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Protocol of networks using energy sharing collisions of bright solitons

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Abstract. It is well known that solitons in integrable systems recover their original profiles after their mutual collisions. This is not true in the case of optical fibre arrays, governed by a set of integrable coupled nonlinear Schrödinger (CNLS) equations. We consider the Manakov- and mixed-type 'two-component' CNLS systems. The most important characteristics of these systems are: (1) The polarizations of the two-component solitons are changed through their mutual collisions (Manakov system) and (2) the energy (intensity) switching occurs through the head-on collision (mixed system). By placing the above solitons on the primary star graph (PSG), we see that soliton collisions give rise to interesting phase changes in PSG: (a) The transition in PSG from its depolarized state to polarized one; (b) a state with selectively amplified bond is generated on PSG from its homogeneous state. These results will be applicable to network protocols using optical fibre arrays.

Keywords. Soliton network; coupled nonlinear Schrödinger system; bright soliton; soliton collision.

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1. Introduction

Solitons possess remarkable stability and intriguing collision properties. Their interesting dynamics enables them to have profound applications in various fields of science and technology. In optical fibres, soliton signals with 100 giga-bits/s (with each pulse-width = 10 ps) can be transmitted over 10,000 km without discernible errors. Soliton dynamics in optical fibre arrays is governed by the multimode (component) coupled nonlinear Schrödinger (CNLS) equations [1,2]. Multicomponent CNLS equations describe several fascinating dynamics which is not possible using their single-component counterpart [1]. For the past few decades, the dynamics of various CNLS-type systems have been explored and the importance of soliton collision dynamics is investigated in the context of optical switches, solitons collision-based computing, construction of optical gates, etc. [3–5].

In two-component CNLS type systems, the state-changing character of solitonic collisions led to the idea of the soliton-net (see Jakubouski *et al* [6] and others [7,8]). However, this idea was concerned with a single optical fibre and not with a spatial network of optical fibres. In fact, the soliton-net proposed by Jakubouski *et al* [6] was constructed in the fictitious two-dimensional plane spanned by the distance along the fibre and the retarded time. Thus, the soliton-net is not a real network developed in the real space.

As for the relationship between solitons and networks, the network provides a nice playground, where one can see interesting soliton propagations and nonlinear dynamics through the network [9–12], namely through an assembly of continuum line segments connected at vertices. Although there exist important analytical studies on the semi-infinite and finite chains [13–16], we find little exact analytical treatment of soliton propagation through networks within a framework of nonlinear Schrödinger equation (NLSE) [17,18]. The subject is difficult due to the presence of vertices where the underlying chain should bifurcate or multifurcate, in general.

Recently, with a suitable boundary condition at each vertex, Sobirov *et al* [19] and Nakamura *et al* [20] developed an exact analytical treatment of soliton propagation through networks within a framework of NLSE. Under an appropriate relationship among values of nonlinearity at individual bonds, they found nonlinear dynamics of solitons with no reflection at the vertex. They also showed that an infinite number of constants of motion are available for NLSE on networks, namely the mapping of Zakharov–Shabat (ZS)'s scheme [21] to networks was achieved.

As long as one stays in a single-component NLSE, no state-changing character of solitonic collisions is expected and therefore multiple soliton dynamics leads to a trivial transport through networks, i.e., splitting (fragmentation) of solitons, exchange of solitonic positions among bonds, etc. If one moves to optical solitons in the integrable two-mode (component) CNLS equations and manage the shape-changing solitons, much more generic phenomena can be expected in networks.

Let us consider the soliton dynamics in the optical fibre arrays, governed by a set of integrable two-component coupled nonlinear Schrödinger (CNLS) equations of Manakov type:

$$i\frac{\partial\psi^{(j)}}{\partial z}+\frac{\partial^2\psi^{(j)}}{\partial t^2}+2\mu(|\psi^{(a)}|^2+\sigma|\psi^{(b)}|^2)\psi^{(j)}=0, \qquad j=a,b. \tag{1}$$

Here $\psi^{(a)}$ and $\psi^{(b)}$ are the two complex modes of the beam, z and t represent respectively the normalized distance along the fibre and transverse coordinate and μ indicates the nonlinearity coefficient. The systems with $\sigma=1$ and -1 are called Manakov and mixed ones, respectively. The most important discovery of these systems is the existence of two- and multisoliton solutions in general, where the solitons change their state after the head-on collision. That is, the polarization of two-component soliton changes by their mutual collisions in the Manakov system [3,4] and the amplification of the intensity of soliton occurs through the head-on collision in the mixed system [5]. The important difference between the Manakov and mixed CNLS systems is the nature of nonlinearity which contributes to the dynamics of the resulting solitons. Particularly, the nature of energy switching for a given soliton in both components is opposite in the case of Manakov system, whereas, it is of the same kind in mixed CNLS system. In view of the conservation, the total energy among the two components is conserved in the former, while the difference between the two components is conserved in the latter.

In this paper, we consider the dynamics of two solitons of integrable two-component CNLS equations of both Manakov and mixed types, which are placed on the primary star graph (PSG) in figure 1. We shall show how the nature of PSG will change through two-soliton collisions.

The remaining part of this paper is as follows. In §2, using the PSG we shall analyse the energy and momentum conservation rules, showing how they lead to a connection formed at a vertex. We shall address the boundary condition to guarantee the connection formula, finding the sum rule for the strength of nonlinearity at each bonds. We shall reveal the general solution of CNLS equations on PSG expressed in terms of the bond-independent solution of CNLS on one-dimensional chain. In §3, after summarizing the two-soliton solution obtained by Hirota's bilinearization method [22] for both Manakov and mixed CNLS equations, we shall explore the dynamics of solitons in simple networks, i.e., PSG. In §4, the conclusion is given.

