



Periodic and solitary wave solutions of cubic–quintic nonlinear reaction-diffusion equation with variable convection coefficients

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Abstract. Attempts have been made to explore the exact periodic and solitary wave solutions of nonlinear reaction diffusion (RD) equation involving cubic–quintic nonlinearity along with time-dependent convection coefficients. Effect of varying model coefficients on the physical parameters of solitary wave solutions is demonstrated. Depending upon the parametric condition, the periodic, double-kink, bell and antikink-type solutions for cubic–quintic nonlinear reaction-diffusion equation are extracted. Such solutions can be used to explain various biological and physical phenomena.

Keywords. Variable coefficient reaction-diffusion equation; solitary wave solution; cubic–quintic nonlinearity.

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1. Introduction

The investigation of periodic and solitary wave solutions to nonlinear evolution equations play an important role in the qualitative description of various nonlinear physical phenomena [1] in different streams of science. Recently, such phenomena have been studied in a variety of nonlinear partial differential equations (NLPDEs). The exact solutions to such equations are helpful to understand the underlying mechanism and provide better knowledge of their physical contents [2]. The reaction-diffusion (RD) equations have attracted considerable attention to model the evolution systems in the real world [3] and play important roles in explaining processes like heat conduction in plasma [4], image processing [5], flow in porous media [6], population genetics [7] etc. The study of solitary wave-type

solutions of RD equations becomes more interesting because experimental findings suggest that RD systems may have spatially well-localized solitary patterns which behave like particles and retain their identity while interacting [8].

The literature discussing the RD equations is massive but results assume that the environment is temporally and spatially homogeneous. However, this idea may be a rough approximation to many systems because most of the physical and biological systems are heterogeneous due to fluctuations in environmental conditions and non-uniform media. Hence, most of the real nonlinear physical equations possess variable coefficients, both in space and time [9,10]. The effect of spatial inhomogeneities on RD systems [11] has been addressed by many authors but the effect of temporal inhomogeneities remained unexplored. So, keeping in view the importance of temporal inhomogeneities, we have chosen the nonlinear reaction-diffusion equation with cubic–quintic terms. The dimensionless form of the variable coefficients in nonlinear reaction-diffusion (NLRD) equation is

$$u_t + v(t)u_x = Du_{xx} + \alpha u - \beta u^3 + \gamma u^5, \quad (1)$$

where $u = u(x, t)$ is the concentration or density variable depending on the phenomena under study; D is the diffusion coefficient; v is the convection term coefficient; α , β and γ are the reaction term coefficients. The first term on the right-hand side of eq. (1) represents diffusion due to concentration gradient and last two terms characterize the cubic and quintic nonlinearities of the system, whereas on the left-hand side of eq. (1), first term shows the evolution of the system with time and second term represents the convective flux term. The homogeneous analog of eq. (1) is the real version of Newell–Whitehead–Segel (NWS) equation which describes the slow spatial modulation of the stripe patterns in usual pattern forming various systems [12] and optical systems [13]. For $v(t) = 0$, eq. (1) reduces to Fisher–Kolmogorov (FK) equation with cubic–quintic nonlinearity [14] having constant coefficients. The FK equation arises in the study of pattern formation in bistable systems [15]. In this paper, we have obtained travelling wave solutions of nonlinear reaction-diffusion equations involving cubic–quintic nonlinearity. The said nonlinearity supports fractional transform and solitary wave solutions with periodic, antikink, bell and double-kink-type solutions.

The paper is organized as follows: §2 describes the mathematical formalism of quintic elliptic equation. Travelling wave solutions of nonlinear reaction-diffusion equations are addressed in §3 and finally concluding remarks are made in §4.

2. Quintic elliptic equation

Consider the Galilean frame of ref. [16],

$$\xi = kx + \eta(t), \quad (2)$$

where k is a constant. Let us assume that the solution of eq. (1) in consonance with eq. (2) is

$$u(\xi) = a(t)\phi(\xi). \quad (3)$$

After employing eq. (3) in eq. (1), we get

$$a_t\phi + (a\eta_t + vak)\phi_\xi = Da^2k^2\phi_{\xi\xi} + \alpha a\phi - \beta a^3\phi^3 + \gamma a^5\phi^5. \quad (4)$$

By selecting $a = \text{constant}$ and $\eta(t) = -k \int v(t)dt$, eq. (4) turns out to be

$$\phi_{\xi\xi} + a_1\phi + a_2\phi^3 + a_3\phi^5 = 0, \quad (5)$$

where $a_1 = \alpha/Dk^2$, $a_2 = \beta a^2/Dk^2$ and $a_3 = \gamma a^4/Dk^2$.

