PRAMANA
— journal of
physics

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Vol. 85, No. 5
November 2015
pp. 807–821

Analytic methods to generate integrable mappings

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DOI: 10.1007/s12043-015-1102-9; ***e*Publication:** 22 October 2015

Abstract. Systematic analytic methods of deriving integrable mappings from integrable nonlinear ordinary differential, differential-difference and lattice equations are presented. More specifically, we explain how to derive integrable mappings through four different techniques namely, (i) discretization technique, (ii) Lax pair approach, (iii) periodic reduction of integrable nonlinear partial difference equations and (iv) construction of sufficient number of integrals of motion. The applicability of methods have been illustrated through Ricatti equation, a scalar second-order nonlinear ordinary differential equation with cubic nonlinearity, 2- and 3-coupled second-order nonlinear ordinary differential equations with cubic nonlinearity, lattice equations of Korteweg–de Vries, modified Korteweg–deVries and sine-Gordon types.

Keywords. Discrete integrable systems; integrable mappings; discretization; integrals of motion; Lax pair.

PACS No. 02.30.Ik

1. Introduction

Integrable systems have emerged as one of the significant research areas in mathematics with applications in different areas of science in the 21st century. Integrable systems form a special class of mathematical models with different kinds of forms and shapes, such as nonlinear partial differential equations (PDEs), special types of nonlinear ordinary differential equations (ODEs), nonlinear differential-difference equations, mappings and lattice equations [1–4]. Integrable systems are studied for various reasons: for their rich algebraic and geometric structure, for their intrinsic mathematical and physical interest. Although important in their own right, these systems form an archipelago of solvable models in a sea of unknown, and can be used as stepping stones to investigate properties of nearby non-integrable systems. Their study has led to the development of new mathematical techniques, such as the inverse scattering transform method, finite-gap integration techniques and the application of Riemann–Hilbert problems [5,6]. During the 1990s, it has been remarkably shown by several groups that most integrable systems governed by differential equations can be discretized resulting in differential-difference, difference equations or mappings [5,7–10]. Obviously, there exist many ways to find discrete analogue for a given differential equation. However, to find a discretization that preserves the essential integrability features of an integrable differential equation is a challenging and far from trivial enterprise. Recent investigations have conclusively shown that such discretizations are possible and the resulting differential-difference and difference equations or mappings not only possess all the hallmarks of integrability, but in fact turn out to be richer and more transparent than their continuous counterparts which led to the birth of discrete integrable systems [6,8,11–17]. In the literature only a handful of discrete nonlinear integrable systems governed by higher order or coupled mappings or lattice equations exist. The main objective of this article is to present brief details of four distinct analytic methods and explain how integrable mappings can be derived. The usefulness of the methods has been illustrated through Ricatti equation, a scalar second-order nonlinear ODE with cubic nonlinearity, 2- and 3-coupled second-order nonlinear ODEs with cubic nonlinearity, lattice equations of Korteweg–de Vries (KdV), modified Korteweg–de Vries (mKdV) and sine-Gordon (s-G) types.

Rest of this paper is organized as follows. In §2, we present some basic definitions and concepts related with integrability of discrete systems including mappings which are required for the remaining part of the paper. In §3 and 4, we explain in detail how integrable mappings can be derived using four main approaches namely, discretization of integrable differential equations, Lax pair method, reduction of lattice equations and by constructing sufficient number of integrals of motion. In §4 we present a summary of our results.

2. Preliminaries

To start with, we present a few basic definitions required for studying the integrability of mappings. Consider an Nth-order ordinary difference equation ($O\Delta E$)

$$x_{n+N} = F(x_n, \ldots, x_{n+N-1}), \quad x_{n+i} = x(n+i), \quad i = 0, 1, \ldots, N. \tag{1}$$

(i) *Integral*: An integral (or conserved quantity) for the above $O\Delta E$ is a function

$$I(n) = I(x_n, \ldots, x_{n+N-1})$$

that is not identically constant but is constant on all solutions of the $O\Delta E$, that is if

$$I(x_n, \ldots, x_{n+N-1}) = I(x_{n+1}, \ldots, x_{n+N})$$

holds. A set of integrals $\{I_1(n), I_2(n), \ldots, I_j(n)\}$ $(2 \leq j \leq N)$ of an Nth-order $O\Delta E$ is said to be functionally independent if the Jacobian of the integrals is of maximal rank.
(ii) *Measure preserving*: A mapping

$$L\colon (x_1, \ldots, x_n) \to (y_1, \ldots, y_n)$$

is said to be measure preserving with density $m(x_1, \ldots, x_n)$ if the Jacobian J

$$J(x_1, \ldots, x_n) = \det \mathrm{d}L(x_1, \ldots, x_n) = \pm m(x_1, \ldots, x_n)/m(y_1, \ldots, y_n).$$

