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# Thermal evolution of the Kramer radiating star

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**Abstract.** The Kramer radiating star uses the interior Schwarzschild solution as a seed solution to generate a model of dissipative collapse. We investigate the thermal behaviour of the radiating star by employing a causal heat transport equation. The causal temperature is explicitly determined for the first time by integrating the transport equation. We further show that the dissipation of energy to the exterior space-time renders the core more unstable than the cooler surface layers.

**Keywords.** Schwarzschild solution; gravitational collapse; heat flow.

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## 1. Introduction

The study of relativistic radiating stars has revealed a rich spectrum of physics for different epochs of the gravitational process. Of particular interest is the end state of gravitational collapse where it is widely debated whether the final outcome is that of a black hole or naked singularity [1]. The first serious attempt in investigating the outcome of gravitational collapse can be attributed to Oppenheimer and Snyder [2] who considered a spherically symmetric dust cloud collapsing under its own gravity. This is a highly idealized model and there is a need to incorporate more realistic effects such as pressure anisotropy, shear, heat flow and bulk viscosity. The study of dissipative systems became possible with the discovery of the Vaidya solution [3] which allowed for energy loss from the stellar interior into the surrounding space-time. For a collapse scenario, space-time is divided into two distinct regions, the interior of the star and the exterior space-time. The smooth matching of the interior geometry to the exterior space-time across a time-like hypersurface was first obtained by Santos [4]. The important result obtained from these junction conditions is the nonvanishing of the pressure at the boundary of the collapsing star. With Santos junction conditions, it became possible to model more realistic collapse scenarios incorporating bulk viscosity [5], shear viscosity [6], pressure anisotropy [7–9] and electromagnetic field [10–12] within the stellar core. The departure from spherical symmetry of a radiating body was also investigated [13]. These general models allowed for a transparent appreciation of the physics at play during the collapse process. The so-called Euclidean star model first proposed by Herrera and Santos [14] analysed the behaviour of a radiating sphere in which the areal radius is always equal to the proper radius. Euclidean stars have close relationship with Lie symmetries and algebras as shown by Abebe *et al* [15,16]. Govinder and Govender [17] showed that these stars are reasonably well-behaved and obey a barotropic equation of state. By assuming the vanishing of the expansion scalar, Di Prisco *et al* [18] showed that this permits the existence of a vacuum cavity surrounding the centre of the stellar configuration.

We are guided by an approach adopted first by Kramer [19] in which the static interior constant density Schwarzschild solution is transformed into a nonstatic solution by allowing the mass function to become time-dependent. The energy–momentum tensor for this Schwarzschild-like radiating solution is a perfect fluid with heat flow. This allows for a spherically symmetric, shear-free star undergoing gravitational collapse because of the radial heat dissipation with the interior Schwarzschild solution being the static limit. A first integral of the boundary condition was provided by Kramer [19]. The full temporal evolution of the Kramer radiating star was obtained by Maharaj and Govender [20]. The time dependence of this particular model is given in terms of Li integrals which are special functions. The complicated nature of the temporal evolution of metric functions did not warrant a comprehensive study of the physical properties of the Kramer radiating star. The physical behaviour of density, pressure and heat flux were analysed only in the asymptotic limit. In this paper, we investigate the full dynamical nature of this particular collapsing model by treating all the thermodynamical variables as functions of a single temporal function. For the first time we are able to show that the Kramer ansatz leads to a physically viable model of dissipative gravitational collapse. We further study the stability of the collapsing core and evolution of the temperature profile by employing a causal heat transport equation [21–25]. Our results agree with the earlier findings that relaxational effects can predict temperatures which are vastly different from their noncausal counterparts.

This paper is organized as follows. In §2 we introduce the space-time geometry and matter content for the interior and exterior of our stellar model. The Kramer ansatz is reviewed in §3. In §4, we investigate the evolution of the temperature profile of the Kramer radiating star within the framework of extended irreversible thermodynamics.

