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Interplay between grand unification and supersymmetry in $SU(5)$ and E_6

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Abstract. Some aspects of minimal supersymmetric renormalizable grand unified theories are reviewed here. These include some constraints on the model parameters from the Higgs and light fermion masses in $SU(5)$, and the issues of symmetry breaking, doublet–triplet splitting and fermion masses in E_6.

Keywords. Grand unification; supersymmetry.

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1. Introduction

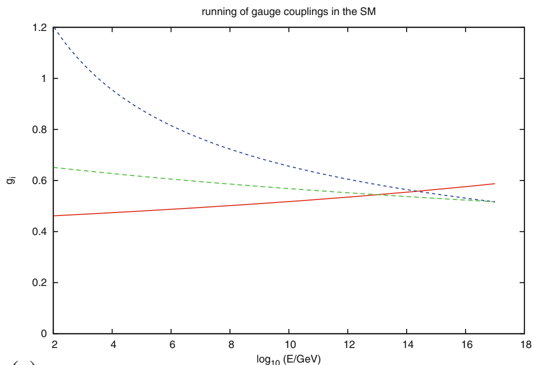
This work is based on the papers [1–5].

I had the pleasure to work with Charan Aulakh (3 papers together) and it was a very fruitful experience (the average citation per paper is, at the day of writing, 112), from which I learned a lot. These were (for me) magic years spent together in ICTP. We have been discussing mainly $SO(10)$, one of Charan's strongholds. He had given a very nice review on the subject. So I will try to cover the other two realistic groups, $SU(5)$ and E_6.

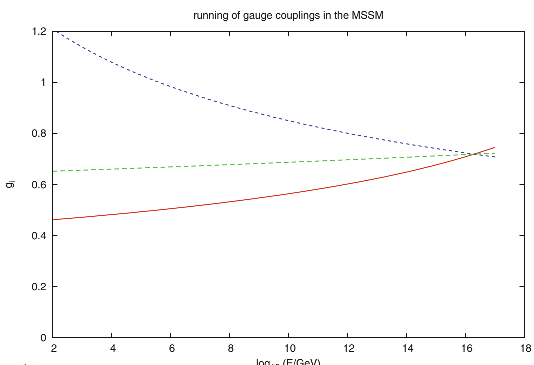
The best known example of interplay between supersymmetry (SUSY) and grand unified theories (GUTs) is the gauge coupling unification. Renormalization group equations make the SM gauge coupling run as shown in figure 1a. The unification is not perfect, although we are pretty close. Nevertheless, the experimental value and the theoretical knowledge is so good, that new states are needed for unification to occur.

If we add the minimal superymmetric Standard Model (MSSM) partners at $\approx$1 TeV and run at 1-loop, we get unification at $M_{\mathrm{GUT}} \approx 10^{16}$ GeV [6–9], as shown in figure 1b. This solution is of course not unique, but enough to motivate supersymmetry.

Usually GUTs do not give new ingredients in the search for dark matter candidates. MSSM has its own candidate, the light neutralino, provided we assume R-parity conservation. But, R-parity is just a subgroup of $SO(10)$. So, taking large representation (126) to



(a)



(b)

Figure 1. The 1-loop RGE running of the three SM gauge couplings in (**a**) SM and (**b**) low-energy MSSM.

break the rank, Aulakh and his collaborators [10–12] have showed that R-parity is exact all the way down to low energies. In this case, grand unification tells us something about supersymmetry and even dark matter.

In this article, the interplay between supersymmetry and grand unification will be studied in the following two cases:

(1) In minimal $SU(5)$, the requirement of unification of couplings, Higgs mass, proton decay bounds, perturbativity and correct fermion masses, put constraints on SUSY parameters like sfermion spectrum.
(2) In E_6, the relation is only tiny, the usual one: the renormalizable superpotential gives a restricted potential and the search of vacua is simplified.

2. Minimal supersymmetric $SU(5)$

The usual reaction here is: hasn't this been ruled out long ago? The argument goes as follows [13]. On one side unification constraint of the gauge couplings at 2-loop order needs light colour triplet $m_T \lesssim 10^{15}$ GeV. On the other side, proton decay constraint needs heavy colour triplet $m_{\mathrm{T}} \gtrsim 10^{17}$ GeV. So we arrive at a contradiction. But, this is true only if

(1) we employ renormalizable couplings;
(2) Gaugini, Higgsino and third generation superpartners' masses are $\mathcal{O}$ (TeV).

Renormalizability is crucial for this conclusion [14]. In fact in general

(1) triplet mass can get large threshold correction from the colour octet (m_8) and weak triplet (m_3) in $SU(5)$ adjoint [15,16]:

$$m_{\mathrm{T}} \approx \left(\frac{m_3}{m_8}\right)^{5/2} 10^{15}\ \mathrm{GeV}. \tag{1}$$

In the renormalizable case $m_3 = m_8$, but in general m_3/m_8 is arbitrary;
(2) non-renormalizable contributions to the superpotential change the relation between Higgs doublet Yukawa and colour triplet Yukawa, which can have a crucial impact on the proton decay estimates [17,18];
(3) these terms can also change the relations between fermion and sfermion mixings without endangering the flavour changing neutral current (FCNC) constraints [19].