(iii) *Symplectic map*: A mapping say L: $R^{2N} \rightarrow R^{2N}$ is said to be symplectic, if there exists an antisymmetric $(2N \times 2N)$ matrix $\Omega(n)$ satisfying the following conditions:

- $J(n)\Omega(n)J(n)^T = \Omega(n+1)$,
- $\Omega(n)$ has maximal rank,
- Jacobi identity,

where $J(n)$ is the Jacobian of the mapping L.

2.1 *Integrability of $O\Delta Es$*

It is known that the concept of complete integrabillity of nonlinear mappings governed by $O\Delta E$s is not clearly defined as for differential equations. In fact there exists no unique definition for integrability of mappings. However, the following working definitions are widely used (list is not exhaustive).

- An Nth-order $O\Delta E$ is said to be integrable if it is measure preserving and possesses $(N-1)$ independent integrals [18–20].
- A $2N$th-order $O\Delta E$ is said to be completely integrable in the sense of Liouville [12,21]

  (i) if it is symplectic,
  (ii) there exists functionally N independent integrals $I_1(n), \ldots, I_N(n)$, such that

  $$\{I_m, I_r\} = \sum_{i,j} \frac{\partial I_m}{\partial x_i} \Omega_{i,j} \frac{\partial I_r}{\partial x_j} = 0,$$

  for each pair $(m, r), m, r = 1, \ldots, N$.

- An Nth-order $O\Delta E$ is said to be integrable if it passes the singularity confinement criterion test and has zero algebraic entropy [13,14,22,23].
- An Nth-order $O\Delta E$ is said to be integrable in the sense of Lax if it arises from the compatibility condition of a system of linear equations [24].

3. Methods to generate integrable mappings

Though the methods to derive integrable mappings is delicate and not very systematic, we explain here four different techniques that have been used over the years to construct integrable mappings and can be considered as systematic methods.

3.1 *Discretization of integrable ODEs and integrable mappings*

Given an integrable ODE, there exist many ways to derive its discretized version. One of the best ways, is to derive a discretized version preserving most of the fundamental characteristics of the integrable ODE [9,25]. With this in mind, we consider the following four specific examples for discretization:

3.1.1 *Ricatti equation.* One of the celebrated integrable first-order ODE is the Ricatti equation given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a(t)x^2 + b(t)x + c(t), \tag{2}$$

where coefficients $a(t)$, $b(t)$ and $c(t)$ are arbitrary functions of t. The Ricatti equation (2) has the following fundamental characteristics:

- It is form-invariant under the linear fractional transformation (Mobius transformations), i.e.,

$$x(t) \to Y(t) = \frac{\alpha x(t) + \beta}{\gamma x(t) + \delta} \Leftrightarrow x(t) = \frac{\delta Y(t) - \beta}{-\gamma Y(t) + \alpha}, \tag{3}$$

  where $\alpha,\ \beta,\ \gamma,\ \delta$ are constants such that $\alpha\delta - \beta\gamma \neq 0$. In fact, it is easy to check that implementing the transformation (3) turns (2) into an equation for $Y(t)$ of the form

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = A(t)Y^2 + B(t)Y + C(t) \tag{4}$$

  with new coefficients $A(t)$, $B(t)$ and $C(t)$ expressed in terms of $a(t),\ b(t),\ c(t)$ and their derivatives.
- It can be transformed into a second-order linear ODE with variable coefficients through a transformation

$$x(t) = -\frac{1}{a(t)}\frac{y'(t)}{y(t)}.$$

- Any four distinct particular solutions $x_i(t)$, $i = 1, 2, 3, 4$ of the Riccati equation (2) are related through the Cross ratio

$$[x_1, x_2, x_3, x_4] = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)} = \text{constant}.$$