## 2. Shear-free space-times

We model a spherically symmetric radiating star in the limit of vanishing shear. The interior metric in co-moving coordinates for shear-free matter is

$$\mathrm{d}s^2 = -A^2\mathrm{d}t^2 + B^2[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)], \tag{1}$$

where $A$ and $B$ are functions of both the temporal coordinate $t$ and the radial coordinate $r$. The energy–momentum tensor for the interior matter distribution is described by a perfect fluid with heat flux

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \tag{2}$$

where $\rho$ is the energy density, $p$ is the isotropic pressure and $q^a$ is the heat flow vector. For the line element (1) and matter distribution (2) the coupled Einstein field equation

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \tag{3}$$

becomes

$$\rho = 3\frac{1}{A^2}\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{4}$$

$$\begin{aligned} p = {} & \frac{1}{A^2}\left(-2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right) \\ & + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + 2\frac{A'}{A}\frac{B'}{B} + \frac{2}{r}\frac{A'}{A} + \frac{2}{r}\frac{B'}{B}\right), \end{aligned} \tag{5}$$

$$\begin{aligned} p = {} & -2\frac{1}{A^2}\frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A^3}\frac{\dot{B}}{B} - \frac{1}{A^2}\frac{\dot{B}^2}{B^2} + \frac{1}{r}\frac{A'}{A}\frac{1}{B^2} \\ & + \frac{1}{r}\frac{B'}{B^3} + \frac{A''}{A}\frac{1}{B^2} - \frac{B'^2}{B^4} + \frac{B''}{B^3}, \end{aligned} \tag{6}$$

$$q = -\frac{2}{AB^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2} + \frac{A'}{A}\frac{\dot{B}}{B}\right), \tag{7}$$

where the heat flux $q^a = (0, q, 0, 0)$ consists of only the nonvanishing radial component. The condition of pressure isotropy is obtained by equating (5) and (6) yielding

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{A_r}{A} + \frac{B_r}{B}\right). \tag{8}$$

We note that (8) does not contain any time derivatives. This implies that any static solution in isotropic, co-moving coordinates will satisfy (8). The exterior space-time is taken to be the Vaidya's outgoing solution given by [3]

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(v)}{R}\right)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}R + R^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{9}$$

where $m(v)$ is the Newtonian mass of the star as measured by an observer at infinity. The junction conditions required for the smooth matching of the interior space-time (1) to the exterior space-time (9) have been extensively utilized as the atmosphere of a collapsing star. We present the main results that are necessary for modelling a radiating star. The continuity of the intrinsic and extrinsic curvature components of the interior and exterior space-times across a time-like hypersurface are

$$m(v) = \left(\frac{r^3 B}{2A^2} B_t^2 - r^2 B_r - \frac{r^3}{2B} B_r^2\right)_\Sigma, \tag{10}$$

$$(p)_\Sigma = (qB)_\Sigma. \tag{11}$$

Relation (11) determines the temporal evolution of the collapsing star.

## 3. Kramer ansatz

Kramer started off by writing the static interior Schwarzschild solution in isotropic and co-moving coordinates. The metric takes the following form:

$$\begin{aligned} \mathrm{d}s^2 =& -\frac{(1+2yr^2-2y-r^2y^2)^2}{(1+y)^2(1+yr^2)^2}\mathrm{d}t^2 \\ &+\frac{(1+y)^6}{(1+yr^2)^2}\left[\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\right], \end{aligned} \tag{12}$$

where $y = M/2r_0$ is a constant. Kramer then allowed $y = y(t)$, i.e., he allowed the mass function to become time-dependent. Equation (8) is satisfied and the resulting nonstatic metric is also a solution of the Einstein field eq. (3). The matter variables for the interior of the Kramer star are

$$\rho = \frac{12y}{(1+y)^6} + 3\frac{(1-y)^2}{(1+y)^6}\left(\ln\frac{1-y}{1-y_0}\right)^2\left(\frac{2yr^2-r^2+3}{1+2yr^2-2y-y^2r^2}\right)^2, \tag{13}$$