Is the second requirement – $\mathcal{O}$ (TeV) spartners – also crucial to rule out the model? This is discussed below based on [1,2]. We shall be considering

(1) renormalizable minimal supersymmetric $SU(5)$ with superfield content

$$3 \times (10_F + \bar{5}_F) + (24_H + 5_H + \bar{5}_H) + 24_V, \tag{2}$$

(2) soft terms $SU(5)$ symmetric at M_{GUT} but otherwise arbitrary; to keep small FCNC effects, we shall assume equality between the first and second generation soft masses:

$$\tilde{m}_1 \approx \tilde{m}_2. \tag{3}$$

We have to take into account several constraints:

(1) Higgs mass,
(2) fermion masses,
(3) perturbativity (couplings $\lesssim 1$),
(4) vacuum metastability (no tachyons, UFB, CCB),
(5) proton decay (decay width $\Gamma_p \propto \sin\beta\cos\beta$, so small $\tan\beta \lesssim 5$ preferred),
(6) unification constraints ($g_1 = g_2 = g_3$, $y_b = y_\tau$).

Let us now go through some of them in greater detail.

2.1 *Higgs mass*

It is defined as

$$m_h^2 = 2\lambda(m_h)v^2. \tag{4}$$

The matching scale between SM and MSSM is $M_{\text{EWSB}} \equiv m_{\tilde{t}}$. The self-coupling is

$$\lambda(\tilde{m}_t) = \underbrace{\lambda_0(\tan\beta)}_{\text{tree level}} + \underbrace{\lambda_1\left(y_t, \frac{X_t}{\tilde{m}_t}\right)}_{>0} + \underbrace{\lambda_1\left(y_b, \frac{X_b}{\tilde{m}_b}\right)}_{<0} + \cdots, \tag{5}$$

where

$$\tilde{m}_t = M_{\text{EWSB}} \equiv \sqrt{\tilde{m}_{t_L}\tilde{m}_{t_R}}, \tag{6}$$

$$X_t = A_t/y_t - \mu/\tan\beta, \tag{7}$$

$$X_b = A_b/y_b - \mu\tan\beta. \tag{8}$$

The Higgs mass is thus a function of $\tan\beta$, stop mass $\tilde{m}_t$ and trilinear couplings $X_{t,b}$:

$$m_h = m_h\left(\tan\beta, \tilde{m}_t, \frac{X_t}{\tilde{m}_t}, \frac{X_b}{\tilde{m}_b}\right). \tag{9}$$

More precisely, eq. (5) is

$$\begin{aligned}\lambda(\tilde{m}_t) = & \underbrace{\frac{m_Z^2}{2v^2}(\tilde{m}_t)\cos^2(2\beta)}_{\text{small for } \tan\beta=\mathcal{O}(1)} \\ & + \underbrace{\frac{6(y_t\sin\beta)^4}{(4\pi)^2}\left(\frac{X_t}{\tilde{m}_t}\right)^2\left[1-\frac{1}{12}\left(\frac{X_t}{\tilde{m}_t}\right)^2\right]}_{\text{maximally positive for } |X_t/\tilde{m}_t|=\sqrt{6}} \\ & + \underbrace{\frac{6(y_b\cos\beta)^4}{(4\pi)^2}\left(\frac{X_b}{\tilde{m}_b}\right)^2\left[1-\frac{1}{12}\left(\frac{X_b}{\tilde{m}_b}\right)^2\right]}_{\text{maximally negative for } |X_b/\tilde{m}_b|\approx 1/y_b} + \cdots.\end{aligned} \tag{10}$$

We can see form figure 2a which values among the input parameters M_{EWSB} and $\tan\beta$ are allowed by the Higgs mass.

2.2 *Fermion masses*

$SU(5)$ constraints them at M_{GUT}: $y_b = y_\tau$, $y_s = y_\mu$, $y_d = y_e$. At low energy, we must correct them to be in accord with data. Assuming that lepton masses are exact, we need

$$\frac{\delta m_d}{m_d} \approx 2, \quad \frac{\delta m_s}{m_s} \approx -3, \quad \frac{\delta m_b}{m_b} \approx -0.3. \tag{11}$$

1-loop finite SUSY threshold corrections (for leptons we would have $\alpha_{1,2}$ instead of α_3, this is why we neglect them) give

$$\frac{\delta m_i}{m_i} = -\frac{\alpha_3}{3\pi}\frac{X_i}{\tilde{m}_i} I\left(\frac{m_{\tilde{g}}}{\tilde{m}_i}\right). \tag{12}$$