2. Primary star graph: Conservation rules, connection formula and sum rules

We consider a primary star graph (PSG) in figure 1 with single vertex at O and three semi-infinite bonds b_1, b_2 and b_3. The retarded time t (see below) at each bond is defined in the region $-\infty < t_1 < 0$, $0 < t_2 < \infty$ and $0 < t_3 < \infty$. Here t_1, t_2 and t_3 may be regarded as pseudospace variables. Let us define each bond described by the following two-component CNLS equations:

$$i\frac{\partial \psi_k^{(a)}}{\partial z} + \frac{\partial^2 \psi_k^{(a)}}{\partial t_k^2} + 2\mu_k(|\psi_k^{(a)}|^2 + \sigma|\psi_k^{(b)}|^2)\psi_k^{(a)} = 0, \tag{2a}$$

$$i\frac{\partial \psi_k^{(b)}}{\partial z} + \frac{\partial^2 \psi_k^{(b)}}{\partial t_k^2} + 2\mu_k(|\psi_k^{(a)}|^2 + \sigma|\psi_k^{(b)}|^2)\psi_k^{(b)} = 0, \tag{2b}$$

where $\psi_k^{(j)}$, $j = a, b$, $k = 1, 2, 3$, is the envelope in the jth component, k represents the bond index, μ_k indicates the strength of nonlinearity at bond k and the partial derivatives of $\psi_k^{(j)}$ are with respect to the normalized distance (z) and retarded time (t_k). This CNLS system is the extension of the Manakov-type ($\sigma = 1$) and mixed-type ($\sigma = -1$) CNLS

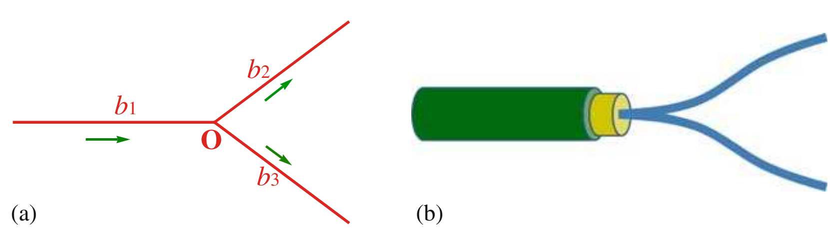

Figure 1. Primary star graph (PSG) with three semi-infinite bonds connected by a vertex: symbolic PSG (**a**) and realistic one constructed from optical fibre arrays (**b**).

systems on the ideal 1D chain to PSG. In the following, first we explore the conditions and connecting formulae which guarantee the integrability of the Manakov system.

2.1 *Conservation of energy*

In the 2-CNLS system (2) with $\sigma = 1$, the total energy along the three bonds is

$$E = \sum_{k=1}^{3} E_k = \sum_{k=1}^{3} \int_k (|\psi_k^{(a)}|^2 + |\psi_k^{(b)}|^2)\mathrm{d}t_k$$
$$= \int_{-\infty}^{0} (|\psi_1^{(a)}|^2 + |\psi_1^{(b)}|^2)\mathrm{d}t_1 + \sum_{k=2,3} \int_0^{\infty} (|\psi_k^{(a)}|^2 + |\psi_k^{(b)}|^2)\mathrm{d}t_k. \tag{3}$$

The requirement for the energy to be conservative is

$$\frac{\mathrm{d}E}{\mathrm{d}z} = \int_{-\infty}^{0} \left(\frac{\partial |\psi_1^{(a)}|^2}{\partial z} + \frac{\partial |\psi_1^{(b)}|^2}{\partial z} \right) \mathrm{d}t_1$$
$$+ \sum_{k=2,3} \int_0^{\infty} \left(\frac{\partial |\psi_k^{(a)}|^2}{\partial z} + \frac{\partial |\psi_k^{(b)}|^2}{\partial z} \right) \mathrm{d}t_k = 0. \tag{4}$$

We can write the continuity equations from (2) as

$$\frac{\partial |\psi_k^{(a)}|^2}{\partial z} = -2 \frac{\partial}{\partial t_k} \mathrm{Im}\left(\psi_k^{(a)*} \frac{\partial \psi_k^{(a)}}{\partial t_k} \right) \equiv -\frac{\partial}{\partial t_k} J_k^{(a)}(z, t_k), \tag{5a}$$

$$\frac{\partial |\psi_k^{(b)}|^2}{\partial z} = -2 \frac{\partial}{\partial t_k} \mathrm{Im}\left(\psi_k^{(b)*} \frac{\partial \psi_k^{(b)}}{\partial t_k} \right) \equiv -\frac{\partial}{\partial t_k} J_k^{(b)}(z, t_k), \tag{5b}$$

where $k = 1, 2, 3$, Im represents the imaginary part and $J_k^{(j)}$ with $j = a$ and b is the current density for each component. Using eq. (5) in eq. (4) and assuming $J_1^{(j)}(z, -\infty) = J_2^{(j)}(z, \infty) = J_3^{(j)}(z, \infty) = 0$, we obtain the condition (rule) for energy conservation

$$\sum_{j=a,b} \mathrm{Im}\left(\psi_1^{(j)*} \frac{\partial \psi_1^{(j)}}{\partial t_1} \right)\Bigg|_{t_1=0} = \sum_{j=a,b} \mathrm{Im}\left(\psi_2^{(j)*} \frac{\partial \psi_2^{(j)}}{\partial t_2} \right)\Bigg|_{t_2=0}$$
$$+ \sum_{j=a,b} \mathrm{Im}\left(\psi_3^{(j)*} \frac{\partial \psi_3^{(j)}}{\partial t_3} \right)\Bigg|_{t_3=0}. \tag{6}$$

