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Group formalism of Lie transformations to time-fractional partial differential equations

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Abstract. A systematic method is given to derive Lie point symmetries of time-fractional partial differential equation with Riemann–Liouville fractional derivative and its applicability illustrated through (i) time-fractional diffusive equation and (ii) time-fractional cylindrical Korteweg–de Vries equation. Using the Lie point symmetries obtained, we show that each of them has been transformed into ordinary differential equation of fractional order with a new independent variable. We also explain how exact or invariant solutions can be derived from the obtained point symmetries.

Keywords. Lie symmetry analysis; Fractional partial differential equation; Riemann–Liouville fractional derivative; Mittag–Leffler function; Erdélyi–Kober operators.

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1. Introduction

Generalization has always been an interesting subject in mathematics. Fractional differential equations (FDEs) are generalization of differential equations to arbitrary (non-integer) order. FDEs can also be viewed as alternative models to nonlinear differential equations. Even though the first step of the theory of FDEs itself was initiated in the first half of the nineteenth century, the subject really came to life only in the last few decades [1–4]. Recent investigations reveal that new fractional-order models are more appropriate than the existing integer-order models due to the exact description of nonlinear phenomena [5–7].

FDEs occur in a surprising number of real-world models in different areas of applied science and engineering. It is known that while formulating the diffusion equation with one space and time variables

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

it was assumed that the mean squared displacement is linear and time-dependent. It is not necessarily the case always (for example, anomalous diffusion). In such situations the mean squared displacement is nonlinear and time-dependent, i.e., $\langle x^2 \rangle \sim t^\alpha$, $\alpha \neq 1$. Then the governing equation of anomalous diffusion process can be written as [6]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{1}$$

In literature, the above equation is known as the fractional diffusion equation. If $0 < \alpha < 1$, then (1) describes a subdiffusion phenomenon (slow movement of particles) whereas $\alpha > 1$ represents a superdiffusion phenomenon (fast movement of particles). Moreover, if $1 < \alpha < 2$, then (1) represents the fractional diffusion-wave equation which is useful to study the intermediate process between diffusion and wave phenomena. Hence we observe that FDE provides a tool to study both PDEs of parabolic and hyperbolic type processes simultaneously.

Let us consider another real-world problem, namely, stress $\sigma(t)$ and strain $\varepsilon(t)$ relationship of the viscoelastic material. If one deals with pure elastic solids, Hooke's law yields

$$\sigma(t) = E\varepsilon(t) = E\frac{\mathrm{d}^0\varepsilon}{\mathrm{d}t^0},$$

where E is a constant. On the other hand, if one considers pure viscous liquids, Newton's law gives

$$\sigma(t) = \eta\frac{\mathrm{d}\varepsilon}{\mathrm{d}t},$$

where η is a constant. However, in reality the viscoelastic materials exhibit a behaviour somewhere between the pure elastic solid and pure viscous liquid. It is possible to model the stress–strain relationship for such a viscoelastic material via a FDE [7]

$$\sigma(t) = \nu\frac{\mathrm{d}^\alpha\varepsilon}{\mathrm{d}t^\alpha}, \quad 0 < \alpha < 1, \tag{2}$$

where ν is a constant.

More FDEs describing real-world problems are illustrated in [1–9]. Thus, FDEs have the potential to accomplish what integer-order differential equations cannot. In recent years, different ad-hoc methods applicable to FDEs have been proposed by several research groups. Among them, the Lie group transformation technique provides an effective tool to analyse FDEs.

Basically, Lie symmetry analysis of differential equations is a continuous transformation group theory, originally advocated by the Norwegian mathematician, Sophus Lie in the beginning of the 19th century and was further developed by Ovsiannikov [10] and others [11–13]. The fundamental idea of Lie symmetry analysis is to find one or several parameter continuous transformations leaving the equation invariant. The effectiveness of the Lie point symmetry approach has widely been demonstrated in a variety of nonlinear differential equations occurring in different areas of applied science [11,14].