MSSM vacuum stability requires [20]

$$\left|\frac{X_i}{\tilde{m}_i}\right| \lesssim 1. \tag{13}$$

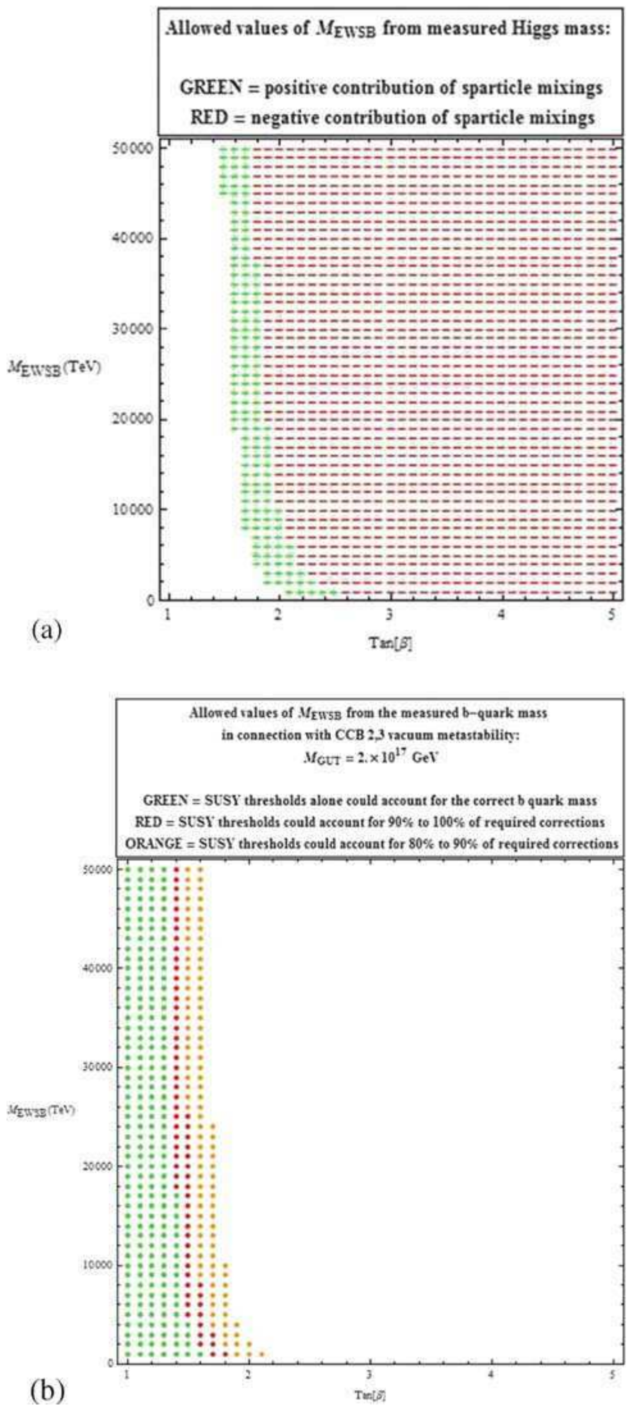


Figure 2. Allowed parameter space for (**a**) Higgs mass and (**b**) bottom quark mass. The white region is excluded.

We need large $|X_i/\tilde{m}_i|$ which leads to a metastable vacuum. The requirement that the Universe is long-lived enough means that (see for example [21])

$$\left|\frac{X_i}{\tilde{m}_i}\right| \lesssim \frac{1}{y_i}. \tag{14}$$

From here we see that it is harder to get corrections for b than for s or d, in spite of the fact that the strange quark and down quark need in percentage larger corrections. So only the bottom quark could be a problem.

The function $I(x)$ in (12) peaks around $x = 2$ ($I_1(2) \approx 1$), and so to maximize corrections we shall take $m_{\tilde{g}} \approx \tilde{m}_{b_1}$, i.e., gluino and heaviest sbottom masses comparable.

Putting together constraints on Higgs mass and fermion masses, we get figure 3. From left to right: the black dots denote the forbidden region due to non-perturbative y_t, the green (red, orange) dots show the parameter space that can account for the correct (90–100%, 80–90%) b quark mass, while the orange crosses satisfy both Higgs mass and 80–90% of the bottom mass.