It is of interest to find a discretization of the Ricatti equation (2) that preserves all the characteristics mentioned above. Discretization of the Ricatti equation can be done at least in two different ways. For example, if one replaces $x(t)^2$ by x_n^2, then the discretized Ricatti equation reads as

$$x_{n+1} = A_n x_n^2 + B_n x_n + C_n,$$

where A_n, B_n and C_n are functions of n which is a well-known logistic map displaying chaotic motion. On the other hand if one discretizes $x(t)^2$ as $x_n x_{n+1}$, then eq. (2) can be written as

$$x_{n+1} = \frac{A_n x_n + B_n}{C_n x_n + D_n}, \tag{5}$$

where $A_n,\ B_n,\ C_n$ and D_n are functions of n, which is usually referred to as discrete Ricatti equation. It is easy to check that the above equation is invariant under Mobius transformation. Similarly if one introduces a new dependent variable

$$x_n = \frac{D_n}{C_n}\left(\frac{Y_{n+1} - Y_n}{Y_n}\right),$$

the discrete Ricatti equation (5) can be transformed into a linear second-order $O\Delta E$ with variable coefficients

$$\frac{D_{n+1}}{C_{n+1}}Y_{n+2}-\left(\frac{D_{n+1}}{C_{n+1}}\right)Y_{n+1}+\left(\frac{A_n}{C_n}+\frac{B_n}{D_n}\right)Y_n=0.$$

We would like to mention that eq. (5) also possesses the remaining characteristics of the Ricatti equation (2). Note that eq. (5) also passes the notion known as preimage nonproliferation criterion of rational mappings [26].

3.1.2 *Second-order integrable ODE with cubic nonlinearity.* Consider a second-order nonlinear integrable ODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+ax+bx^3=0, \tag{6}$$

where a and b are arbitrary constants, which can be written as

$$\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\frac{ax^2}{2}+\frac{bx^4}{4}=\text{constant}=H. \tag{7}$$

Equation (7) is a Hamiltonian system with one degree of freedom and hence it is integrable. It is appropriate to mention here that Quispel, Roberts and Thompson (QRT) have discovered a symmetric mapping in the plane with 12 parameters having the form

$$x_{n+1}x_{n-1}f_3(x_n)-(x_{n+1}+x_{n-1})f_2(x_n)+f_1(x_n)=0, \tag{8}$$

where f_i's are specific quartic polynomials [4,27]. QRT has also shown that eq. (8) is a symplectic mapping (that is time-discrete Hamiltonian system) and admits an integral $I(n)$ expressed as a ratio of biquadratic polynomials in x_n and x_{n+1} satisfying $I(n+1)-I(n)=0$. Thus, QRT mapping (8) is integrable in the sense of Liouville. Note that eq. (8) can also be viewed as discretized version of a second-order autonomous ODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=F\left(x,\frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{9}$$

To derive a discretized version of a scalar second-order nonlinear ODE there exists no unique prescription. However, the following working procedures are widely used [28–33]:

(i) The order of the discretized equation remains the same as that of ODE.
(ii) The derivative terms of ODE induce nontrivial denominators in the discretized equation.
(iii) Nonlinear terms of ODE have been replaced by nonlocal representations in discrete equation.
(iv) The discretized equation preserves the basic characteristics of the original equation.

With this in mind we first write $x^3(t)$ as a cubic polynomial in (x_{n-1},x_n,x_{n+1}) and then demand under what choice of coefficients the discretized equation belongs to QRT family. As a result we obtain the following discretized equation:

$$x_{n-1}-2x_n+x_{n+1}+ax_n+b\alpha(x_{n-1}+x_{n+1})x_n^2=0$$

which can be rewritten as

$$x_{n+2} = -x_n + \frac{(2-a)x_{n+1}}{1+b\alpha x_{n+1}^2} \tag{10}$$

which is the well-known McMillan mapping [34].