This rule is nothing but the conservation of current density at the vertex:

$$\sum_{j=a,b} J_1^{(j)}(z, 0) = \sum_{j=a,b} J_2^{(j)}(z, 0) + \sum_{j=a,b} J_3^{(j)}(z, 0). \tag{7}$$

2.2 *Conservation of momentum*

Here we examine the momentum conservation of 2-CNLS system (2) with $\sigma = 1$ in our PSG. The expression for momentum can be written as

$$\begin{aligned} P &= \sum_{k=1}^{3} \int_k i \left[\psi_k^{(a)*} \frac{\partial \psi_k^{(a)}}{\partial t_k} - \psi_k^{(a)} \frac{\partial \psi_k^{(a)*}}{\partial t_k} + \psi_k^{(b)*} \frac{\partial \psi_k^{(b)}}{\partial t_k} - \psi_k^{(b)} \frac{\partial \psi_k^{(b)*}}{\partial t_k} \right] \mathrm{d}t_k, \\ &= -2 \sum_{k=1}^{3} \int_k \mathrm{Im} \left[\psi_k^{(a)*} \frac{\partial \psi_k^{(a)}}{\partial t_k} + \psi_k^{(b)*} \frac{\partial \psi_k^{(b)}}{\partial t_k} \right] \mathrm{d}t_k. \end{aligned} \tag{8}$$

Differentiating this equation with respect to z, we get

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}z} &= -2 \sum_{k=1}^{3} \int_k \mathrm{Im} \left[\frac{\partial \psi_k^{(a)*}}{\partial z} \frac{\partial \psi_k^{(a)}}{\partial t_k} + \psi_k^{(a)*} \frac{\partial^2 \psi_k^{(a)}}{\partial z \partial t_k} \right. \\ &\quad \left. + \frac{\partial \psi_k^{(b)*}}{\partial z} \frac{\partial \psi_k^{(b)}}{\partial t_k} + \psi_k^{(b)*} \frac{\partial^2 \psi_k^{(b)}}{\partial z \partial t_k} \right] \mathrm{d}t_k. \end{aligned} \tag{9}$$

We substitute the expressions for $\partial \psi_k^{(j)} / \partial z$ from eq. (2) in the above equation

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}z} &= -2 \sum_{k=1}^{3} \int_k \mathrm{Im} \left(\frac{\partial^3 \psi_k^{(a)}}{\partial t_k^3} \psi_k^{(a)*} - \frac{\partial \psi_k^{(a)}}{\partial t_k} \frac{\partial^2 \psi_k^{(a)*}}{\partial t_k^2} \right. \\ &\quad \left. + \frac{\partial^3 \psi_k^{(b)}}{\partial t_k^3} \psi_k^{(b)*} - \frac{\partial \psi_k^{(b)}}{\partial t_k} \frac{\partial^2 \psi_k^{(b)*}}{\partial t_k^2} \right) \mathrm{d}t_k \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}z} &= -2i\ \mathrm{Im} \left(\frac{\partial^2 \psi_1^{(a)}}{\partial t_1^2} \psi_1^{(a)*} + \frac{\partial^2 \psi_1^{(b)}}{\partial t_1^2} \psi_1^{(b)*} \right)_{t_1=-\infty}^{0} \\ &\quad -2i\ \mathrm{Im} \sum_{k=2,3} \left(\frac{\partial^2 \psi_k^{(a)}}{\partial t_k^2} \psi_k^{(a)*} + \frac{\partial^2 \psi_k^{(b)}}{\partial t_k^2} \psi_k^{(b)*} \right)_{t_k=0}^{\infty} \\ &= -2i\ \mathrm{Im} \left(\frac{\partial^2 \psi_1^{(a)}}{\partial t_1^2} \psi_1^{(a)*} + \frac{\partial^2 \psi_1^{(b)}}{\partial t_1^2} \psi_1^{(b)*} \right)_{t_1=0} \\ &\quad +2i\ \mathrm{Im} \sum_{k=2,3} \left(\frac{\partial^2 \psi_k^{(a)}}{\partial t_k^2} \psi_k^{(a)*} + \frac{\partial^2 \psi_k^{(b)}}{\partial t_k^2} \psi_k^{(b)*} \right)_{t_k=0}. \end{aligned} \tag{10}$$

Thus the conservation of momentum $((\mathrm{d}P/\mathrm{d}z) = 0)$ results in the following condition at the vertex:

$$\begin{aligned} \mathrm{Im} \left(\frac{\partial^2 \psi_1^{(a)}}{\partial t_1^2} \psi_1^{(a)*} + \frac{\partial^2 \psi_1^{(b)}}{\partial t_1^2} \psi_1^{(b)*} \right)_{t_1=0} &= \mathrm{Im} \sum_{k=2,3} \left(\frac{\partial^2 \psi_k^{(a)}}{\partial t_k^2} \psi_k^{(a)*} \right. \\ &\quad \left. + \frac{\partial^2 \psi_k^{(b)}}{\partial t_k^2} \psi_k^{(b)*} \right)_{t_k=0}. \end{aligned} \tag{11}$$

