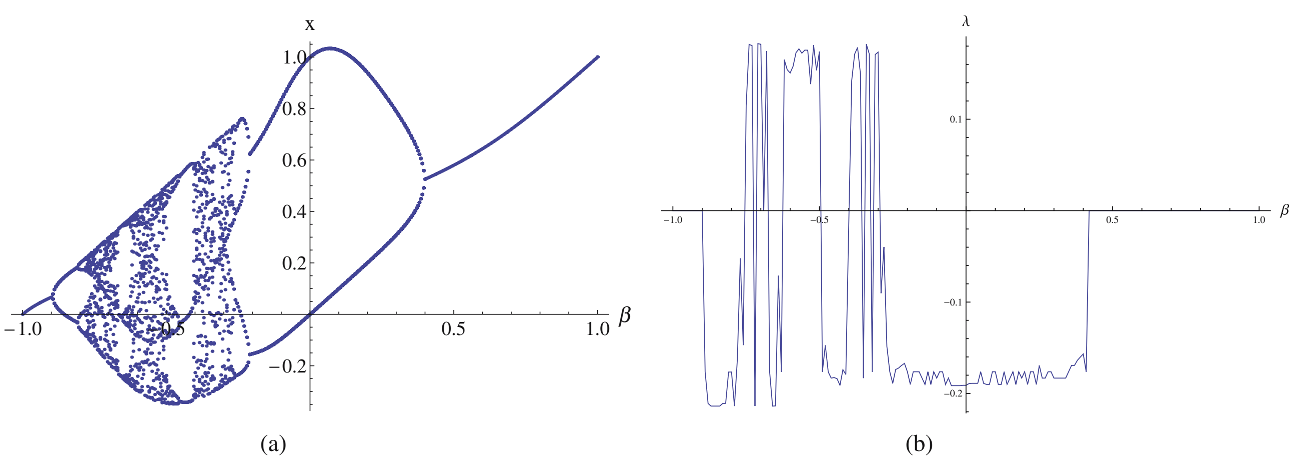
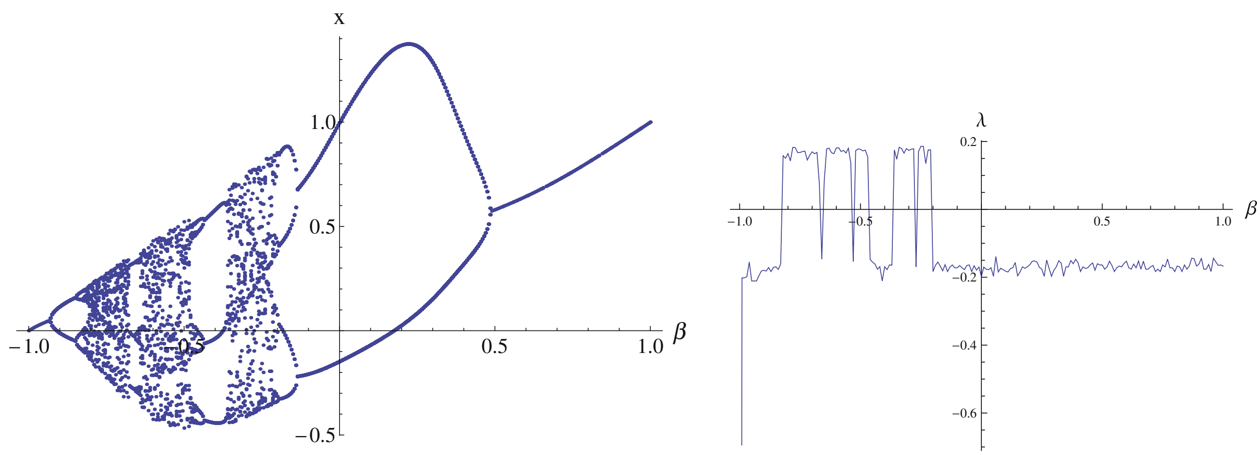


Figure 4. The bifurcation diagram (**a**) and Lyapunov exponent distribution (**b**) for integer-order difference equation of the Gauss map with $u(0) = 0$ and $\alpha = 1$.