3.1.3 *2-coupled second-order ODEs.* Lakshmanan and Sahadevan [1] have reported that the following Hamiltonian system with two degrees of freedom given by

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2) + Ax^2 + By^2 + \alpha(x^2+y^2)^2, \tag{11}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and A, B and α's are arbitrary parameters is integrable in the sense of Liouville. The associated Hamilton's equation read as

$$\begin{aligned} &\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + Ax + 4\alpha x^3 + 4\alpha xy^2 = 0, \\ &\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + By + 4\alpha y^3 + 4\alpha yx^2 = 0. \end{aligned} \tag{12}$$

We wish to discretize the above such that the discretized version is symplectic, measure preserving and admits two functionally independent integrals of motion. Proceeding as before, we obtain the following discretized equations:

$$\begin{aligned} x_{n+1} &+ x_{n-1} - 2x_n \\ &= -2Ax_n - 2\alpha\left[[x_{n+1}+x_{n-1}]x_n^2 + \frac{(1-A)}{(1-B)}[y_{n+1}+y_{n-1}]x_ny_n\right], \\ y_{n+1} &+ y_{n-1} - 2y_n \\ &= -2By_n - 2\alpha\left[[y_{n+1}+y_{n-1}]y_n^2 + \frac{(1-B)}{(1-A)}[x_{n+1}+x_{n-1}]x_ny_n\right], \end{aligned} \tag{13}$$

which can be rewritten as

$$\begin{aligned} x_{n+2} &= -x_n + \frac{2(1-A)x_{n+1}}{1+2\alpha y_{n+1}^2+2\alpha x_{n+1}^2}, \\ y_{n+2} &= -y_n + \frac{2(1-B)y_{n+1}}{1+2\alpha y_{n+1}^2+2\alpha x_{n+1}^2}. \end{aligned} \tag{14}$$

It is straightforward to check that the above mappings are measure preserving and symplectic.

3.1.4 *3-coupled second-order ODEs.* Lakshmanan and Sahadevan [1] have also reported the following Hamiltonian system with three degrees of freedom given by

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) + Ax^2 + By^2 + Cz^2 + \alpha(x^2+y^2+z^2)^2, \tag{15}$$

where A, B, C and α are arbitrary parameters. The equations of motion are given by

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + Ax + 4\alpha x^3 + 4\alpha xy^2 + 4\alpha xz^2 = 0,$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + By + 4\alpha y^3 + 4\alpha yx^2 + 4\alpha yz^2 = 0,$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} + Cz + 4\alpha z^3 + 4\alpha zx^2 + 4\alpha zy^2 = 0. \tag{16}$$

Following the procedure outlined above, one can find a discretization of the above differential equations (16) leading to the following system of three second-order $O\Delta E$s:

$$x_{n+2} = -x_n + \frac{2(1-A)x_{n+1}}{1 + 2\alpha y_{n+1}^2 + 2\alpha x_{n+1}^2 + 2\alpha z_{n+1}^2},$$

$$y_{n+2} = -y_n + \frac{2(1-B)y_{n+1}}{1 + 2\alpha y_{n+1}^2 + 2\alpha x_{n+1}^2 + 2\alpha z_{n+1}^2},$$

$$z_{n+2} = -z_n + \frac{2(1-C)z_{n+1}}{1 + 2\alpha y_{n+1}^2 + 2\alpha x_{n+1}^2 + 2\alpha z_{n+1}^2}. \tag{17}$$

We have checked that the above 3-coupled second-order mappings are measure preserving and symplectic. Proceeding along similar lines, one can also discretize a Hamiltonian system with N degrees of freedom whose equations of motion read as

$$\frac{\mathrm{d}^2x_i}{\mathrm{d}t^2} + Ax_i + 4\alpha x_i^3 + 4\alpha \sum_{i\neq j} x_i x_j^2 = 0, \qquad i = 1, 2 \ldots, N. \tag{18}$$

3.2 *Lax pair approach and derivation of mappings of order* ≥ 3

A nonlinear $O\Delta E$ is said to be integrable in the sense of Lax if it arises from the compatibility condition of a system of linear equations called Lax equations. In matrix notation this can be written as

$$M_n\phi_n = \lambda\phi_n, \tag{19}$$

$$\phi_{n+1} = L_n\phi_n, \tag{20}$$

where L_n and M_n are matrices whose entries are difference operators E and E^{-1} and λ-spectral parameter. For nontrivial ϕ_n the compatibility of (19) and (20) gives rise to the equation

$$M_{n+1}L_n = L_nM_n \tag{21}$$

which is usually referred to as Lax equation for autonomous $O\Delta E$'s or mappings [24,35]. We wish to confine our attention to third-order mappings here. Conventionally, the derivation of integrable mappings from the Lax pair approach proceeds in the following manner:

Fix the entries of the Lax matrix L_n and express the entries of the other matrix M_n as a polynomial in the spectral variable. For example we consider L_n as

$$L_n = \begin{pmatrix} 0 & 1 \\ \lambda x_n x_{n+1} - L_4(n) & x_{n+1} \end{pmatrix}, \tag{22}$$

where $L_4(n) = L_4(x_n, x_{n+1}, x_{n+2})$. Let us assume the other Lax matrix M_n as

$$M_n = \begin{pmatrix} M_1(n,\lambda) & M_2(n,\lambda) \\ M_3(n,\lambda) & M_4(n,\lambda) \end{pmatrix}, \tag{23}$$

where $M_i(n,\lambda) = M_i(x_n, x_{n+1}, x_{n+2}, \lambda)$, for $n = 1, 2, 3, 4$. Then the compatibility condition (21) gives rise to the following:

(i) $M_3(n,\lambda) - M_2(n+1,\lambda)(\lambda x_n x_{n+1} - L_4(n)) = 0,$ (24)

(ii) $M_4(n,\lambda) - M_1(n+1,\lambda) - x_{n+1} M_2(n+1,\lambda) = 0,$ (25)

(iii) $(\lambda x_n x_{n+1} - L_4(n))(M_4(n+1,\lambda) - M_1(n,\lambda)) - x_{n+1} M_3(n,\lambda) = 0,$ (26)

(iv) $M_3(n+1,\lambda) + x_{n+1}(M_4(n+1,\lambda) - M_4(n,\lambda)) - M_2(n,\lambda)(\lambda x_n x_{n+1} - L_4(n)) = 0.$ (27)

To solve these functional equations we first express $M_i(n,\lambda)$ as

$$M_i(n,\lambda) = M_{i1}(n) + \lambda M_{i2}(n), \quad i = 1, 2, 3, 4,$$

and then expand $L_4(n)$, $M_{i1}(n)$ and $M_{i2}(n)$ as rational functions of (x_n, x_{n+1}, x_{n+2}). After a simple manipulation we find that some of the known third-order mappings admit Lax pair. For example when

$$\begin{aligned} M_1(n,\lambda) &= a_6(x_n + x_{n+2}) + a_5 x_n x_{n+2} \\ &\quad + \frac{a_2 + a_1 x_n + a_1 x_{n+2} + a_6 x_n x_{n+2}}{x_{n+1}}, \\ L_4(n) &= -M_1(n,\lambda) x_n x_{n+1}, \\ M_2(n,\lambda) &= \frac{1}{x_n}, \\ M_3(n,\lambda) &= \lambda x_n + M_1(n,\lambda) x_n, \\ M_4(n,\lambda) &= \frac{a_1 x_{n+1}}{x_n} + \frac{a_2}{x_n} + \frac{a_2 x_{n+1}}{x_n x_{n+2}} + \frac{a_4}{x_n x_{n+2}} \\ &\quad + \frac{a_1 x_{n+1}}{x_{n+2}} + \frac{a_2}{x_{n+2}} + a_6 x_{n+1} + 1, \end{aligned}$$

eqs (24)–(27) are compatible if

$$x_{n+3} = \frac{1}{x_n}\left[\frac{a_1 x_{n+1} x_{n+2} + a_2 x_{n+1} + a_2 x_{n+2} + a_4}{a_5 x_{n+1} x_{n+2} + a_6 x_{n+1} + a_6 x_{n+2} + a_1}\right] \tag{28}$$

which is referred to as a s-G mapping. Proceeding in a similar way we have checked that the following third-order mappings have also been derived from Lax pair approach. The explicit forms of the mappings are

$$x_{n+3} = \frac{1}{x_n}\left[\frac{a_1(x_{n+1}x_{n+2} + x_{n+1} + x_{n+2}) + a_4}{a_5 x_{n+1}x_{n+2} + a_1(x_{n+1} + x_{n+2} + 1)}\right], \tag{29}$$

$$x_{n+3} = \frac{x_{n+1}x_{n+2}}{x_n}\left[\frac{a_2(x_{n+1} + x_{n+2}) + a_4}{a_5 x_{n+1}x_{n+2} + a_2(x_{n+1} + x_{n+2})}\right], \tag{30}$$

$$x_{n+3} = x_n\left[\frac{a_1 x_{n+1}x_{n+2} + a_2 x_{n+1} + a_3 x_{n+2} + a_4}{a_1 x_{n+1}x_{n+2} + a_3 x_{n+1} + a_2 x_{n+2} + a_4}\right], \tag{31}$$