2.3 *Sum rule for the strength of nonlinearity*

The bilinear connection formulae in eqs (6) and (11) can be satisfied by the following linear relations at vertex O:

$$\alpha_1 \begin{pmatrix} \psi_1^{(a)} \\ \psi_1^{(b)} \end{pmatrix}_{t_1=-0} = \alpha_2 \begin{pmatrix} \psi_2^{(a)} \\ \psi_2^{(b)} \end{pmatrix}_{t_2=+0} = \alpha_3 \begin{pmatrix} \psi_3^{(a)} \\ \psi_3^{(b)} \end{pmatrix}_{t_3=+0},$$

$$\frac{1}{\alpha_1} \begin{pmatrix} \partial^n \psi_1^{(a)}/\partial t_1^n \\ \partial^n \psi_1^{(b)}/\partial t_1^n \end{pmatrix}_{t_1=-0} = \sum_{k=2,3} \frac{1}{\alpha_k} \begin{pmatrix} \partial^n \psi_k^{(a)}/\partial t_k^n \\ \partial^n \psi_k^{(b)}/\partial t_k^n \end{pmatrix}_{t_k=+0}, \quad n = 1, 2, \tag{12a}$$

or

$$\alpha_1 \begin{pmatrix} \partial^n \psi_1^{(a)}/\partial t_1^n \\ \partial^n \psi_1^{(b)}/\partial t_1^n \end{pmatrix}_{t_1=-0} = \alpha_2 \begin{pmatrix} \partial^n \psi_2^{(a)}/\partial t_2^n \\ \partial^n \psi_2^{(b)}/\partial t_2^n \end{pmatrix}_{t_2=+0} = \alpha_3 \begin{pmatrix} \partial^n \psi_3^{(a)}/\partial t_3^n \\ \partial^n \psi_3^{(b)}/\partial t_3^n \end{pmatrix}_{t_3=+0}, \quad n = 1, 2,$$

$$\frac{1}{\alpha_1} \begin{pmatrix} \psi_1^{(a)} \\ \psi_1^{(b)} \end{pmatrix}_{t_1=-0} = \sum_{k=2,3} \frac{1}{\alpha_k} \begin{pmatrix} \psi_k^{(a)} \\ \psi_k^{(b)} \end{pmatrix}_{t_k=+0}. \tag{12b}$$

In eq. (12), α_k with $k = 1, 2, 3$, are arbitrary real parameters, and $n = 1, 2$, is the degree of differential. The analogous issue is available in the system described by the mixed-type 2-CNLS equations (2) with $\sigma = -1$.

One can obtain various choices of α_k which satisfy eq. (12a) or eq. (12b). There is the most interesting case in which an infinite number of constants of motion can be found and hence the Manakov and the mixed CNLS systems on PSG become completely integrable.

In this case, following the idea of refs [19,20], we obtain

$$\alpha_k \begin{pmatrix} \psi_k^{(a)}(t_k, z) \\ \psi_k^{(b)}(t_k, z) \end{pmatrix}_{t_k=0} = \begin{pmatrix} g^{(a)}(0, z) \\ g^{(b)}(0, z) \end{pmatrix}, \tag{13a}$$

$$\alpha_k \begin{pmatrix} \partial^n \psi_k^{(a)}/\partial t_k^n \\ \partial^n \psi_k^{(b)}/\partial t_k^n \end{pmatrix}_{t_k=0} = \begin{pmatrix} \partial^n g^{(a)}(t, z)/\partial t^n \\ \partial^n g^{(b)}(t, z)/\partial t^n \end{pmatrix}_{t=0}, \qquad n = 1, 2, \tag{13b}$$

together with the constraint on α_k:

$$\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}. \tag{14}$$

In the above equations, $g^{(a,b)}(t, z)$ are the bond-independent universal functions, i.e., the general solution of the integrable Manakov ($\sigma = 1$) and mixed CNLS ($\sigma = -1$) systems with unit nonlinearity ($\mu = 1$) in the ideal one-dimensional chain. To be explicit, the governing equations for bond-independent nonlinearity can be written as

$$i \frac{\partial g^{(a)}}{\partial z} + \frac{\partial^2 g^{(a)}}{\partial t^2} + 2(|g^{(a)}|^2 + \sigma |g^{(b)}|^2) g^{(a)} = 0, \tag{15a}$$

$$i\frac{\partial g^{(b)}}{\partial z}+\frac{\partial^2 g^{(b)}}{\partial t^2}+2(|g^{(a)}|^2+\sigma|g^{(b)}|^2)g^{(b)}=0. \tag{15b}$$

Hence the solution of the above equations can be used to obtain the solution for the CNLS system with bond-dependent nonlinearity (2) and is written as

$$\begin{pmatrix}\psi_k^{(a)}(t_k,z)\\ \psi_k^{(b)}(t_k,z)\end{pmatrix}=\frac{1}{\sqrt{\mu_k}}\begin{pmatrix}g^{(a)}(t_k,z)\\ g^{(b)}(t_k,z)\end{pmatrix}. \tag{16}$$

Equations (13a) and (13b) can be satisfied by choosing α_k as

$$\alpha_k=\sqrt{\mu_k},\quad k=1,2,3. \tag{17}$$

To satisfy eq. (14), μ_k must satisfy the sum rule:

$$\frac{1}{\mu_1}=\frac{1}{\mu_2}+\frac{1}{\mu_3}. \tag{18}$$

In closing this section, it should be noted that in eq. (16), $g^{(a,b)}$ are universal functions defined in the ideal 1D chain with bond-independent nonlinearity ($\mu=1$) and that $\psi_1^{(a,b)}$ is defined in the region $-\infty<t_1<0$, while $\psi_{2,3}^{(a,b)}$ are defined in the region $0<t_{2,3}<+\infty$. The amplitude of $\psi_k^{(a,b)}$ shows a jump at the vertex, owing to the rule in eq. (16). However, $\psi_k^{(a,b)}$s show no singular behaviour like a reflection there, as $g^{(a,b)}$ itself has no singularity at $t=0$.