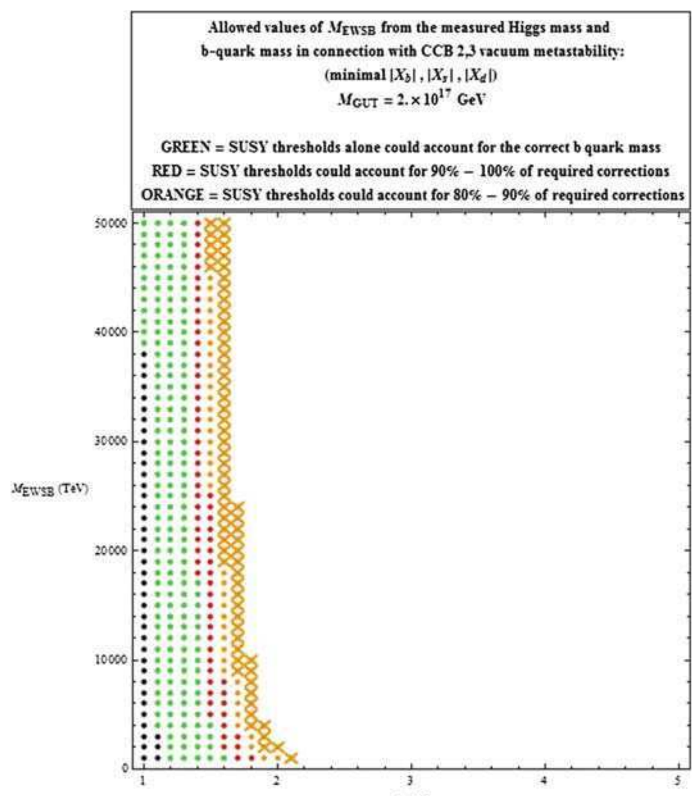


Figure 3. From left to right: the black dots denote the forbidden region due to non-perturbative y_t, the green (red, orange) dots show the parameter space that can account for the correct (90–100%, 80–90%) b quark mass, while the orange crosses satisfy both Higgs mass and 80–90% of the bottom mass.

We can see that very little region survives so that essentially there is a correlation between the MSSM parameters $m_{\tilde{t}}$ and $\tan\beta$.

Anyway, large X_is with different signs are needed to get (12) mean large A_is. In $SO(10)$, this automatically means also large A_t. Aulakh and Garg [22] (based on even earlier papers) used a large value for the soft trilinear $A_0(M_X)$ in a gravity-mediated scenario to fit charged fermion masses from 10 + 120 VEVs combined with large $\tan\beta$ driven threshold corrections at $M_{\rm SUSY}$ to down and strange quark yukawa couplings. This also made it easy for them to obtain large radiative corrections to the Higgs mass: like the one measured four years later by ATLAS and CMS.

2.3 *Summary of SU(5) results*

(1) fermion masses make the MSSM vacuum metastable,
(2) from correction to b, mass $\tilde{m}_b \approx m_{\tilde{g}}$ follows,
(3) $SU(5)$ implies $m_{\tilde{g}} \approx m_{\tilde{w}}$,
(4) Higgs mass and correction to b mass lead to $\tilde{m}_t = \tilde{m}_t(\tan\beta)$,
(5) corrections to s and d quarks much easier ($X/\tilde{m}$ allowed to be much larger).

3. Minimal supersymmetric E_6

Until recently only a few explicit examples of renormalizable realistic Higgs sectors have been considered. What is known is that renormalizable supersymmetric E_6 with 78, 27, $\overline{27}$ could be spontaneously broken only to $SO(10)$ [23].

Here 1-step unification, i.e., $m_{\rm SUSY} \approx 1$ TeV will be assumed.

3.1 *Generic Yukawa sector in E_6*

In all generality, there are three types of Yukawas:

$$W = 27_i\left(Y_{27}^{ij}\, 27_H \;+ Y_{\overline{351}'}^{ij}\, \overline{351}'_H + Y_{\overline{351}}^{ij}\, \overline{351}_H\right)27_j, \tag{15}$$

where

$$Y_{27,\overline{351}'} = Y^T_{27,\overline{351}'} \text{ symmetric}, \quad Y_{\overline{351}} = -Y^T_{\overline{351}} \text{ antisymmetric.}$$

This is completely analogous to $SO(10)$ where

$$W = 16_i\left(Y_{10}^{ij}\, 10_H + Y_{\overline{126}}^{ij}\, \overline{126}_H + Y_{120}^{ij}\, 120_H\right)16_j \tag{16}$$

with

$$Y_{10,\overline{126}} = Y^T_{10,\overline{126}} \text{ symmetric}, \quad Y_{120} = -Y^T_{120} \text{ antisymmetric.}$$

The antisymmetric 351, similar to 120 in $SO(10)$, is less promising. So it will be removed in the following. What remains can be decomposed in the $SO(10)$ language as

$$W = \begin{pmatrix} 16 & 10 & 1 \end{pmatrix} Y_{27} \begin{pmatrix} 10 & 16 & 0 \\ 16 & 1 & 10 \\ 0 & 10 & 0 \end{pmatrix}_H \begin{pmatrix} 16 \\ 10 \\ 1 \end{pmatrix}$$

$$+ \begin{pmatrix} 16 & 10 & 1 \end{pmatrix} Y_{\overline{351}'} \begin{pmatrix} \overline{126}+10 & 144 & \overline{16} \\ 144 & 54 & 10 \\ \overline{16} & 10 & 1 \end{pmatrix}_H \begin{pmatrix} 16 \\ 10 \\ 1 \end{pmatrix}. \tag{17}$$