$$q = \frac{4r(1+yr^2)^2}{(1+y)^4(1+2yr^2-2y-y^2r^2)^2}\frac{1-y}{(1+y)^3}\ln\frac{1-y}{1-y_0}, \tag{14}$$

$$\begin{aligned} p =& \frac{12(1-r^2)y^2}{(1+y)^6(1+2yr^2-2y-y^2r^2)} + \frac{2(1+y)(1+yr^2)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^2} \\ &\times\frac{1-y}{(1+y)^6}\ln\frac{1-y}{1-y_0}\left(1+2\frac{(2-y)}{1+y}\ln\frac{1-y}{1-y_0}\right) \\ &-\left[\left(\frac{4[3(1+yr^2)^2-r^2(1+y)(2yr^2-r^2+3)]+(2yr^2-r^2+3)^2}{(1+2yr^2-2y-y^2r^2)^2}\right)\right. \\ &\left.+\left(\frac{2(3y^2r^4-y^2r^2+4yr^2-r^2+3)(2yr^2-r^2+3)}{(1+2yr^2-2y-y^2r^2)^3}\right)\right] \\ &\times\left[\frac{(1-y)^2}{(1+y)^6}\left(\ln\frac{1-y}{1-y_0}\right)^2\right]. \end{aligned} \tag{15}$$

The physical analysis that follows is the first complete treatment for the Kramer model. We require $y > 0$ for the energy density to be positive in the interior of the star. It must be observed that $(\mathrm{d}y/\mathrm{d}t) < 0$ which ensures that $q > 0$ and that the heat flow is directed outwards. The constant of integration $y_0$ gives the initial value of the function $y(t)$ for $t \to -\infty$ when the solution approaches the static Schwarzschild limit. The mass parameter is given by

$$\frac{m(v)}{2r_0} = y + \left(\ln \frac{1-y}{1-y_0}\right)^2,$$

which coincides with mass $m(v)$ in the Vaidya solution (9). When $y = y_0$ this expression becomes

$$\frac{m}{2r_0} = y_0,$$

which coincides with the interior Schwarzschild mass $M$. The requirement that $(\mathrm{d}y/\mathrm{d}t) < 0$ and singularity in the metric at $y = -1$ places the following restriction on $y(t)$:

$$y_0 \geq y > -1,$$

for a consistent model. In order to generate the plots of the thermodynamical quantities displayed in figures 1–4 we assumed that $0 \leq r \leq 1$, where $r = 1$ defines the boundary of the star at a particular snapshot of the collapse process. Kramer also fixed the boundary at $r = 1$ which, without any loss of generality, simplifies the boundary condition obtained from (11). The temporal function $y(t)$ is chosen such that we avoid the singularity at $y = -1$ and furthermore ensures that the physical quantities such as density, pressure and temperature are well-behaved during this epoch. Figure 1 shows