3. Dynamics of solitons in networks

Soliton solution of the integrable CNLS systems (2) can be obtained by using various analytical methods. Hirota's bilinearization method [22] is one of the prominent analytical tools which found advantage over other methods due to its algebraic nature.

3.1 *Bright one- and two-soliton solutions on primary star graph (PSG)*

The exact form of single bright soliton solution of eq. (2) on PSG available from the bond-independent ($\mu=1$) solution $g^{(a,b)}$ is given as

$$\psi_k^{(j)}=\frac{A^{(j)}\kappa_{1R}}{\sqrt{\mu_k}}\operatorname{sech}(\eta_{1R}+R/2)\,e^{i\eta_{1I}},\qquad j=a,b;\,k=1,2,3, \tag{19}$$

where

$$A^{(j)}=\frac{\beta_1^{(j)}}{(|\beta_1^{(a)}|^2+\sigma|\beta_1^{(b)}|^2)^{1/2}},\qquad e^R=\frac{|\beta_1^{(a)}|^2+\sigma|\beta_1^{(b)}|^2}{(\kappa_1+\kappa_1^*)^2},$$

$$\eta_1=\kappa_1(t_k+i\kappa_1 z)\equiv\eta_{1R}+i\eta_{1I}\quad\text{and}\quad\kappa_1\equiv\kappa_{1R}+i\kappa_{1I}.$$

This one-soliton solution is characterized by three arbitrary complex parameters $\beta_1^{(a)}$, $\beta_1^{(b)}$ and κ_1. The amplitudes of the soliton at the bond $k=1,2,3$ in components a and b are

$$\bar{A}_1^{(a)}(k)\equiv\frac{A^{(a)}\kappa_{1R}}{\sqrt{\mu_k}},$$

$$\bar{A}_1^{(b)}(k)\equiv\frac{A^{(b)}\kappa_{1R}}{\sqrt{\mu_k}}. \tag{20}$$

The soliton velocity in both components is $2\kappa_{1I}$. It is important to note that here in the 2-CNLS system (2), the soliton amplitude is independent of its velocity.

Explicit form of bright two-soliton solution [3–5] on PSG is also available from the bond-independent ($\mu = 1$) one and can be written as

$$\begin{aligned}
\psi_k^{(j)} &= \frac{1}{\sqrt{\mu_k}}\frac{Q^{(j)}}{W},\\
Q^{(j)} &= \beta_1^{(j)}e^{\eta_1} + \beta_2^{(j)}e^{\eta_2} + e^{\eta_1+\eta_1^*+\eta_2+\delta_{1j}} + e^{\eta_1+\eta_2+\eta_2^*+\delta_{2j}},\\
W &= 1 + e^{\eta_1+\eta_1^*+R_1} + e^{\eta_1+\eta_2^*+\delta_0} + e^{\eta_1^*+\eta_2+\delta_0^*}\\
&\quad + e^{\eta_2+\eta_2^*+R_2} + e^{\eta_1+\eta_1^*+\eta_2+\eta_2^*+R_3}, \quad j = a, b; k = 1, 2, 3,
\end{aligned} \tag{21}$$

where, by using individual soliton indices ($l, m = 1, 2$) of the two-soliton solution, we have defined

$$\begin{aligned}
\eta_l &= \kappa_l(t_k + i\kappa_l z), \quad e^{\delta_0} = \frac{\kappa_{12}}{\kappa_1 + \kappa_2^*},\\
e^{R_1} &= \frac{\kappa_{11}}{\kappa_1 + \kappa_1^*}, \quad e^{R_2} = \frac{\kappa_{22}}{\kappa_2 + \kappa_2^*},\\
e^{\delta_{1j}} &= \frac{(\kappa_1 - \kappa_2)(\beta_1^{(j)}\kappa_{21} - \beta_2^{(j)}\kappa_{11})}{(\kappa_1 + \kappa_1^*)(\kappa_1^* + \kappa_2)},\\
e^{\delta_{2j}} &= \frac{(\kappa_2 - \kappa_1)(\beta_2^{(j)}\kappa_{12} - \beta_1^{(j)}\kappa_{22})}{(\kappa_2 + \kappa_2^*)(\kappa_1 + \kappa_2^*)},\\
e^{R_3} &= \frac{|\kappa_1 - \kappa_2|^2(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}{(\kappa_1 + \kappa_1^*)(\kappa_2 + \kappa_2^*)|\kappa_1 + \kappa_2^*|^2},\\
\kappa_{lm} &= \frac{\beta_l^{(a)}\beta_m^{(a)*} + \sigma\beta_l^{(b)}\beta_m^{(b)*}}{\left(\kappa_l + \kappa_m^*\right)}.
\end{aligned} \tag{22}$$