There are some differences with respect to $SO(10)$:

(1) several new Higgs doublets (not only in 10_H and $\overline{126}_H$);
(2) some fields have large $\mathcal{O}(M_{\rm GUT})$ VEVs, which means

- mixing between $\bar{5} \in 16$ and $\bar{5} \in 10$ (d^c, L),
- mixing between $1 \in 1$ and $1 \in 16$ (ν^c);

(3) mass matrices are typically bigger and to get the light fermion masses one needs to integrate out the heavy vector-like states: from the original $M^U_{3\times 3}$, $M^D_{6\times 6}$, $M^E_{6\times 6}$, $M^N_{15\times 15}$, we end up with light $(M_{U,D,E,N})_{3\times 3}$.

We can ask ourselves now:

(1) What are the large VEVs that produce family mixings with vector-like extra matter?
(2) Where are the MSSM Higgs doublets?

To answer this we need the full model.

3.2 *Higgs sector with* $351' + \overline{351}' + 27 + \overline{27}$

The minimal Higgs sector with $E_6 \to$ SM is composed of $351'_H + \overline{351}'_H + 27_H + \overline{27}_H$ [3,4], with the superpotential

$$\begin{aligned} W = {} & m_{351'}\overline{351}'_H 351'_H + \lambda_1 351'^3_H + \lambda_2 \overline{351}'^3_H \\ & + m_{27}\overline{27}_H 27_H + \lambda_3 27_H 27_H \overline{351}'_H + \lambda_4 \overline{27}_H \overline{27}_H 351'_H \\ & + \lambda_5 27^3_H + \lambda_6 \overline{27}^3_H . \end{aligned} \tag{18}$$

There are 14 SM singlets denoted as

$$27_H : c_{1,2}, \quad \overline{27}_H : d_{1,2}, \quad 351'_H : e_{1,2,3,4,5}, \quad \overline{351}'_H : f_{1,2,3,4,5}. \tag{19}$$

There is more than one solution. As an example [3,4]

$$c_2 = e_2 = e_4 = 0, \qquad d_2 = f_2 = f_4 = 0 \tag{20}$$

$$d_1 = \frac{m_{351'} m_{27}}{2\lambda_3\lambda_4 c_1} \tag{21}$$

$$e_1 = -\frac{m_{351'}}{6\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_1 = -\frac{m_{351'}}{6\lambda_1^{1/3}\lambda_2^{2/3}} \tag{22}$$

$$e_3 = -\lambda_3 c_1^2/m_{351'}, \qquad f_3 = -\frac{m_{351'} m_{27}^2}{4\lambda_3^2\lambda_4 c_1^2} \tag{23}$$

$$e_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{1/3}\lambda_2^{2/3}} \tag{24}$$

with

$$\begin{aligned} 0 = {} & |m_{351'}|^4 |m_{27}|^4 + 2|m_{351'}|^4 |m_{27}|^2 |\lambda_3|^2 |c_1|^2 \\ & - 8|m_{351'}|^2 |\lambda_3|^4 |\lambda_4|^2 |c_1|^6 - 16|\lambda_3|^6 |\lambda_4|^2 |c_1|^8 . \end{aligned} \tag{25}$$

This case seems really minimal: 27_H and $\overline{351}'_H$ that participate in symmetry breaking could in principle contribute to Yukawa terms. This is however not automatic even though correct quantum numbers are available. Hence we have to answer the following question: Can linear combinations of the weak doublets with $Y = \pm 1$ in 27_H and $\overline{351}'_H$ be the Higgses H, $\bar{H}$ of the MSSM? As E_6 is a GUT, the question is: Can we make the doublet–triplet splitting with the massless eigenvector living in both 27_H and $\overline{351}'_H$?

3.3 *The doublet–triplet splitting*

This issue is present in all GUTs. The prototype example is $SU(5)$, where the MSSM Higgses H and $\bar{H}$ live in the same multiplet as the colour triplets T and $\bar{T}$:

$$5_H = \begin{pmatrix} T \\ H \end{pmatrix}, \qquad \bar{5}_H = \begin{pmatrix} \bar{T} \\ \bar{H} \end{pmatrix}. \tag{26}$$

When we decompose the renormalizable $SU(5)$ invariant Yukawa sector into the SM fields, we get

$$\begin{aligned} W_{\text{Yukawa}} &= Y_{\bar{5}}^{ij} \bar{5}_i 10_j \bar{5}_H + Y_{10}^{ij} 10_i 10_j 5_H \\ &\to Y_{\bar{5}}^{ij} \left(d_i^c Q_j + L_i e_j^c \right) \bar{H} + Y_{10}^{ij} u_i^c Q_j H \\ &+ Y_{\bar{5}}^{ij} \left(L_i Q_j + d_i^c u_j^c \right) \bar{T} + Y_{10}^{ij} \left(Q_i Q_j + u_i^c e_j^c \right) T. \end{aligned} \tag{27}$$

The doublet–triplet splitting problem appears because on one side $H, \bar{H}$ Higgses of MSSM are light, $M_H \approx m_Z$, but on the other side eq. (27) makes the triplets $T, \bar{T}$ mediate proton decay with $\tau \propto M_T^2$. So in order to be long-lived, we need $M_T \approx M_{\text{GUT}} \gg m_Z$.