(a) (b)

Figure 5. The bifurcation diagram (**a**) and Lyapunov exponent distribution (**b**) for fractional-order difference equation of the Gauss map with $u(0) = 0$ and $\alpha = 0.8$.

point increases rapidly. Thus, the rate of convergence decreases as α decreases. For $\alpha \leq 0.2$, this rate is ultraslow.

(4) No significant change is observed by changing the initial conditions.

In figure 3, we plot the chaotic orbit (t vs. $u(t)$) of the tent map for $\alpha = 1$ and $\alpha = 0.6$.

4. Chaos in the discrete Gauss map

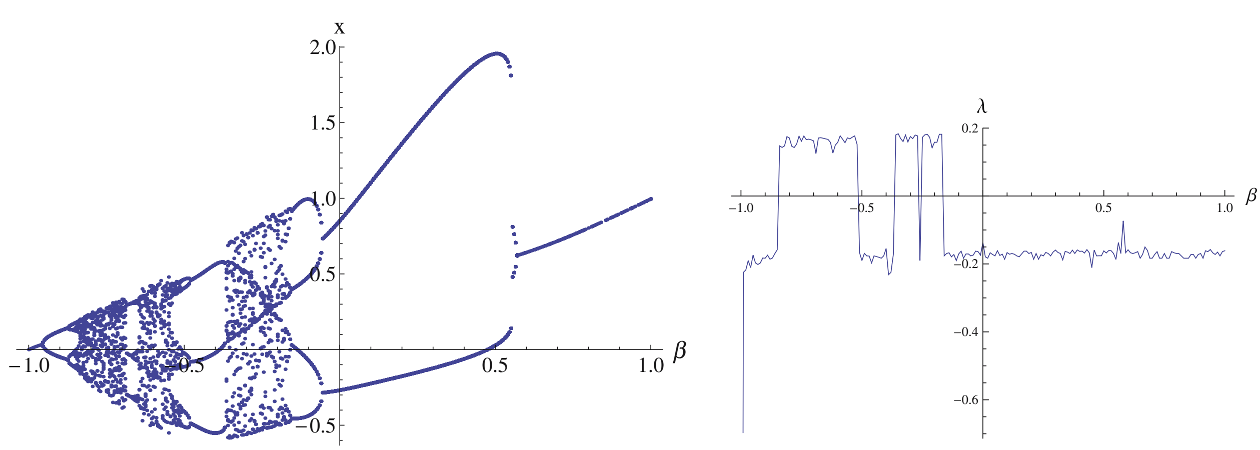
Gauss map is defined as

$$g(x) := \exp(-\nu x^2) + \beta, \quad x \in \mathbb{R}, \tag{13}$$

where ν and β are parameters. The graph of the Gauss map is a Gaussian curve. Parameter ν determines the width of the peak, while β controls its position. Hereafter, we fix $\nu = 7.5$ and study the features by varying β.

Integer-order difference equation for the Gauss map is given as

$$\begin{aligned} &u(t+1) = \exp(-7.5(u(t))^2) + \beta, \quad u(0) = c, \\ &u(t+1) - u(t) = \exp(-7.5(u(t))^2) + \beta - u(t), \\ &\quad u(0) = c, \\ &\Delta u(t) = \exp(-7.5(u(t))^2) + \beta - u(t), \\ &\quad u(0) = c. \end{aligned} \tag{14}$$



(a) (b)

Figure 6. The bifurcation diagram (**a**) and Lyapunov exponent distribution (**b**) for fractional-order difference equation of the Gauss map with $u(0) = 0$ and $\alpha = 0.6$.