The Lie symmetry analysis of differential equations has been extended to FDEs by Gazizov *et al* [15] (see also [16–19]). It is appropriate to mention here that most often FDEs with Riemann–Liouville and Caputo fractional derivatives are considered for discussions in the literature. In [15], Gazizov *et al* have considered FDEs with Riemann–Liouville fractional derivative and derived a prolongation formula for it, enabling one to determine its Lie point symmetries. The usefulness of this has been illustrated in [15,18]. Recently, we have considered a time-fractional nonlinear generalized Korteweg–de Vries and Burgers equations with Riemann–Liouville fractional derivative and derived its point symmetries and also reduced it into a nonlinear fractional ordinary differential equation with Erdelyi–Kober fractional derivative by means of similarity transformation [19].

To the best of our knowledge, only a limited number of FDEs with Riemann–Liouville derivative has been investigated through Lie symmetry approach. The main objective of this paper is to present the prolongation formula for time-fractional PDE with Riemann–Liouville fractional derivative and illustrate its applicability through the following FDEs:

(1) Time-fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = k\frac{\partial^2 u}{\partial x^2}, \quad \alpha \in (0, 1), \tag{3}$$

where k is a constant.

(2) Time-fractional cylindrical KdV equation with Riemann–Liouville fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{u}{2t^\alpha} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad \alpha \in (0, 1) \tag{4}$$

which occurs in different contexts in mathematical physics, for example in physics of plasmas [20]. We also explain how exact or invariant solutions can be derived from the obtained point symmetries.

The plan of the paper is as follows. In §2, for clarity of presentation we present basic definitions and some properties of the fractional operators which are required for the remaining part of the paper. In §3, to be self-contained, we explain how to derive the prolongation formula for nonlinear time-fractional partial differential equations with Riemann–Liouville fractional derivative. In §4, we illustrate the effectiveness of Lie symmetry analysis for time-fractional partial differential equations by finding Lie point symmetries of time-fractional diffusion and cylindrical Korteweg–de Vries equations. In §5, we give a brief summary of our results and concluding remarks.

2. Basic definitions

To be self-contained, we briefly provide some basic definitions of fractional calculus below.

(i) *Riemann–Liouville fractional integral*

It is well known that the Cauchy formula for n-fold integrations is

$$J^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\, \mathrm{d}s, \quad n \in \mathbb{N}.$$

Replacing the integer n with real number $\alpha > 0$ and the discrete factorial $(n-1)!$ with the continuous gamma function $\Gamma(n)$, the Riemann–Liouville fractional integral is defined by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\, \mathrm{d}s, \quad \alpha > 0$$

and

$$J^0 f(t) = f(t),$$

where $\Gamma(\lambda) = \int_0^{+\infty} x^{\lambda-1} \mathrm{e}^{-x} \mathrm{d}x,\ \lambda > 0,$ is the Euler gamma function.

(ii) *Riemann–Liouville fractional derivative*
In the literature several definitions of fractional derivative such as the Riemann–Liouville [1–4], the Grunwald–Letnikov [1–4], the Weyl [2,4], the Caputo [3,4], and the Riesz [2] have been adopted by different researchers. Among them the Riemann–Liouville and the Caputo fractional derivatives have been widely used.

The Riemann–Liouville fractional derivative of a continuous function $f(t)$ is obtained by splitting its fractional derivative operator into an integer-order derivative and a fractional integral operator. The Riemann–Liouville fractional differential operator of order $\alpha > 0$, denoted by $D^{\alpha}_{a^+}$, is defined by [2–4]

$$\begin{aligned} D^{\alpha}_{a^+} f(t) &= D^n J_a^{n-\alpha} f(t), \quad n = [\alpha] + 1, \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \quad t > a \end{aligned}$$

and

$$D^0_{a^+} f(t) = f(t),$$

where $[\alpha]$ is the integral part of α.

(iii) *Riemann–Liouville partial fractional derivative*

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \alpha = n \in N; \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_a^t \frac{u(\tau, x)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, & n-1 < \alpha < n,\ n \in N. \end{cases} \tag{5}$$

Note that the above-mentioned operators satisfy the following properties for the suitable functions $f(t)$ and $g(t)$:

$$J^\alpha (f(t) + g(t)) = J^\alpha f(t) + J^\alpha g(t),$$

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t),$$

$$D^\alpha (f(t) + g(t)) = D^\alpha f(t) + D^\alpha g(t),$$

$$J^\alpha (D^\alpha f(t)) = f(t) - \sum_{r=0}^{n-1} \frac{f^{(r)}(0)}{r!} t^r, \quad n-1 < \alpha \le n,$$

$$D^\alpha (J^\alpha f(t)) = f(t),$$

$$J^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} t^{\lambda+\alpha}, \quad \alpha > 0,\ \lambda > -1,$$

$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, \quad \alpha > 0,\ \lambda > -1.$$

It is appropriate to mention here that the fractional-order operators are nonlocal, i.e., the value of the fractional derivative at a point in the domain depends on values of the function throughout the domain. Hence fractional-order models incorporate nonlocal and system memory effects respectively through its fractional-order space and time derivatives [5]. Thus the study of fractional nonlinear differential equations is important and challenging.