The dynamics of the soliton collision can be explored clearly by carrying out asymptotic analysis. In general, the velocity of solitons is considered to be arbitrary and opposite to admit head-on collision. By using eq. (20), the amplitude of solitons at the bond $k+$ after collision can be related to that at the bond $k'-$ before collision as

$$\begin{aligned}
\bar{A}_{l+}^{(j)}(k+) &= \sqrt{\frac{\mu_{k'-}}{\mu_{k+}}}T_l^{(j)}\bar{A}_{l-}^{(j)}(k'-), \quad l = 1, 2; \quad j = a, b;\\
&\quad k'-, k+ = 1, 2, 3,
\end{aligned} \tag{23}$$

where $\bar{A}_{l-}^{(j)}(k'-)$ and $\bar{A}_{l+}^{(j)}(k+)$ represent the amplitude of the lth soliton in the jth component at the bond $k'-$ and $k+$ before and after interaction, respectively. $\sqrt{(\mu_{k'-}/\mu_{k+})}T_l^{(j)}$ stands for the transition amplitude, where $T_l^{(j)}$ are the bond-independent ($\mu = 1$) transition amplitudes defined as

$$\begin{aligned}
T_1^{(j)} &= \left(\frac{(\kappa_1 - \kappa_2)(\kappa_2 + \kappa_1^*)}{(\kappa_1^* - \kappa_2^*)(\kappa_1 + \kappa_2^*)}\right)^{1/2}\left(\frac{1 - \Lambda_2}{\sqrt{1 - \Lambda_1\Lambda_2}}\right),\\
T_2^{(j)} &= -\left(\frac{(\kappa_2 + \kappa_1^*)(\kappa_1^* - \kappa_2^*)}{(\kappa_1 - \kappa_2)(\kappa_1 + \kappa_2^*)}\right)^{1/2}\left(\frac{\sqrt{1 - \Lambda_1\Lambda_2}}{1 - \Lambda_1}\right), \qquad j = a, b.
\end{aligned} \tag{24}$$

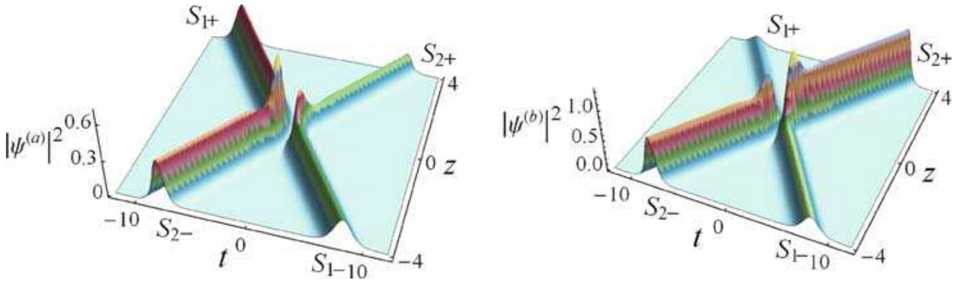

Figure 2. Type-I inelastic collision of bright solitons in Manakov system ($\sigma = 1$) for $k_1 = 1 - i$, $k_2 = 1.5 + i$, $\beta_1^{(a)} = 0.8 - 2i$, $\beta_2^{(a)} = 1$, $\beta_1^{(b)} = 1 - i$ and $\beta_2^{(b)} = 1 + i$.

Here

$$\Lambda_1 = \frac{\kappa_{21}}{\kappa_{11}}\frac{\beta_1^{(j)}}{\beta_2^{(j)}}, \quad \Lambda_2 = \frac{\kappa_{12}}{\kappa_{22}}\frac{\beta_2^{(j)}}{\beta_1^{(j)}},$$

$$\kappa_{lm} = \frac{\beta_l^{(a)}\beta_m^{(a)*} + \sigma\beta_l^{(b)}\beta_m^{(b)*}}{(\kappa_l + \kappa_m^*)}, \quad l, m = 1, 2.$$

It is evident from the above equations that the solitons undergo inelastic collision in general. Only for special choice of parameters ($\beta_1^{(a)}/\beta_2^{(a)} = \beta_1^{(b)}/\beta_2^{(b)}$) the solitons exhibit elastic collision without any change in their amplitudes, namely $|T_l^{(j)}|^2 = 1$ for all j and l, but suffer a phase-shift.

Before studying the state change in networks through soliton collisions, we shall come back to the case of the ideal 1D chain and demonstrate the inelastic (energy sharing) collision of two bright solitons in figures 2 and 3 corresponding in Manakov- and mixed CNLS-type systems, respectively.

The figures clearly depict the nature of energy switching of solitons. In the Manakov system, the amplitude of soliton S_1 increases after collision with soliton S_2 in the $\psi^{(a)}$ component, while the reverse scenario takes place in the $\psi^{(b)}$ component. That is, polarization of individual soliton changes through the head-on collision, where the polarization

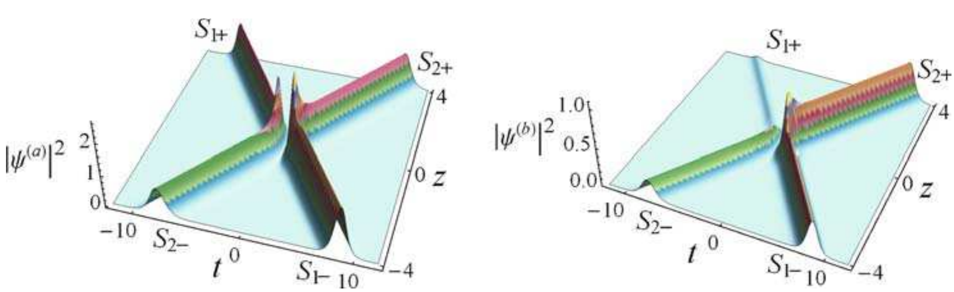

Figure 3. Type-II inelastic collision of bright solitons in mixed CNLS system ($\sigma = -1$) for $k_1 = 1.5 - i$, $k_2 = 1.1 + i$, $\beta_1^{(a)} = 2.5$, $\beta_2^{(a)} = 1.3$, $\beta_1^{(b)} = 0.5$ and $\beta_2^{(b)} = 0.7$.