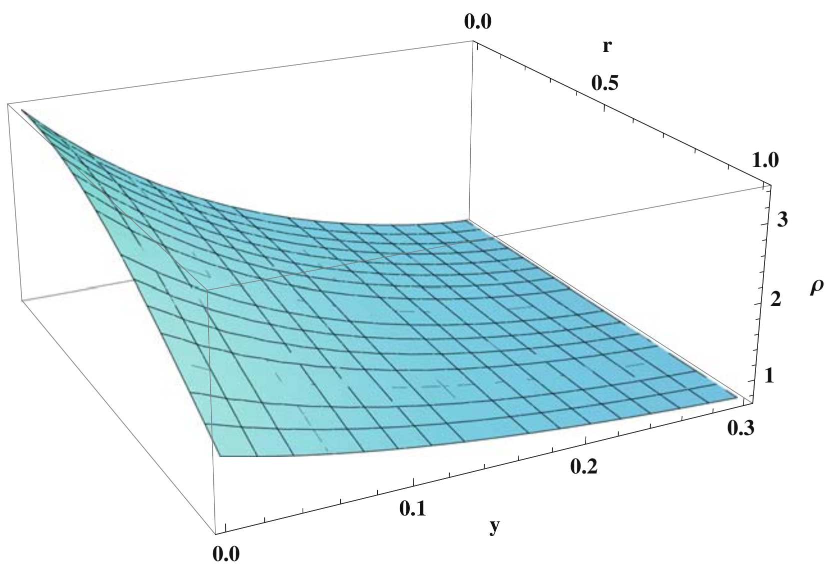


**Figure 1.** Density profile as a function of the radial coordinate and the temporal function $y(t)$.

the behaviour of density of the collapsing fluid sphere as a function of the radial coordinate $r$ and the temporal function $y(t)$. It is clear that the central density is higher than each interior point of the stellar fluid and drops off towards the boundary. It can also be noted that the density increases at late times. This is expected for a collapse model. As the collapse proceeds, the proper radius of the star decreases thus squeezing more matter into a smaller volume. Note that at $y = -1$ the density is undefined. The pressure for the interior is displayed in figure 2. The central core pressure is higher than all interior points of the collapsing star. This is expected as the central density is highest at the centre of the star. The generation of heat flux is associated with the high central core pressure. The heat production at the centre is greatest and decreases towards the cooler surface layers of the star. Figure 3 shows the plot of $\Gamma_{\text{centre}} - \Gamma_{\text{surface}}$ as a function of $y(t)$. It is clear that $\Gamma_{\text{centre}} < \Gamma_{\text{surface}}$ for all times. This indicates that the central regions of the collapsing object are more unstable than the outer regions, close to the surface of the star. This result confirms the earlier findings by Maharaj and Govender [26] for a conformally flat radiating sphere.

## 4. Causal heat transport

To investigate the physical plausibility of the Kramer ansatz we shall now consider the evolution of the temperature profile within the framework of extended irreversible thermodynamics. Early investigations of dissipative collapse in the form of radial heat flux employed a noncausal heat transport equation to determine the behaviour of temperature within the stellar core. The Fourier heat transport equation suffers several drawbacks. For example, it violates causality by predicting superluminous propagation velocities for the thermodynamical fluxes and the equilibrium states are unstable. The general framework for employing extended irreversible thermodynamics to study heat dissipation during gravitational collapse was first considered by Herrera and co-workers [27–32]. Various studies have shown that relaxational effects are significant during the later stages