How can we get such a large splitting from components of the same multiplet? The renormalizable superpotential is

$$W = \mu \bar{5}_H 5_H + \eta \bar{5}_H 24_H 5_H \tag{28}$$

and as the adjoint breaks $SU(5)$ spontaneously

$$\langle 24_H \rangle = M_{\text{GUT}} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}, \tag{29}$$

the masses get split:

$$W = \bar{H}(\mu - 3\eta M_{\text{GUT}}) H + \bar{T}(\mu + 2\eta M_{\text{GUT}}) T. \tag{30}$$

What we phenomenologically need is

$$M_H = \mu - 3\eta M_{\text{GUT}} \approx 0, \tag{31}$$

$$M_T = \mu + 2\eta M_{\text{GUT}} \approx M_{\text{GUT}}, \tag{32}$$

i.e.,

$$\mu = 3\eta M_{\text{GUT}} \approx M_{\text{GUT}}. \tag{33}$$

The conclusion is that fine-tuning is unavoidable in minimal models.

In our E_6 case doublets and triplets live in $351'_H$, $\overline{351}'_H$, 27_H, $\overline{27}_H$. More precisely, $351'_H$ has 8 doublets (9 triplets), $\overline{351}'_H$ has 8 doublets (9 triplets), 27_H has 3 doublets (3 triplets) and $\overline{27}_H$ has 3 doublets (3 triplets).

Altogether there are 22 doublets (11 with $Y = +1$ and 11 with $Y = -1$), i.e., the doublet matrix M_D is 11×11 and 24 triplets (12 with $Y = +2/3$ and 12 with $Y = -2/3$). i.e., the triplet matrix M_T is 12×12.

The breaking of E_6 into SM gives rise to $78 - 12 = 66$ would-be-Goldstones. Among them, there are also $16 + \overline{16} \in 78$ and so the mass matrices $M_{T,D}$ have automatically one zero eignevalue. We need thus the determinant without these zero modes:

$$\mathrm{Det}(M) \equiv \prod_{i=2}^{n} m_i. \tag{34}$$

Doublet–triplet splitting means

$$\mathrm{Det}(M_D) = 0, \quad \mathrm{Det}(M_T) \neq 0. \tag{35}$$

But after a long calculation the result is [3,4]:

$$\mathrm{Det}(M_T) = \#\mathrm{Det}(M_D), \tag{36}$$

i.e., doublet–triplet splitting is impossible!

This is a bizarre situation: although the symmetry breaking was successful, we failed on the doublet–triplet splitting. And not because we do not like fine-tuning, we cannot even fine-tune!

We can think of two (simplest) solutions:

(1) add another $27 + \overline{27}$ pair with couplings

$$\begin{aligned} W_{DT} &= m_{27}\, 27\, \overline{27} + \kappa_1\, 27\, 27\, \overline{351}'_H + \kappa_2\, \overline{27}\, \overline{27}\, 351'_H \\ &\quad + \kappa_3\, 27\, 27\, 27_H + \kappa_4\, \overline{27}\, \overline{27}\, \overline{27}_H \end{aligned} \tag{37}$$

with $\langle 27\rangle$, $\langle\overline{27}\rangle = \mathcal{O}(m_Z)$.

DT splitting is now possible: MSSM Higgs live only in 27, $\overline{27}$. The bad point is that three Yukawa matrices are involved, which makes the model too easily realistic and so not predictable [3,4].

(2) add another 78_H: although it does not contribute to Yukawas, it changes the symmetry breaking pattern (not being needed) thus relaxing constraints on DT.

DT is now possible in the old sector: MSSM Higgses live also in $\overline{351}'_H$ and 27_H! This possibility is more minimal, only two Yukawas appear [5]. Let us now study this case in more detail.

3.4 *Higgs sector with* $351' + \overline{351}' + 27 + \overline{27} + 78$

$$\begin{aligned} W &= m_{351'}\overline{351}'_H 351'_H + \lambda_1 351'^3_H + \lambda_2 \overline{351}'^3_H \\ &\quad + m_{27}\overline{27}_H 27_H + \lambda_3 27^2_H \overline{351}'_H + \lambda_4 \overline{27}^2_H 351'_H \\ &\quad + \lambda_5 27^3_H + \lambda_6 \overline{27}^3_H \\ &\quad + m_{78} 78^2_H + \lambda_7 27_H 78_H \overline{27}_H + \lambda_8 351'_H 78_H \overline{351}'_H. \end{aligned} \tag{38}$$

In addition to (19), we now have other SM singlets:

$$78_H: \quad a_1, a_2, a_3, a_4, a_5. \tag{39}$$

Solution with $a_i \neq 0$ are explicitly shown to be possible. They are disconnected with the previous one (20)–(24), i.e. no limit gives the previous solution with $a_i \to 0$.