$$x_{n+3} = \frac{x_n x_{n+1}}{x_{n+2}}\left[\frac{a_1 x_{n+1}x_{n+2} + a_3 x_{n+2} + a_4}{a_1 x_{n+1}x_{n+2} + a_3 x_{n+1} + a_4}\right], \tag{32}$$

$$x_{n+3} = -x_n - \frac{2a_1 x_{n+1}x_{n+2} + a_2 x_{n+1} + a_2 x_{n+2} + a_3}{a_4 x_{n+1}x_{n+2} + a_1 x_{n+1} + a_1 x_{n+2} + a_5}, \tag{33}$$

$$x_{n+3} = -x_n - \frac{a_1(x_{n+1} + x_{n+2})^2 + a_2(x_{n+1} + x_{n+2}) + a_3}{a_1 x_{n+1} + a_1 x_{n+2} + a_4}. \tag{34}$$

Here a_i's are constants. The mappings given in eqs ((29,30), (31,32) and (33,34)) are respectively called s-G, mKdV and KdV mappings. The entries of $M_i(n, \lambda)$, $i = 1, 2, 3, 4$ and $L_4(n)$ for each of the mappings (29)–(34) are given in Appendix A.

3.3 *Reduction from $P\Delta\Delta Es$ to $O\Delta Es$*

Consider an integrable partial difference equation having the form

$$x_{m+1}^{l+1} = F(x_m^l, x_{m+1}^l, x_m^{l+1}), \; x_m^l = x(l, m), \tag{35}$$

where F is a well-defined function. Now consider a solution x_m^l of the nonlinear $P\Delta\Delta E$ satisfying the periodicity property $x_{m+z_1}^{l-z_2} = x_m^l = x_n$, where $gcd(z_1, z_2) = 1$, $z_1, z_2 \in Z$. Here $n = mz_1 + lz_2$ and so

$$x_{m+1}^l = x_{n+z_1}, \quad x_m^{l+1} = x_{n+z_2}, \quad x_{m+1}^{l+1} = x_{n+z_1+z_2}.$$

For convenience we choose $z_1 = 1$ and $z_2 = z$ and so the given integrable $P\Delta\Delta E$ is transformed into $(z + 1)$th-order ordinary difference equation, that is

$$x_{n+z+1} = F(x_n, x_{n+1}, x_{n+z})$$

from which one can derive higher-order $O\Delta E$s [2,12]. We explain the above through an integrable $P\Delta\Delta E$ given by

$$x_{m+1}^{l+1} = \frac{x_{m+1}^l x_m^{l+1} - x_m^l[a x_m^{l+1} + (1-a)x_{m+1}^l]}{[a x_{m+1}^l + (1-a)x_m^{l+1}] - x_m^l}, \tag{36}$$

where a is an arbitrary parameter. The $(z+1)$th-order $O\Delta E$ obtained from the periodicity condition reads as

$$x_{n+z+1} = \frac{x_{n+1}x_{n+z} - x_n[a x_{n+z} + (1-a)x_{n+1}]}{[a x_{n+1} + (1-a)x_{n+z}] - x_n}. \tag{37}$$

For $z = 1$, eq. (37) becomes trivial. Next if $z = 2$, we obtain a third-order $O\Delta E$ given by

$$x_{n+3} = \frac{x_{n+1}x_{n+2} - x_n[ax_{n+2} + (1-a)x_{n+1}]}{[ax_{n+1} + (1-a)x_{n+2}] - x_n}. \tag{38}$$

Next for $z = 3$, we obtain a fourth-order $O\Delta E$ given by

$$x_{n+4} = \frac{x_{n+1}x_{n+3} - x_n[ax_{n+3} + (1-a)x_{n+1}]}{[ax_{n+1} + (1-a)x_{n+3}] - x_n}. \tag{39}$$

In a similar manner one can derive higher-order $O\Delta E$s. The above third- and fourth-order mappings (38) and (39) are measure preserving and each admits two functionally independent integrals whose explicit forms are given in the next section.