3. Prolongation formula for nonlinear time-fractional partial differential equation with Riemann–Liouville derivative

Consider a scalar time-fractional PDE having the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},u_{xxx},\dots), \quad \alpha > 0, \tag{6}$$

where subscripts denote partial derivatives. Let us assume that the above time-fractional PDE (6), is invariant under a one-parameter (ϵ) continuous point transformations

$$\begin{aligned}
\bar{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\
\bar{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\
\bar{u} &= u + \epsilon\eta(x,t,u) + O(\epsilon^2),\\
\frac{\partial^n \bar{u}}{\partial \bar{t}^n} &= \frac{\partial^n u}{\partial t^n} + \epsilon\eta_t^{(n)} + O(\epsilon^2),\\
\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_t^{(\alpha)} + O(\epsilon^2),\\
\frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \epsilon\eta_x^{(1)} + O(\epsilon^2),\\
\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\eta_x^{(2)} + O(\epsilon^2),\\
\frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\eta_x^{(3)} + O(\epsilon^2),\\
&\vdots
\end{aligned} \tag{7}$$

provided any solution $u(x,t)$ satisfies (6). It is known that the nth ($n \in N$) extended infinitesimal $\eta_t^{(n)}$ satisfies [10,12]

$$\eta_t^{(n)} = D_t^n(\eta - \xi u_x - \tau u_t) + \xi D_t^n u_x + \tau D_t^{n+1} u, \tag{8}$$

where D_t is the total derivative operator given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + \cdots .$$

The nth ($n \in N$) extended infinitesimal $\eta_x^{(n)}$ satisfies a similar expression. As the lower limit $t = a$ of the integral in the definition of Riemann–Liouville fractional partial derivative (5) is fixed, it should be invariant with respect to the transformations (7). Such invariance condition arrives at

$$\tau(x,t,u)|_{t=a} = 0. \tag{9}$$

Theorem 3.1. *The* α*th* ($\alpha \in R^+$) *extended infinitesimal related to Riemann–Liouville fractional partial derivative reads*

$$\eta_t^{(\alpha)} = {}_aD_t^\alpha(\eta - \xi u_x - \tau u_t) + \xi\ {}_aD_t^\alpha u_x + \tau\ {}_aD_t^{\alpha+1}u. \tag{10}$$

Here the operator ${}_aD_t^\alpha$ *denotes the total fractional derivative operator with respect to* t.

Proof. We recall the generalized Leibnitz rule given by

$${}_aD_x^\alpha(f(x)g(x)) = \sum_{n=0}^{\infty}\binom{\alpha}{n}{}_aD_x^{\alpha-n}f(x)\ g^{(n)}(x),\ \alpha > 0, \tag{11}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-n+1)\Gamma(n+1)}.$$

Using the Leibnitz rule (11), we write

$${}_a\partial_{\bar{t}}^\alpha\bar{u} = \frac{\partial^\alpha\bar{u}(\bar{x},\bar{t})}{\partial\bar{t}^\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(\bar{t}-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}\frac{\partial^n\bar{u}(\bar{x},\bar{t})}{\partial\bar{t}^n}. \tag{12}$$

From (7) and applying (12), we have

$$\begin{aligned}\eta_t^{(\alpha)} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\frac{\partial^\alpha\bar{u}(\bar{x},\bar{t})}{\partial\bar{t}^\alpha}\right]_{\epsilon=0}\\ &= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(t-a)^{n-\alpha}\eta_t^{(n)} + (n-\alpha)(t-a)^{n-\alpha-1}\tau D_t^n u}{\Gamma(n-\alpha+1)}.\end{aligned} \tag{13}$$