Equation (5) is a quintic elliptic equation and admits a variety of solutions such as periodic, double kink, antikink and bell-type solutions [17]. In general, all travelling wave solutions of eq. (5) can be expressed in a generic form by means of the Weierstrass function [18]. When $a_1 > 0$, eq. (5) admits periodic solutions and for $a_1 < 0$, it yields double-kink solutions. On keeping a_1, a_2 fixed and for a particular value of a_3 , eq. (5) yields antikink and bell-type solutions.

3. Travelling wave solutions

Depending upon the choice of parameters, a_1, a_2 and a_3 , appearing in eq. (5), the travelling wave solutions are classified as fractional transform and solitary wave solutions and discussed as follows.

3.1 Fractional transform soliton solutions

The existence of fractional transform soliton solutions can be understood if all the parameters, a_1, a_2 and a_3 , appearing in eq. (5) are non-zero. For a large value of free parameter ϵ , and with parameters a_1, a_2 as fixed, one can obtain double-kink-type solitary wave solutions and periodic solution depending upon the choice of parametric condition a_3 .

Case I. Double-kink solutions

If $a_1 < 0$, then eq. (5) admits double-kink-type solitary wave solutions [19,20] of the form

$$\phi(\xi) = \frac{p \sinh(q\xi)}{\sqrt{\epsilon + \sinh^2(q\xi)}}. \quad (6)$$

Here, p, q and ϵ are wave parameters which depend on the model parameters. After substituting eq. (6) in eq. (5), one obtains various wave parameters in terms of model parameters and it is observed that ϵ behaves as a free parameter under the following conditions:

$$p = \sqrt{\frac{-2a_1(2\epsilon - 3)}{a_2(\epsilon - 3)}}, \quad q = \sqrt{\frac{-a_1\epsilon}{\epsilon - 3}} \quad \text{and} \quad a_3 = \frac{-3q}{p} \left(\frac{\epsilon - 1}{\epsilon} \right). \quad (7)$$

Here, ϵ is a free parameter which controls the width of solitary wave solutions. By selecting $a_1 = -1, a_2 = 2, k = 1$ and $v = \text{sech}(t)$, the amplitude profile for different values of ϵ is shown in figure 1a and 3D profile is shown in figure 1b. From the figure, one can see that the double-kink nature of the solution exists only for sufficiently large value of ϵ . It can also be seen that as the value of ϵ changes, the width of the wave is affected but amplitude of the wave remains the same.

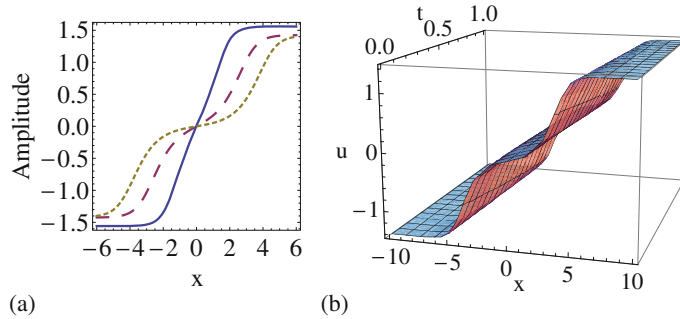


Figure 1. (a) Amplitude profile for double-kink solutions of eq. (6) with different ϵ along x -direction at $t = 0$; (b) the corresponding 3D profile for $\epsilon = 1000$ along x - and t -directions for values mentioned in the text.

Case II. Periodic solutions

For $a_1 > 0$, eq. (5) exhibits the periodic solutions [14] as shown by figure 2a. However, the evaluation of 3D profile for $a_1 = 1, a_2 = 2, k = 1, \epsilon = 1000$ and $v = \text{sech}(t)$ is shown in figure 2b.