4. Integrable mappings and integrals of motion

Consider an autonomous Nth-order $O\Delta E$ having the form

$$\begin{aligned} x_{n+N} &= F(x_n, \ldots, x_{n+N-1}), \\ N &\geq 3,\ x_{n+i} = x(n+i), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{40}$$

where F is a smooth function. Let us assume that the above $O\Delta E$ admits integral having the form

$$I(n) = \frac{P(n)}{Q(n)} = \frac{\sum_{i=1}^{3}[A_{i1}(n)x_n^2 + A_{i2}(n)x_n + A_{i3}(n)]x_{n+N-1}^{3-i}}{\sum_{i=1}^{3}[a_{i1}(n)x_n^2 + a_{i2}(n)x_n + a_{i3}(n)]x_{n+N-1}^{3-i}}, \tag{41}$$

where

$$A_{ij}(n) = A_{ij}(x_{n+1}, \ldots, x_{x+N-2}), \quad a_{ij}(n) = a_{ij}(x_{n+1}, \ldots, x_{n+N-2})$$

are unknown functions to be determined. The integrability condition $I(n+1) - I(n) = 0$ leads to a quadratic equation in x_{n+N}. Again by expanding the coefficients $A_{ij}(n)$ as a quadratic polynomial in the dependent variables, we find under what conditions on the coefficients, the quadratic equation in x_{n+N} has real and distinct roots which in turn lead to the derivation of $O\Delta E$ with integrals [12,15,20,21,36]. We explain the above for a specific example say, 2-coupled second-order $O\Delta E$ given in (14). For clarity of presentation, we consider an integral $I(n) = I(x_n, y_n, x_{n+1}, y_{n+1})$ expressed as a quadratic polynomial in x_{n+1} and y_{n+1}, i.e.,

$$\begin{aligned} I(n) &= A_1(n)x_{n+1}^2 + A_2(n)y_{n+1}^2 + A_3(n)x_{n+1} + A_4(n)y_{n+1} \\ &\quad + A_5(n)x_{n+1}y_{n+1} + A_6(n), \end{aligned}$$

where $A_i(n) = A_i(x_n, y_n)$. Demanding that $I(n)$ be an integral for (14) leads to a quadratic equation in the variables x_{n+2} and y_{n+2}, i.e.,

$$\begin{aligned} A_1(n+1)x_{n+2}^2 &+ A_2(n+1)y_{n+2}^2 + A_3(n+1)x_{n+2} \\ &+ A_4(n+1)y_{n+2} + A_5(n+1)x_{n+2}y_{n+2} + A_6(n+1) \\ &- [A_1(n)x_{n+1}^2 + A_2(n)y_{n+1}^2 + A_3(n)x_{n+1} \\ &+ A_4(n)y_{n+1} + A_5(n)x_{n+1}y_{n+1} + A_6(n)] = 0. \end{aligned} \tag{42}$$

We then expand each $A_i(n),\ i = 1, 2, \ldots, 6$ as a quadratic polynomial in x_n and y_n, i.e.,

$$A_i(n) = \sum_{j=1}^{3}[A_{ij1}x_n^2 + A_{ij2}x_n + A_{ij3}]y_n^{3-j},$$

where A_{ijk}'s are constants. Substituting x_{n+2} and y_{n+2} given in eq. (14) in (42) and after a tedious calculation we find that it satisfies for two distinct sets of parametric values which in turn lead to the following integrals:

$$\begin{aligned} I_1(n) &= x_{n+1}^2 + x_n^2 + 2\alpha x_{n+1}^2 x_n^2 - 2(1-A)x_n x_{n+1} \\ &\quad + \frac{4\alpha(1-A)(B-A)x_n y_n x_{n+1} y_{n+1}}{(1-A)^2 - (1-B)^2} \\ &\quad + \frac{2\alpha(1-A)^2(x_{n+1}y_n - y_{n+1}x_n)^2}{(1-A)^2 - (1-B)^2}, \end{aligned} \tag{43}$$

$$\begin{aligned} I_2(n) &= y_{n+1}^2 + y_n^2 + 2\alpha y_{n+1}^2 y_n^2 - 2(1-B)y_n y_{n+1} \\ &\quad + \frac{4\alpha(1-B)(A-B)x_n y_n x_{n+1} y_{n+1}}{(1-B)^2 - (1-A)^2} \\ &\quad + \frac{2\alpha(1-B)^2(x_{n+1}y_n - y_{n+1}x_n)^2}{(1-B)^2 - (1-A)^2}. \end{aligned} \tag{44}$$