3.5 *Yukawa sector in the minimal E_6 model*

As an example of what happens let us see the down sector:

$$\left(d^{cT}\ d'^{cT}\right)\begin{pmatrix} \bar{v}_2 Y_{27} + \left(\frac{1}{2\sqrt{10}}\bar{v}_4 + \frac{1}{2\sqrt{6}}\bar{v}_8\right) Y_{\overline{351}'} & c_2 Y_{27} \\ -\bar{v}_3 Y_{27} - \left(\frac{1}{2\sqrt{10}}\bar{v}_9 + \frac{1}{2\sqrt{6}}\bar{v}_{11}\right) Y_{\overline{351}'} & \frac{1}{\sqrt{15}} f_4 \widetilde{Y}_{\overline{351}'} \end{pmatrix}\begin{pmatrix} d \\ d' \end{pmatrix}, \tag{40}$$

where $\bar{v}_{2,3,4,8,9,11} = \mathcal{O}(m_Z)$, while $c_2,\ f_4 = \mathcal{O}(M_{\text{GUT}})$.

The different states are:

$$\left.\begin{pmatrix} d^c \in \bar{5}_{SU(5)} \in 16_{SO(10)} \\ d'^c \in \bar{5}_{SU(5)} \in 10_{SO(10)} \end{pmatrix}\right\}\text{mix}, \tag{41}$$

$$d \in 10_{SU(5)} \in 16_{SO(10)}, \tag{42}$$

$$d' \in 5_{SU(5)} \in 10_{SO(10)} \ldots \text{ heavy}. \tag{43}$$

The 6×6 matrix above has the form

$$\mathcal{M} = \begin{pmatrix} m_1 & M_1 \\ m_2 & M_2 \end{pmatrix} \tag{44}$$

with the 3×3 matrices $m_{1,2} = \mathcal{O}(m_Z)$ and $M_{1,2} = \mathcal{O}(M_{\text{GUT}})$. The idea is to find a 6×6 unitary matrix $\mathcal{U}$ that projects the heavy states into the lower block:

$$\mathcal{U}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \text{something} \end{pmatrix}. \tag{45}$$

The solution is

$$\mathcal{U} = \begin{pmatrix} \left(1 + XX^\dagger\right)^{-1/2} & -\left(1 + XX^\dagger\right)^{-1/2} X \\ X^\dagger\left(1 + XX^\dagger\right)^{-1/2} & \left(1 + X^\dagger X\right)^{-1/2} \end{pmatrix} \tag{46}$$

with

$$X = M_1 M_2^{-1} \tag{47}$$

so that

$$\mathcal{U}\mathcal{M} = \begin{pmatrix} \underbrace{\mathcal{O}(m_Z)}_{\text{light sector}} & 0 \\ \mathcal{O}(m_Z) & \mathcal{O}(M_{\text{GUT}}) \end{pmatrix}. \tag{48}$$

The mass matrices for the light-charged fermions turn out to be

$$M_U = -v_1 Y_{27} + \left(\frac{1}{2\sqrt{10}} v_5 - \frac{1}{2\sqrt{6}} v_7\right) Y_{\overline{351}'}, \tag{49}$$

$$\begin{aligned} M_D^T = &\left(1 + XX^\dagger\right)^{-1/2} (\bar{v}_2 - \bar{v}_3 X) Y_{27} \\ &+ \left(\frac{1}{2\sqrt{10}} (\bar{v}_4 - \bar{v}_9 X) + \frac{1}{2\sqrt{6}} (\bar{v}_8 - \bar{v}_{11} X)\right) Y_{\overline{351}'}\Big), \end{aligned} \tag{50}$$

$$\begin{aligned} M_E = &\left(1 + \frac{4}{9} XX^\dagger\right)^{-1/2} \left(\left(-\bar{v}_2 - \frac{2}{3}\bar{v}_3 X\right) Y_{27}\right. \\ &+ \left.\left(-\frac{1}{2\sqrt{10}} (\bar{v}_4 + \frac{2}{3}\bar{v}_9 X) + \sqrt{\frac{3}{8}} \left(\bar{v}_8 + \frac{2}{3}\bar{v}_{11} X\right)\right) Y_{\overline{351}'}\right) \end{aligned} \tag{51}$$

with

$$X = -3\sqrt{\frac{5}{3}} \frac{c_2}{f_4} Y_{27} Y^{-1}_{\overline{351}'}, \tag{52}$$

$X \to 0$ gives minimal $SO(10)$, but this limit is not available here ($c_2 \neq 0$).