Substituting (8) in (13), we get

$$\begin{aligned}\eta_t^{(\alpha)} &= \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(t-a)^{n-\alpha}[D_t^n(\eta-\xi u_x-\tau u_t)+\xi D_t^n u_x+\tau D_t^n u_t]}{\Gamma(n-\alpha+1)}\\ &\quad+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(n-\alpha)(t-a)^{n-\alpha-1}\tau D_t^n u}{\Gamma(n-\alpha+1)}.\end{aligned} \tag{14}$$

Using the Leibnitz rule (11) in (14) we have

$$\begin{aligned}\eta_t^{(\alpha)} &= {}_aD_t^\alpha(\eta-\xi u_x-\tau u_t)+\xi\ {}_aD_t^\alpha u_x + \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}\tau D_t^{n+1}u\\ &\quad+\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{(n-\alpha)(t-a)^{n-\alpha-1}\tau D_t^n u}{\Gamma(n-\alpha+1)}.\end{aligned} \tag{15}$$

Replace n by $n-1$ in the third term and substitute $n=0$ in the last term of eq. (15)

$$\begin{aligned}\eta_t^{(\alpha)} &= {}_aD_t^\alpha(\eta-\xi u_x-\tau u_t)+\xi\ {}_aD_t^\alpha u_x + \sum_{n=1}^{\infty}\binom{\alpha}{n-1}\frac{(t-a)^{n-\alpha-1}\tau D_t^n u}{\Gamma(n-\alpha)}\\ &\quad-\frac{\alpha(t-a)^{-\alpha-1}\tau u}{\Gamma(1-\alpha)}+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{(t-a)^{n-\alpha-1}\tau D_t^n u}{\Gamma(n-\alpha)}.\end{aligned} \tag{16}$$

Using the relation $\binom{\alpha}{n-1} + \binom{\alpha}{n} = \binom{\alpha+1}{n}$, we obtain

$$\begin{aligned} \eta_t^{(\alpha)} &= {}_aD_t^\alpha(\eta - \xi u_x - \tau u_t) + \xi\, {}_aD_t^\alpha u_x + \sum_{n=0}^{\infty} \binom{\alpha+1}{n} \frac{(t-a)^{n-\alpha-1}}{\Gamma(n-\alpha)} \tau D_t^n u, \\ &= {}_aD_t^\alpha(\eta - \xi u_x - \tau u_t) + \xi\, {}_aD_t^\alpha u_x + \tau\, {}_aD_t^{\alpha+1} u, \\ &= {}_aD_t^\alpha(\eta) - {}_aD_t^\alpha(\xi u_x) - {}_aD_t^\alpha(\tau u_t) + \xi\, {}_aD_t^\alpha u_x + \tau\, {}_aD_t^{\alpha+1} u \end{aligned} \tag{17}$$

which is the required prolongation formula. Note that when $\alpha \to n \in N$, we recovered the classical prolongation formula (8). □

We wish to mention that (17) can be further simplified which will be helpful for the computation of infinitesimals. It is known that [2,4]

$$ {}_aD_x^\alpha(y^{(1)}) = {}_aD_x^{\alpha+1} y - \frac{(x-a)^{-\alpha-1} y(a)}{\Gamma(-\alpha)}. \tag{18}$$

Making use of (9), that is, $\tau(a) = 0$ along with (18), the third term in the RHS of (17) can be rewritten as

$$ {}_aD_t^\alpha(\tau u_t) = {}_aD_t^\alpha(D_t(\tau u) - u D_t \tau) = {}_aD_t^{\alpha+1}(\tau u) - {}_aD_t^\alpha(u D_t \tau). \tag{19}$$

Substituting (19) in (17) we obtain

$$\begin{aligned} \eta_t^{(\alpha)} &= {}_aD_t^\alpha(\eta) - {}_aD_t^\alpha(\xi u_x) + {}_aD_t^\alpha(u D_t \tau) - {}_aD_t^{\alpha+1}(\tau u) \\ &\quad + \xi\, {}_aD_t^\alpha u_x + \tau\, {}_aD_t^{\alpha+1} u. \end{aligned} \tag{20}$$