Fractional difference equation equivalent to (14) is given as

$$\Delta_0^\alpha u(t) = \exp(-7.5(u(t+\alpha-1))^2)+\beta-u(t+\alpha-1),$$

$$u(0)=c, \quad t \in \mathbb{N}_{1-\alpha}, \quad 0<\alpha\le 1. \tag{15}$$

In view of (8), the integral form of (15) is

$$u(t) = c + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{t}\frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} \times [\exp(-7.5(u(j-1))^2)+\beta-u(j-1)], \quad t\in\mathbb{N}_1. \tag{16}$$

In figure 4, bifurcation diagram along with Lyapunov exponent distribution has been plotted for integer-order difference equation with initial condition $u(0)=0$.

Here, we consider $-1\le\beta\le 1$, as chaotic behaviour and bifurcations are observed in this region. We observe the following features:

(1) Series of period doubling bifurcations leading to chaos.
(2) Two prominently visible period-3 windows around $\beta=-0.65$ and $\beta=-0.45$.
(3) Sequence of period undoubling and again single stable fixed point $\beta\ge 0.45$.

Corresponding fractional counterparts for $\alpha=0.8$, 0.6, 0.4 are plotted in figures 5, 6, 7 and 8.

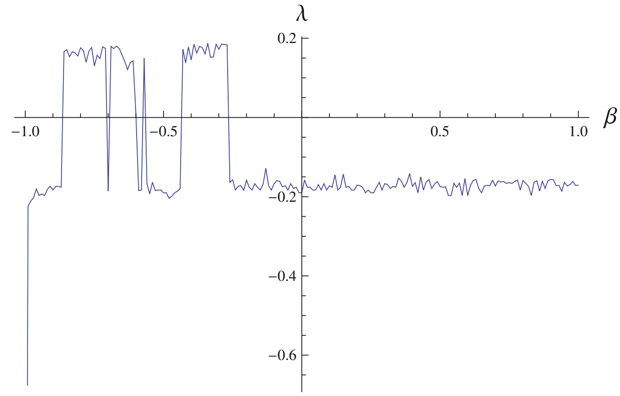


Figure 8. The Lyapunov exponent distribution for fractional-order difference equation of the Gauss map with $\alpha=0.4$ and $u(0)=0$.

From the figures, we make the following observations:

(1) All the features present in the integer-order bifurcation diagram can also be observed in fractional-order bifurcation diagrams.
(2) We observe period doubling bifurcations leading to chaos in all bifurcation diagrams.
(3) For $\alpha=0.8, 0.6$ we observe two prominently visible period-3 windows. As the value of α decreases, we observe that the first window becomes smaller in width, while this window completely disappears in the $\alpha=0.4$ bifurcation diagram. The second period-3 window (around $\beta=-0.45$) increases in width as α decreases. For $\alpha=0.4$, we observe another period doubling and

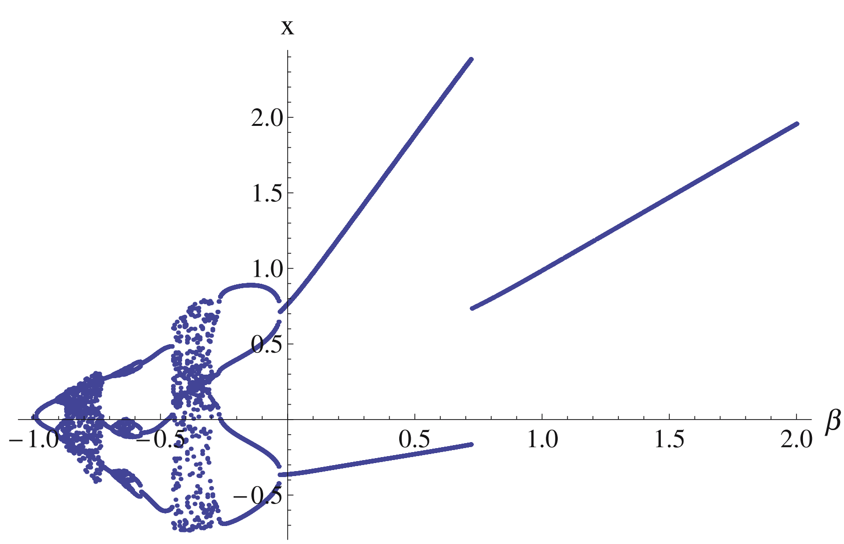


Figure 7. The bifurcation diagram for fractional-order difference equation of the Gauss map with $\alpha=0.4$ and $u(0)=0$.