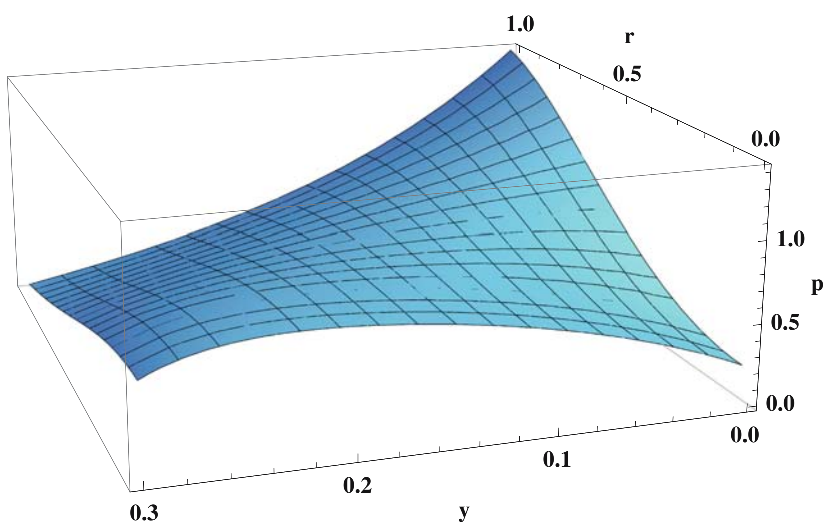


**Figure 2.** Pressure as a function of the radial coordinate and the temporal function $y(t)$.

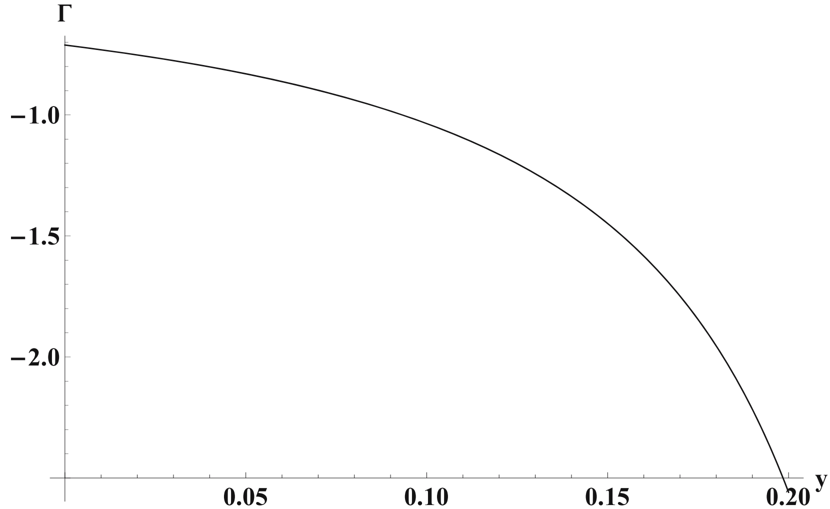


**Figure 3.** Adiabatic index ($\Gamma = (\mathrm{d}p/\mathrm{d}\rho)$) as a function of $y(t)$.

of collapse leading to higher core temperatures. These findings hold true in both the shear-free and shearing cases [33–35]. The relativistic Cattaneo equation for the heat transport takes the following form:

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa(D_a T + T\dot{u}_a), \tag{16}$$

where $\tau(\rho, n)$ is the relaxation time for the heat flux. For the line element (1) the causal heat transport eq. (16) becomes

$$\tau(qB)_{,t} + A(qB) = -\kappa \frac{(AT)_{,r}}{B}, \tag{17}$$

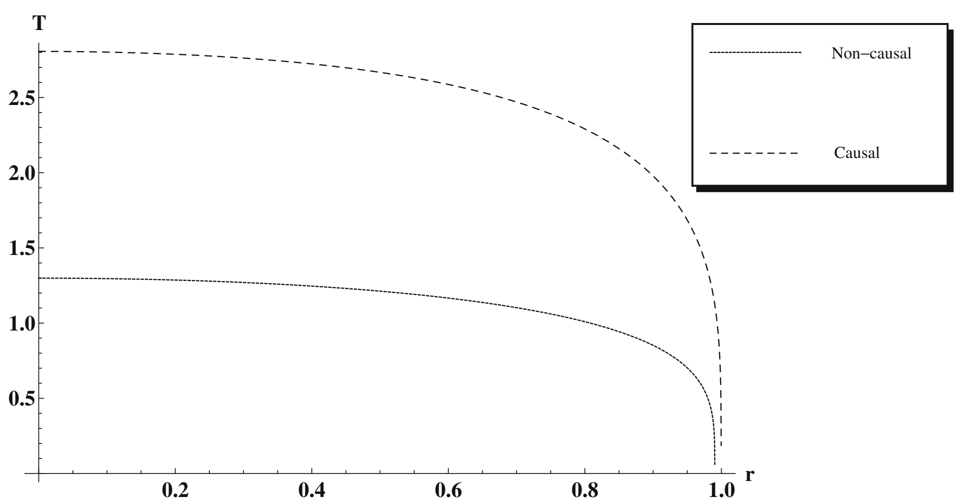


**Figure 4.** Causal and noncausal temperature profiles.

which governs the temperature behaviour. Substituting $\tau = 0$ in (17) we obtain the familiar Fourier heat transport equation

$$A(qB) = -\kappa \frac{(AT)_{,r}}{B}, \tag{18}$$

which predicts reasonable temperatures when the fluid is close to quasistationary equilibrium.