Applying the generalized Leibnitz rule (11) in (20) and using the relation $\binom{\alpha+1}{n+1} = \binom{\alpha}{n} \frac{\alpha+1}{n+1}$, we obtain

$$\begin{aligned} \eta_t^{(\alpha)} &= {}_aD_t^\alpha(\eta) + \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{n-\alpha}{n+1}\, D_t^{n+1} \tau \\ &\quad \times {}_aD_t^{\alpha-n} u - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n \xi\, {}_aD_t^{\alpha-n} u_x. \end{aligned} \tag{21}$$

4. Time-fractional partial differential equations

In this section, we illustrate the effectiveness of Lie symmetry analysis by finding Lie point symmetries through time-fractional diffusion and cylindrical Korteweg–de Vries equations. For clarity, we consider them separately.

4.1 *Time-fractional diffusion equation*

The symmetry analysis of the diffusion equation ($\alpha = 1$) is well known (see [11,12]). In [21], Wyss considered the time-fractional diffusion equation and constructed its closed form solution in terms of Fox functions. Buckwar and Luchko [16], Luchko and Gorenflo [22] constructed the scale invariant solution of the fractional diffusion equation in terms of Wright functions (see also [18,23]). Let us assume that the time-fractional diffusion equation, (3), is invariant under a one-parameter continuous point transformations (7), and so the invariant equation reads as

$$[\eta_t^{(\alpha)} - k\eta_x^{(2)}]\big|_{(3)} = 0 \tag{22}$$

which is not solvable, in general, for $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. In order to solve (22), we assume that the infinitesimal be of the form

$$\eta = p(x,t)u + q(x,t), \tag{23}$$

where $p(x,t)$ and $q(x,t)$ are unknown functions to be determined. After a systematic calculation we obtain the following determining equations:

$$\begin{aligned}
&\xi_u = \xi_t = \tau_u = \tau_x = 0,\\
&\partial_t^n p + \frac{n-\alpha}{n+1}\tau^{(n+1)}(t) = 0,\ n \in N,\\
&2p_x - \xi''(x) = 0,\\
&2\xi'(x) - \alpha\tau'(t) = 0,\\
&{}_0\partial_t^\alpha q - kp_{xx}u - kq_{xx} = 0.
\end{aligned} \tag{24}$$

Solving the system (24) consistently and imposing the condition $\tau(0) = 0$ from (9), we obtain the explicit form of infinitesimals

$$\xi = a_0 x + a_1, \quad \tau = \frac{2a_0 t}{\alpha}, \quad p = b_0, \quad {}_0\partial_t^\alpha q = kq_{xx},$$

where a_0, a_1 and b_0 are arbitrary constants. Hence the infinitesimal operator becomes

$$X = (a_0 x + a_1)\frac{\partial}{\partial x} + \frac{2a_0 t}{\alpha}\frac{\partial}{\partial t} + (b_0 u + q(x,t))\frac{\partial}{\partial u}.$$

We then obtain the following infinitesimal generators:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_\infty = q(x,t)\frac{\partial}{\partial x},$$

where q is a solution of the given equation

$$\frac{\partial^\alpha q}{\partial t^\alpha} = k\frac{\partial^2 q}{\partial x^2}.$$

We now explain how invariant or exact solution can be constructed. Let $u(x,t) = \theta(x,t)$ be an invariant solution associated with the generator $X_1 + \gamma X_3$, i.e.,

$$\left[\frac{\partial}{\partial x} + \gamma u\frac{\partial}{\partial u}\right]\theta = 0.$$

Then the associated characteristic equation reads as

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{\gamma u}$$

from which we obtain $u(x,t) = \mathrm{e}^{\gamma x}\phi(t)$ where $\phi(t)$ satisfies the equation

$$\frac{\mathrm{d}^\alpha \phi}{\mathrm{d}t^\alpha} = k\gamma^2\phi(t).$$

The above linear fractional equation can be solved and its solution can be expressed in terms of Mittag–Leffler function. Thus we obtain a solution of time-fractional diffusive equation (3) as

$$u(x,t) = \mathrm{e}^{\gamma x} t^{\alpha-1} E_{\alpha,\alpha}(k\gamma^2 t^{\alpha}),$$

where $E_{\alpha,\beta}(\cdot)$ is the two-parameter Mittag–Leffler function given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0.$$