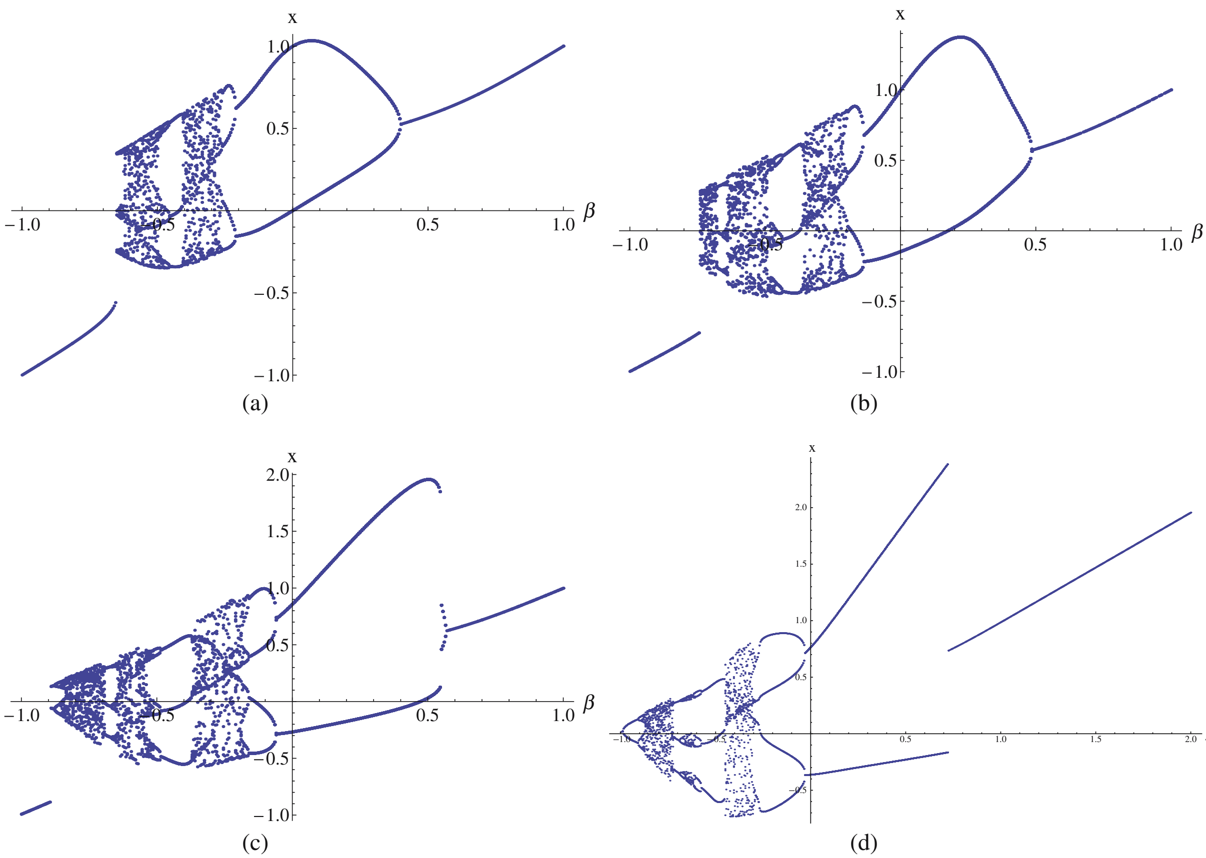


Figure 9. The bifurcation diagram for fractional-order difference equation of the Gauss map with initial condition as $u(0) = 0.5$ and (**a**) $\alpha = 1$, (**b**) $\alpha = 0.8$, (**c**) $\alpha = 0.6$ and (**d**) $\alpha = 0.4$.

undoubling inside this period-3 window (around $\beta = -0.61$).