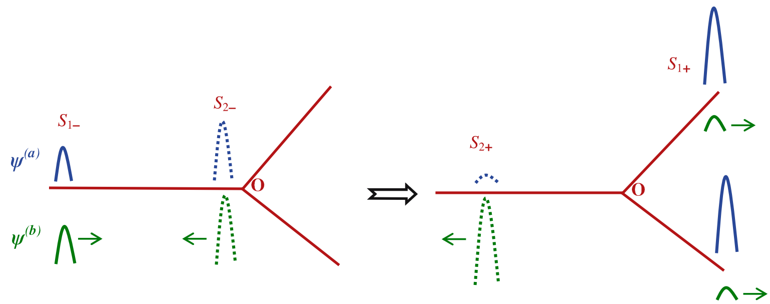

Figure 4. Inelastic collision of bright solitons in Manakov network – Case I.

quantifies the relative intensity. On the other hand, in the mixed CNLS system, the amplitude of soliton S_1 (S_2) decreases (increases) in both components after collision with S_2 (S_1), a remarkable property arising due to the negative nonlinearity coefficient ($\sigma = -1$). Now we shall demonstrate how this soliton collision processes can be advantageously used in PSG.

3.2 *State change in networks through soliton collisions*

We shall now place a two-soliton solution on PSG in various ways and see how this solution will evolve after the collision. Figures 4, 5 and 6 show the initial (left) and final (right) configurations of the two-soliton solution in the Manakov-type system on PSG.

Both solitons are located initially on the bond b_1 and on the bonds b_2, b_3, respectively, in figures 4 and 6. In figure 5, initially one soliton is located on bond b_1 and the other one on bonds b_2, b_3. Also each of the two solitons is assumed to be almost depolarized initially. We find: (1) despite the variety of initial configurations, the final configuration is identical, which reminds us of an attractor in dissipative systems and (2) despite the depolarized nature of the initial configuration, the final configuration shows a clear polarization on each bond. That is, we see the phase change of PSG from depolarized to polarized phase thanks to soliton collisions along PSG.

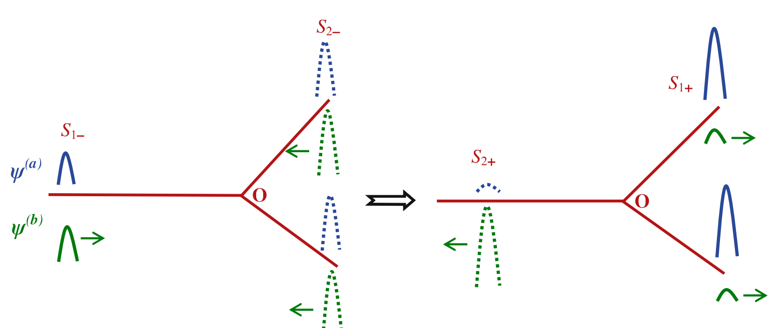

Figure 5. Inelastic collision of bright solitons in Manakov network – Case II.

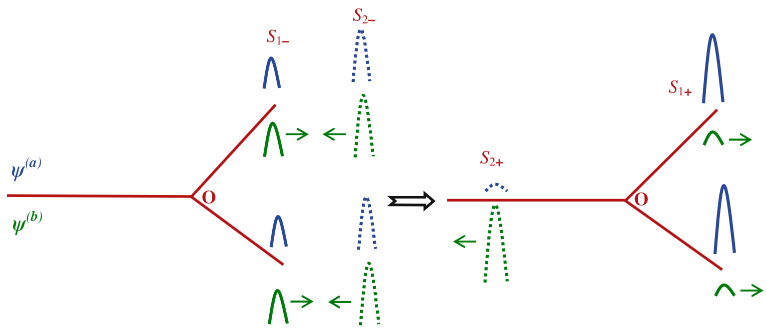

Figure 6. Inelastic collision of bright solitons in Manakov network – Case III.

Figures 7, 8 and 9 show the initial (left) and final (right) configurations of the two-soliton solution in the mixed-type system on PSG. Both the solitons are located initially on the bond b_1 and on the bonds b_2 and b_3, respectively, in figures 7 and 9. In figure 8, initially one soliton is located on bond b_1 and the other one on the bonds b_2, b_3. Also each soliton, which is almost depolarized, has identical intensity initially. We find: (1) A state with a selectively amplified bond is generated in PSG from its homogeneous state. In fact, after collision among solitons, there appears a big soliton with enhanced intensity on the bond b_1 and tiny ones with decreased intensity on the other bonds, while initially solitons on each bond has equal intensity and democratic. (2) Various kinds of initial democratic states finally lead to the uniquely amplified state where only the bond b_1 is intensified, which again reminds us of an attractor in dissipative systems.

Thus, two-soliton dynamics on PSG composed of optical fibre arrays shows the phase change in PSG, and can play a role of protocol of networks. This phenomenon is caused by the inelastic collision among solitons proper to the integrable Manakov and mixed CNLS equations.