4.2 *Time-fractional cylindrical Korteweg–de Vries equation*

Let us assume that the time-fractional cylindrical KdV equation (4), is invariant under a one-parameter transformation (7), and so the invariant equation reads as

$$\left[\eta_t^{(\alpha)} + \frac{\eta}{2t^{\alpha}} - \frac{\alpha u \tau}{2t^{\alpha+1}} + 6(u\eta_x^{(1)} + \eta u_x) + \eta_x^{(3)}\right]\Bigg|_{(4)} = 0 \tag{25}$$

which is not solvable, in general, for $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. On the other hand, if

$$\eta = p(x,t)u + q(x,t), \tag{26}$$

where $p(x,t)$ and $q(x,t)$ are unknown functions to be determined. Then, after a systematic calculation we find that the infinitesimals $\xi(x,t,u)$ is independent of t and u while $\tau(x,t,u)$ is independent of x and u. Then we obtain the following determining equations:

$$\begin{aligned}
&\partial_t^n p + \frac{n-\alpha}{n+1}\tau^{(n+1)}(t) = 0, \quad n \in N,\\
&p_x - \xi''(x) = 0,\\
&3\xi'(x) - \alpha\tau'(t) = 0,\\
&2u\alpha\tau'(t) - 2u\xi'(x) + 2up + 2q + p_{xx} = 0,\\
&0\partial_t^{\alpha} q + \frac{\alpha u \tau'(t)}{2t^{\alpha}} + \frac{q}{2t^{\alpha}} - \frac{\alpha u \tau}{2t^{\alpha+1}} + 6u^2 p_x + 6uq_x + p_{xxx}u + q_{xxx} = 0.
\end{aligned} \tag{27}$$

Solving the system (27) consistently and imposing the condition $\tau(0) = 0$ from (9), we obtain the explicit form of infinitesimals

$$\xi = a_0 x + a_1, \quad \tau = \frac{3a_0 t}{\alpha}, \quad p = -2a_0, \quad q = 0,$$

where a_0 and a_1 are arbitrary constants. Hence the infinitesimal operator becomes

$$X = (a_0 x + a_1)\frac{\partial}{\partial x} + \frac{3a_0 t}{\alpha}\frac{\partial}{\partial t} - 2a_0 u \frac{\partial}{\partial u}$$

and so the underlying symmetry algebra of time-fractional cylindrical Korteweg–de Vries equation is two-dimensional with basis ($X_1 = \partial/\partial x$, $X_2 = x(\partial/\partial x) + (3t/\alpha)(\partial/\partial t) - 2u(\partial/\partial u)$). The similarity variable and similarity transformation corresponding to the infinitesimal generator X_2 can be obtained by solving the associated characteristic equation given by

$$\frac{\mathrm{d}x}{x} = \frac{\alpha \mathrm{d}t}{3t} = \frac{\mathrm{d}u}{-2u}$$

which respectively take the following form

$$z = xt^{-\alpha/3}, \quad u = t^{-2\alpha/3} f(z). \tag{28}$$

Theorem 4.1. *The similarity transformation* $u(x,t) = t^{-2\alpha/3} f(z)$ *along with the similarity variable* $z = xt^{-\alpha/3}$ *reduces the time-fractional cylindrical KdV equation* (4) *to the nonlinear fractional ordinary differential equation of the form*

$$\left(P_{3/\alpha}^{1-\alpha-\frac{2\alpha}{3},\alpha} f\right)(z) + \frac{f}{2} + 6f\frac{\mathrm{d}f}{\mathrm{d}z} + \frac{\mathrm{d}^3 f}{\mathrm{d}z^3} = 0 \tag{29}$$

with Erdélyi–Kober fractional differential operator [24]

$$\begin{aligned} \left(P_\delta^{\tau,\alpha} g\right)(z) &:= \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\delta} z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(K_\delta^{\tau+\alpha,m-\alpha} g\right)(z), \\ &\qquad z>0,\ \delta>0,\ \alpha>0, \\ m &= \begin{cases} [\alpha]+1, & \alpha \notin N, \\ \alpha, & \alpha \in N, \end{cases} \end{aligned} \tag{30}$$

where

$$\left(K_\delta^{\tau,\alpha} g\right)(z) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_1^\infty (v-1)^{\alpha-1} v^{-(\tau+\alpha)} g(zv^{1/\delta})\,\mathrm{d}v, & \alpha > 0, \\ g(z), & \alpha = 0. \end{cases} \tag{31}$$

is the Erdélyi–Kober fractional integral operator.