We utilize the thermodynamic coefficients for radiative transfer as motivated by Govender [36], in which heat dissipation within the core occurs via thermally generated neutrinos. The thermal conductivity, mean collision time and relaxation time are given by

$$\kappa = \chi T^3 \tau_{\mathrm{c}}, \quad \tau_{\mathrm{c}} = \left(\frac{\alpha}{\chi}\right) T^{-\omega}, \quad \tau = \left(\frac{\beta\chi}{\alpha}\right)\tau_{\mathrm{c}}\,, \tag{19}$$

where $\chi$ ($\geq 0$), $\alpha$ ($\geq 0$) and $\omega$ ($\geq 0$) are constants. We further assume a power-law generalization for the mean collision time given by

$$\tau_{\mathrm{c}} = \left(\frac{\alpha}{\chi}\right) T^{-\omega}\,, \tag{20}$$

where $\alpha$ ($\geq 0$) and $\omega$ ($\geq 0$) are constants. We observe that the mean collision time decreases as the temperature increases. A reasonable assumption would be to consider relaxation time proportional to the collision time [37]:

$$\tau = \left(\frac{\beta\chi}{\alpha}\right)\tau_{\mathrm{c}}\,, \tag{21}$$

where $\tau$ ($\geq 0$) is a constant.

Using the above definitions for $\tau$ and $\kappa$ we are in a position to integrate the causal transport eq. (17) for constant collision ($\omega = 0$). The causal temperature profile for this case is

$$\begin{aligned} T^4(r,t) = {} & G(r,t)\left[\frac{F(t)}{4} + \frac{2P(r,t)\dot{y}}{Q(r,t)}\right] \\ & - G(r,t)\left[\frac{\beta\left(R(r,t)\dot{y}^2 - S(r,t)\ddot{y}\right)}{yQ(r,t)}\right], \end{aligned} \tag{22}$$

where we have introduced

$$\begin{aligned} G(r,t) &= \left(\frac{4(1+y)(1+yr^2)}{1-2y+2yr^2-r^2y^2}\right)^4, \\ P(r,t) &= -7 + (13-18r^2)y + (-7+27r^2-12r^4)y^2 \\ &\quad +3r^2(-3+4r^2)y^3 - 3r^4y^4, \\ Q(r,t) &= 3y(1+y)^2(1+yr^2)^3, \\ R(r,t) &= -7 + (4-21r^2)y + (29-6r^2-12r^4)y^2 \\ &\quad +(51r^2-6r^4)y^3 + 24r^4y^4, \\ S(r,t) &= 3y(1+y)(-3+(3-7r^2)y+(5r^2-4r^4)y^2+2r^4y^3). \end{aligned}$$

The integration function $F(t)$ can be calculated from the effective surface temperature of the star as

$$\left(T^4\right)_\Sigma = \left(\frac{1}{r^2B^2}\right)_\Sigma \left(\frac{L_\infty}{4\pi\delta}\right), \tag{23}$$

where $L_\infty$ is the total luminosity at infinity and $\delta$ (>0) is a constant. It is noted that the expression for the casual temperature in (22) is a new result; it is remarkable that an explicit expression can be obtained in terms of elementary functions involving the function $y(t)$. Figure 4 shows both the causal and noncausal temperature profiles as functions of the radial coordinate. As expected, the temperature is highest at the centre of the radiating star and decreases rapidly towards the surface layers of the star. It is also clear that the causal temperature is higher at each interior point of the stellar fluid and is the same as its noncausal counterpart at the surface.

We have shown that the Kramer ansatz is reasonable and produces a model of a radiating star that predicts physically reasonable temperature profiles in both the causal and noncausal theories. Our results are in agreement with the earlier investigations of shear-free collapse, acceleration-free collapse (geodesic flows), conformally flat radiating models (vanishing of Weyl stresses), collapse from an initial static configuration and collapse to a final static core. It is clear that relaxational effects can alter the temperature distribution as the collapse proceeds.

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