(4) We observe period undoubling in all the bifurcation diagrams. Although for lower values of α we see merging happening for larger values of β, for $\alpha = 0.4$, we see the last two threads merge around $\beta = 1.170$. This elongates the bifurcation diagram.

(5) We observe that the rate of convergence of the orbit is very slow for lower values of α and hence the system takes much longer time to obtain a stable state, especially for $\alpha \leq 0.4$.

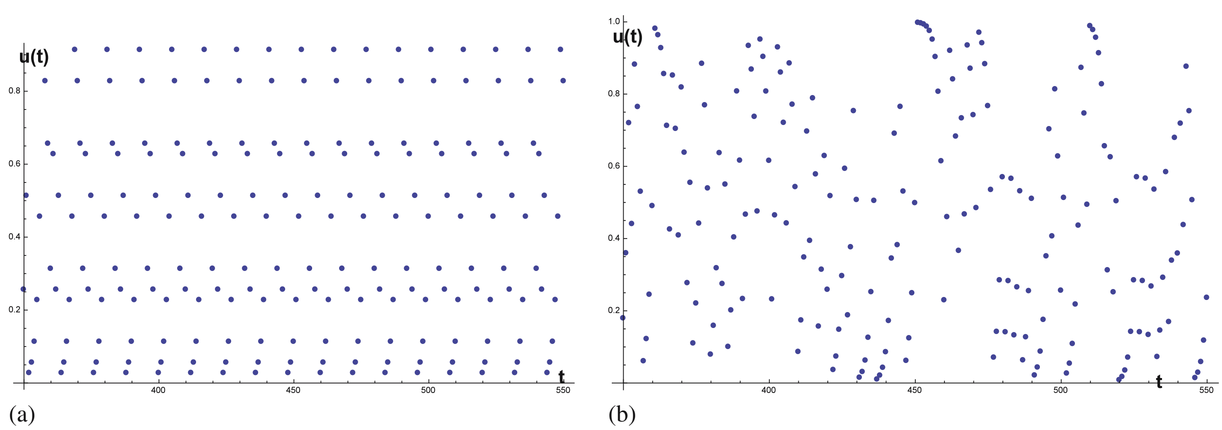


Figure 10. Orbit $\alpha = 1$ with initial condition (**a**) $c = 22/70$ (periodic orbit) and (**b**) $c = \pi/10$ (chaotic orbit).

(6) With change in initial condition from $u(0) = 0$ to $u(0) = 0.5$, we do see a drastic 'jump discontinuity' in the bifurcation diagram of integer order. The same can be observed in each of the fractional bifurcation diagram as well (see figure 9).

5. Chaos in the discrete $2x \pmod 1$ map

The $2x \pmod 1$ map is defined as

$$h(x) := 2x \pmod 1, \quad x \in \mathbb{R}. \tag{17}$$

The successive iterations of the map will shift a digit towards left in the binary representation of x. Thus, the map $h(x)$ is also called a bit-shift map or Bernoulli map.

Integer-order difference equation for the $2x \pmod 1$ map is given as

$$\begin{aligned} &u(t+1) = 2u(t) \pmod 1, \quad u(0) = c, \\ &u(t+1) - u(t) = 2u(t) - u(t) \pmod 1, \\ &\quad u(0) = c, \\ &\Delta u(t) = u(t) \pmod 1, \quad u(0) = c. \end{aligned} \tag{18}$$