Proof. Let $0 < \alpha < 1$. Then the Riemann–Liouville fractional derivative for the similarity transformation (28) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{1-\alpha-1} s^{-2\alpha/3} f(xs^{-\alpha/3})\mathrm{d}s\right].$$

Let $v = t/s$. Then the above equation can be written as

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t}\Bigg[& t^{1-\alpha-(2\alpha/3)} \frac{1}{\Gamma(1-\alpha)} \\ & \times \int_1^\infty (v-1)^{1-\alpha-1} v^{-(1-\alpha-(2\alpha/3)+1)} f(zv^{\alpha/3})\mathrm{d}v\Bigg]. \end{aligned}$$

Following the definition of the Erdélyi–Kober fractional integral operator given in (31), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial t}\left[t^{1-\alpha-(2\alpha/3)}\left(K_{3/\alpha}^{1-(2\alpha/3),1-\alpha} f\right)(z)\right]. \tag{32}$$

In order to simplify the RHS of eq. (32), we consider the relation ($z = xt^{-\alpha/3}$, $\phi \in C^1(0,\infty)$)

$$t\frac{\partial}{\partial t}\phi(z) = tx\left(-\frac{\alpha}{3}\right)t^{-(\alpha/3)-1}\phi'(z) = -\frac{\alpha}{3} z\frac{\mathrm{d}}{\mathrm{d}z}\phi(z)$$

and so, we get

$$\frac{\partial}{\partial t}\left[t^{1-\alpha-(2\alpha/3)}\left(K_{3/\alpha}^{1-(2\alpha/3),1-\alpha}f\right)(z)\right]$$
$$= t^{-\alpha-(2\alpha/3)}\left(1-\frac{2\alpha}{3}-\alpha-\frac{\alpha}{3}z\frac{\mathrm{d}}{\mathrm{d}z}\right)\left(K_{3/\alpha}^{1-(2\alpha/3),1-\alpha}f\right)(z).$$

Now using the definition of the Erdélyi–Kober fractional differential operator given in (30), the above equation can be written as

$$\frac{\partial}{\partial t}\left[t^{1-\alpha-(2\alpha/3)}\left(K_{3/\alpha}^{1-(2\alpha/3),1-\alpha}f\right)(z)\right] = t^{-\alpha-(2\alpha/3)}\left(P_{3/\alpha}^{1-(2\alpha/3)-\alpha,\alpha}f\right)(z).$$

Thus we obtain an expression for the time-fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha-(2\alpha/3)}\left(P_{3/\alpha}^{1-(2\alpha/3)-\alpha,\alpha}f\right)(z). \tag{33}$$

Continuing further we find that the time-fractional cylindrical KdV equation (4) reduces to the nonlinear fractional ODE of the form

$$\left(P_{3/\alpha}^{1-(2\alpha/3)-\alpha,\alpha}f\right)(z) + \frac{f}{2} + 6f\frac{\mathrm{d}f}{\mathrm{d}z} + \frac{\mathrm{d}^3 f}{\mathrm{d}z^3} = 0, \tag{34}$$

□

which is not solvable, in general. The above reduced equation combines both local and nonlocal behaviours of f which prevents it from being easily solved and analysed analytically. In order to obtain some information about the behaviour of f, we need to approximate the Erdélyi–Kober fractional differential operator and solve it numerically.

5. Summary and concluding remarks

In this paper, a systematic Lie symmetry approach was used to derive Lie point symmetries of time-fractional partial differential equations and its applicability was illustrated through the time-fractional diffusion and cylindrical Korteweg–de Vries equations with Riemann–Liouville fractional derivative and its Lie point symmetries were derived. Using the obtained point symmetries, we constructed an invariant solution for time-fractional diffusion equation. We observed that for cylindrical Korteweg–de Vries equation the underlying symmetry algebra is two-dimensional. The reduction of dimension in the symmetry algebra is due to the fact that the time-fractional equation is not invariant under time translation symmetry. Using Lie point symmetries, we have shown that the time-fractional cylindrical Korteweg–de Vries equation can be transformed into a nonlinear ODE of fractional order with a new independent variable.

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