As Y_{27} and $Y_{\overline{351}'}$ are symmetric, so is M_U. This is however not true for X and so not for $M_{D,E}$. Let us now choose a basis with $M_U = M_U^d$ (diagonal). Then we can always parametrize

$$X = M_U^d Y \tag{53}$$

with

$$Y = Y^T \text{ symmetric.} \tag{54}$$

Equations (49)–(51) plus the light neutrino mass can be rewritten as

$$\begin{aligned} M_D^T = &\left(1 + M_U^d Y Y^* M_U^d\right)^{-1/2} \\ &\times \left(a + b\,(M_U^d Y) + c\,(M_U^d Y)^2\right)\left(d + (M_U^d Y)\right)^{-1} M_U^d, \end{aligned} \tag{55}$$

$$\begin{aligned} M_E = &\left(1 + (4/9) M_U^d Y Y^* M_U^d\right)^{-1/2} \\ &\times \left(a' + b'\,(M_U^d Y) + c'(M_U^d Y)^2\right)\left(d + (M_U^d Y)\right)^{-1} M_U^d, \end{aligned} \tag{56}$$

$$\begin{aligned} M_N = &\left(1 + (4/9)\, M_U^d Y Y^* M_U^d\right)^{-1/2} \left(a'' + b''(M_U^d Y)\right. \\ &+ \left. c''(M_U^d Y)^2 + d''(M_U^d Y)^3 + e''(M_U^d Y)^4\right) \\ &\times \left(d + (M_U^d Y)\right)^{-1} M_U^d \left(1 + (4/9)\, M_U^d Y^* Y M_U^d\right)^{-1/2}. \end{aligned} \tag{57}$$

A few comments:

(1) The neutrino mass is a sum of type-I and type-II contributions;
(2) a, b, c, d, a', b', c', a'', b'', c'', d'', e'' are functions of the superpotential parameters m_i, λ_j and VEVs c_a, f_b, v_i, $\bar{v}_j$ which are also functions of the superpotentials parameters;

(3) the relations are highly nonlinear, the analysis seems hopeless (unless numerical).

But things become slightly easier if we remember that (assuming $N_g = 2$)

(1) any (reasonable) function of a 2×2 matrix M can be expanded as

$$f(M) = \alpha + \beta M \tag{58}$$

with α, β written with invariants of M;

(2) any 2×2 matrix A can be written as (with the chosen basis)

$$A = \alpha_1 + \alpha_2 M_U^d + \alpha_3 Y + \alpha_4 M_U^d Y. \tag{59}$$

This simplifies the work and decreases the number of unknowns (combinations):

$$M_D^T = \left(1 + M_U^d Y Y^* M_U^d\right)^{-1/2}\left(\alpha + \beta M_U^d Y\right) M_U^d, \tag{60}$$

$$M_E = \left(1 + (4/9) M_U^d Y Y^* M_U^d\right)^{-1/2}\left(\alpha' + \beta' M_U^d Y\right) M_U^d, \tag{61}$$

$$M_N = \left(1 + (4/9) M_U^d Y Y^* M_U^d\right)^{-1/2}\left(\alpha'' + \beta'' M_U^d Y\right) M_U^d \times \left(1 + (4/9)\, M_U^d Y^* Y M_U^d\right)^{-1/2}. \tag{62}$$

3.6 $N_g = 2$ *case*

The number of unknowns is 9: α, β, α', β', α'', β'', $Y_1 \equiv \mathrm{Tr}(Y)$, $Y_2 \equiv \det(Y)$ and $Z \equiv \mathrm{Tr}(M_U^d Y)$.

We have to fit seven quantities: $m_s, m_b, m_\mu, m_\tau, V_{cb}, \Delta m_{23}^2, \sin^2\theta_{23}$.

The fit is naively possible, and it has been shown to work explicitly in [5].

3.7 $N_g = 3$ *case*

Equation (58) now generalizes to

$$f(M) = \alpha + \beta M + \gamma M^2. \tag{63}$$

There are more unknowns, 15: $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma'', Y_{1,2,3}$ and $Z_{1,2,3}$.

The quantities to fit are 14: $m_d, m_s, m_b, m_e, m_\mu, m_\tau, \theta^q_{1,2,3}, \theta^l_{1,2,3}, \Delta m^2_{23}$ and Δm^2_{12}.

It looks still possible, but harder than before. It has not yet been checked.

3.8 *Summary of E_6*

(1) E_6 is a respectable (although complicated) theory;
(2) we showed examples of (so far) possibly realistic cases ($N_g = 2$).

Some open questions:

(1) Neutrino mass scale should be lower than M_{GUT}. To get it, the full mass spectrum at that scale should be known and included in gauge couplings RGEs;
(2) the Landau pole is very close, just above M_{GUT}. Any possibility to treat it?

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