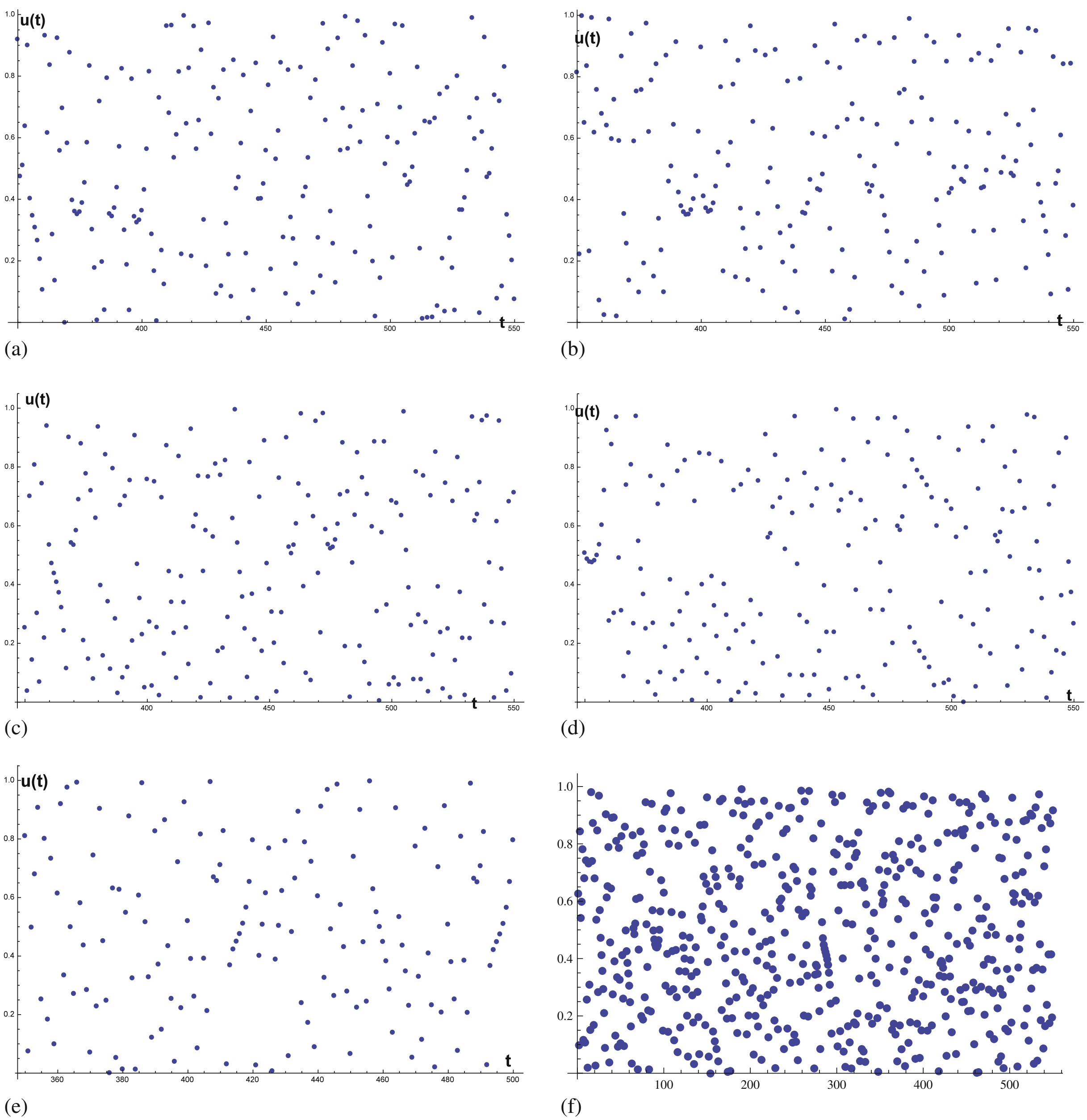
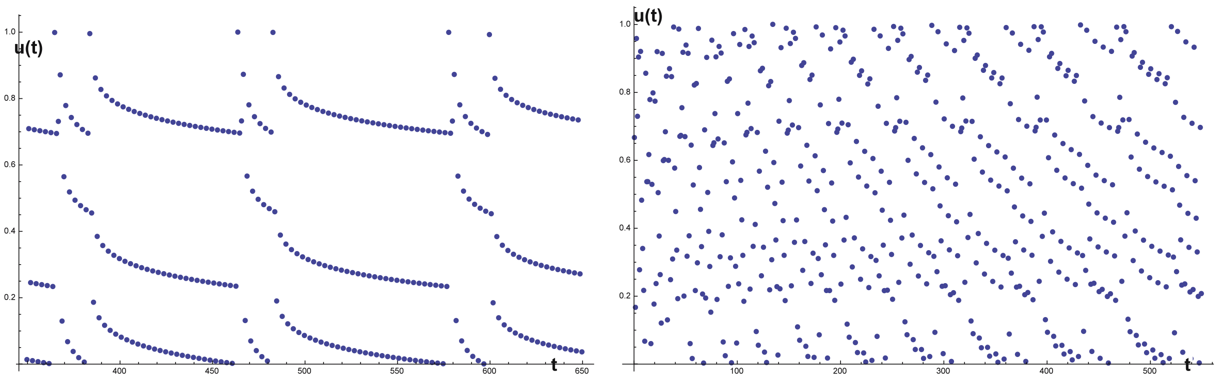