The remaining subject in this section is the quantitative characterization of figures 4–9. Let us concentrate on figure 4. We denote the initial configuration as $(1, 1, 0, 0, 0, 0)$ corresponding to the presence of solitons 1 and 2 (from left to right) on bond 1, and their absence on bonds 2 and 3. Then the final configuration can be denoted as $@(0, 1, 1, 0, 1, 0)$. Now we shall assign d to the depolarized soliton and $p^{(a)}$ $(p^{(b)})$

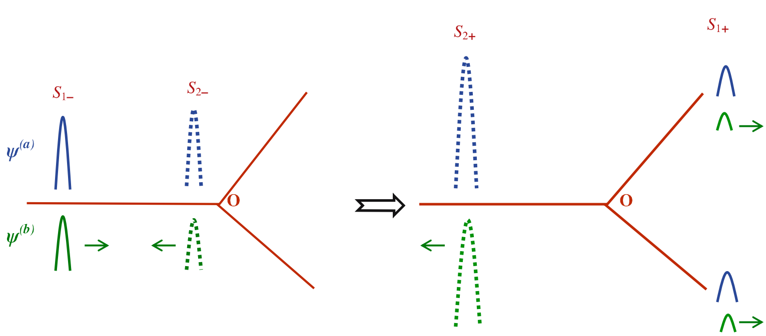

Figure 7. Inelastic collision of bright solitons in mixed CNLS-type network – Case I.

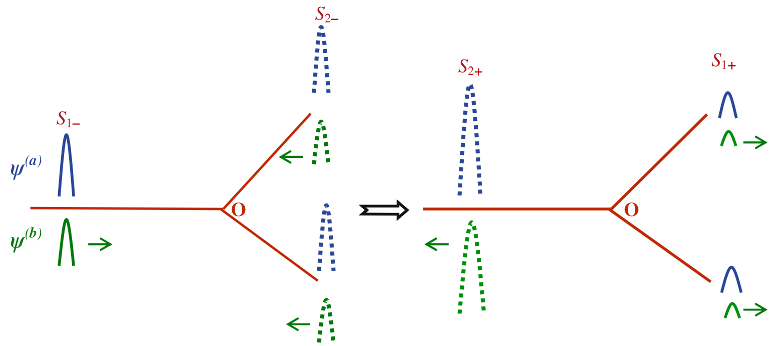

Figure 8. Inelastic collision of bright solitons in mixed CNLS-type network – Case II.

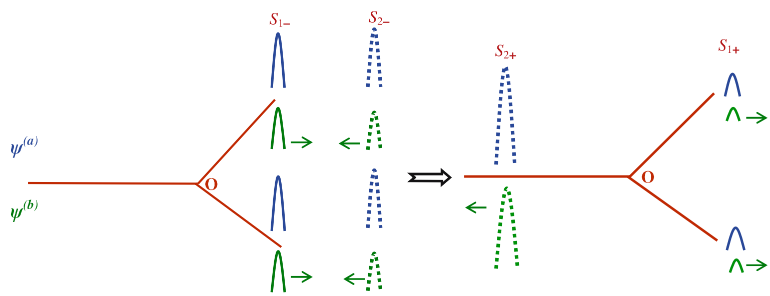

Figure 9. Inelastic collision of bright solitons in mixed CNLS-type network – Case III.

to the polarized soliton with predominant occupancy of $\psi^{(a)}$ ($\psi^{(b)}$) component. Then the phase change in figure 4 is more precisely characterized by: $(d, d, 0, 0, 0, 0) \rightarrow (0, p^{(b)}, p^{(a)}, 0, p^{(a)}, 0)$. The final phase is identical in figures 4–6, as mentioned already. The transition amplitudes for solitons 1 and 2 are given by $\sqrt{(\mu_1/\mu_{2,3})}T_1^{(j)}$ and $\sqrt{(\mu_1/\mu_1)}T_2^{(j)} \equiv T_2^{(j)}$, respectively in figure 4. Here we recognize that the transition probability of soliton 1 from bond 1 to bonds 2 and 3 satisfies the unitarity: $|\sqrt{(\mu_1/\mu_2)}T_1^{(j)}|^2 + |\sqrt{(\mu_1/\mu_3)}T_1^{(j)}|^2 = |T_1^{(j)}|^2$ thanks to the sum rule at the vertex of PSG in eq. (18).

On the other hand, let us move to figures 7–9, where we shall assign 's', 'm' and 'l' to small, middle and large solitons, respectively. Then the phase change in figure 7 is characterized by $(m, m, 0, 0, 0, 0) \rightarrow (0, l, s, 0, s, 0)$. The final phase is again identical in figures 7–9. A systematic quantitative characterization of figures 4–9 is lengthy and will be carried out elsewhere [23].

4. Conclusion

Solitons in the integrable Manakov and mixed coupled NLS equations play vital roles in the network. By placing the above solitons on the network like a primary star graph (PSG), we see that soliton collisions give rise to the dramatic phase changes in the network: (1) The transition in PSG occurs from its depolarized to polarized state (Manakov case). (2) A state with selectively amplified bond is generated on PSG from its homogeneous state (mixed case). The final state is quite unique, despite a variety of initial configurations of solitons on PSG, which is reminiscent of an attractor in the dissipative systems. If the external semi-infinite bonds attached to the network will be taken as transmitters or receivers of messages, the present phase change can be used as protocols of network communication by means of soliton collisions.

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