3.2 Solitary wave solutions

It is very interesting to note that for $a_1 < 0$ and $a_2 > 0$, eq. (5) exhibits bell-type as well as antikink-type solitary wave solutions depending on the choice of a_3 . The corresponding explicit solutions and amplitude profile are discussed as follows:

Case I. Bell-type solitary wave solutions

If $a_3 < 3a_2^2/16a_1$, then solitary wave solutions of eq. (5) are bell-shaped [19,21] and given by

$$\phi(\xi) = \frac{p}{\sqrt{1 + r \cosh(q\xi)}}. \quad (8)$$

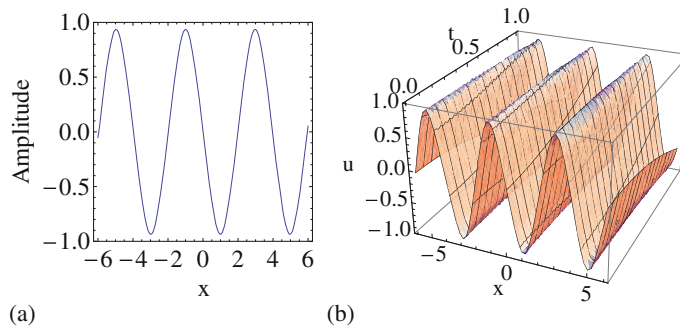


Figure 2. (a) Amplitude profile for periodic solutions of eq. (6) along x -direction at $t = 0$; (b) the corresponding 3D profile along x - and t -directions for values mentioned in the text.

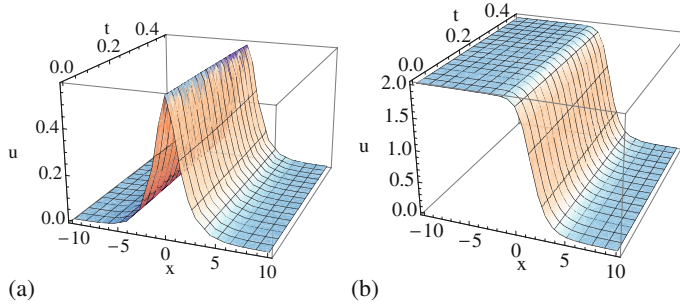


Figure 3. (a) Amplitude profile for bell-type solutions, eq. (8); (b) amplitude profile for antikink-type solutions, eq. (9) for values mentioned in the text.

After making use of eqs (5) and (8), various wave parameters are obtained as

$$p^2 = \frac{-4a_1}{a_2}, \quad q^2 = -4a_1 \quad \text{and} \quad r^2 = \left(1 - \frac{16a_3a_1}{3a_2^2}\right).$$

For p and q to be real numbers, $a_1 < 0$ and $a_2 > 0$, with $a_1 = -1$, $a_2 = 2$, $k = 1.6$, $\epsilon = 1000$ and $v = \tanh[t]$, the corresponding amplitude profile for the bell-type solutions is shown in figure 3a.

Case II. Antikink-type solitary wave solutions

For $a_3 = 3a_2^2/16a_1$, eq. (5) admits antikink-type solutions [19] of the form given by

$$\phi_{\pm}(\xi) = \pm \sqrt{1 \pm \tanh(q\xi)}, \quad (9)$$

where $p^2 = -2a_1/a_2$ and $q^2 = -a_1$. For p and q to be real numbers, $a_1 < 0$ and $a_2 > 0$, with $a_1 = -1$, $a_2 = 2$, $k = 1$, $\epsilon = 1000$ and $v = \sinh(t)$, the corresponding amplitude profile for antikink (corresponding to the negative sign in solution eq. (9)) type solutions is shown in figure 3b.

4. Conclusions

In this paper, we have studied the cubic–quintic NLRD equation with time-dependent coefficients of convection and reaction terms. The investigated solutions are expressed in terms of exact periodic and solitary wave solutions under different parametric conditions. It is to be noted that the solitary wave solutions computed in the present work consist of periodic, double kink, bell and antikink-type solutions. The convection term coefficient can be chosen as any arbitrary function of time which affects the physical parameters i.e., amplitude and velocity of the wave. The travelling wave solutions of nonlinear evolution equation offers new vistas in plasma physics, fluid physics, capillary gravity wave, nonlinear optics, chemical physics, population dynamics and propagation pulse in the double-doped optical fibre.

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