Figure 11. Orbits for the fractional difference equation (**a**) for $c = 22/70$ and $\alpha = 0.8$, (**b**) for $c = \pi/10$ and $\alpha = 0.8$, (**c**) for $c = 22/70$ and $\alpha = 0.6$, (**d**) for $c = \pi/10$ and $\alpha = 0.6$, (**e**) for $c = 22/70$ and $\alpha = 0.5$ and (**f**) for $c = \pi/10$ and $\alpha = 0.5$.



(a) (b)

Figure 12. Orbits for fractional difference equation (**a**) for $c = 1/2$ and $\alpha = 0.5$ and (**b**) for $c = 1/3$ and $\alpha = 0.5$.

Equation (18) is known for chaotic behaviour. For every irrational c, a chaotic orbit is generated, while $c \in \mathbb{Q}$ leads to a periodic orbit (see figure 10 for t vs. $u(t)$ plot).

It may be observed that the map does exhibit sensitive dependence on initial conditions and hence is chaotic. This is also confirmed by Lyapunov exponent which is well known to be $\ln(2)$.

Fractional difference equation equivalent to (18) is given as

$$\Delta_0^{\alpha} u(t) = u(t + \alpha - 1)(\text{mod } 1),$$
$$u(0) = c,\ t \in \mathbb{N}_{1-\alpha}, \quad 0 < \alpha \le 1. \tag{19}$$

In view of (8), the integral form of (19) is

$$u(t) = c + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} u(j-1)(\text{mod } 1),$$
$$t \in \mathbb{N}_1. \tag{20}$$

In figures 11 and 12, we plot the orbits t vs. $u(t)$ for fractional difference equation (20) with various initial conditions and for $\alpha = 0.8, 0.6, 0.5$ respectively.

From the figures we make following observations:

(1) The interpretation of the integer-order map as a bit-shift map no longer holds true for the fractional version.
(2) For an irrational initial condition, a chaotic orbit is generated like its integer version.
(3) In fractional case, the rational initial condition may lead to a periodic orbit (see figure 12) or a chaotic orbit (see figure 11), unlike its integer version.
(4) As the map chops off one digit in each iteration, it is heavily dependent on the accurate representation of a decimal number in the computer program. For all the above figures, we are using precision of upto 500 digits after the decimal point.

6. Conclusions

Fractional difference equations corresponding to the discrete tent map, discrete Gauss map and discrete $2x(\text{mod } 1)$ map have been studied numerically. It is observed that chaotic behaviour varies according to fractional order. Fractional versions of the tent map and the Gauss map are more stable than their integer counterparts. In case of fractional tent map, width of the stability window increases as the fractional order decreases. In case of Gauss map, width of the period-3 window increases as the fractional order decreases. It is further observed that the pattern of chaotic behaviour is altered with changes in the initial conditions in the case of fractional Gauss and $2x(\text{mod } 1)$ maps.

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