It is straightforward to check that the integrals given by (43) and (44) are functionally independent. Following the above procedure we derive three functionally independent integrals for the 3-coupled $O\Delta E$ given by eqs (17). They are

$$\begin{aligned} I_1(n) &= x_{n+1}^2 + x_n^2 + 2\alpha x_{n+1}^2 x_n^2 - 2(1-A)x_n x_{n+1} \\ &\quad + \frac{4\alpha(1-A)(B-A)x_n y_n x_{n+1} y_{n+1}}{(1-A)^2 - (1-B)^2} \\ &\quad + \frac{2\alpha(1-A)^2(x_{n+1}y_n - y_{n+1}x_n)^2}{(1-A)^2 - (1-B)^2} \\ &\quad + \frac{4\alpha(1-A)(C-A)x_n z_n x_{n+1} z_{n+1}}{(1-A)^2 - (1-C)^2} \\ &\quad + \frac{2\alpha(1-A)^2(x_{n+1}z_n - z_{n+1}x_n)^2}{(1-A)^2 - (1-C)^2}, \end{aligned}$$

$$\begin{aligned} I_2(n) &= y_{n+1}^2 + y_n^2 + 2\alpha y_{n+1}^2 y_n^2 - 2(1-B)y_n y_{n+1} \\ &\quad + \frac{4\alpha(1-B)(A-B)x_n y_n x_{n+1} y_{n+1}}{(1-B)^2 - (1-A)^2} \\ &\quad + \frac{2\alpha(1-B)^2(x_{n+1}y_n - y_{n+1}x_n)^2}{(1-B)^2 - (1-A)^2} \\ &\quad + \frac{4\alpha(1-B)(C-B)y_n z_n y_{n+1} z_{n+1}}{(1-B)^2 - (1-C)^2} \\ &\quad + \frac{2\alpha(1-B)^2(y_{n+1}z_n - z_{n+1}y_n)^2}{(1-B)^2 - (1-C)^2}, \end{aligned}$$

$$
\begin{aligned}
I_3(n) &= z_{n+1}^2 + z_n^2 + 2\alpha z_{n+1}^2 z_n^2 - 2(1-C)z_n z_{n+1} \\
&\quad + \frac{4\alpha(1-C)(A-C)x_n z_n x_{n+1} z_{n+1}}{(1-C)^2-(1-A)^2} \\
&\quad + \frac{2\alpha(1-C)^2(x_{n+1}z_n - z_{n+1}x_n)^2}{(1-C)^2-(1-A)^2} \\
&\quad + \frac{4\alpha(1-C)(B-C)y_n z_n y_{n+1} z_{n+1}}{(1-C)^2-(1-B)^2} \\
&\quad + \frac{2\alpha(1-C)^2(y_{n+1}z_n - z_{n+1}y_n)^2}{(1-C)^2-(1-B)^2}.
\end{aligned}
$$

Proceeding in a similar manner we find that (38) possesses two independent integrals. The explicit forms are

$$
I_1(n) = \frac{P_1(n)}{P_2(n)}, \qquad I_2(n) = \frac{P_3(n)}{P_2(n)}, \tag{45}
$$

where

$$
\begin{aligned}
P_1(n) &= (a-1)x_{n+2}^2 - [ax_{n+1} + (a-2)x_n]x_{n+2} \\
&\quad + ax_{n+1}^2 + (a-1)x_n^2 - ax_n x_{n+1}, \\
P_2(n) &= (x_n - x_{n+1})[x_{n+2}^2 - (x_n + x_{n+1})x_{n+2} + x_n x_{n+1}], \\
P_3(n) &= [(3a-2)x_n - ax_{n+1}]x_{n+2}^2 + [(2-a)x_n^2 - 2(3a-2)x_n x_{n+1} \\
&\quad + (3a-2)x_{n+1}^2]x_{n+2} + (3a-4)x_n^2 x_{n+1} + (2-a)x_n x_{n+1}^2.
\end{aligned}
$$

Integrals of (39) are as follows:

$$
J_1(n) = \frac{Q_1(n)}{Q_2(n)}, \qquad J_2(n) = \frac{Q_3(n)}{Q_2(n)}, \tag{46}
$$

where

$$
\begin{aligned}
Q_1(n) &= (a-1)(x_{n+2} - x_n)x_{n+3}^2 + [(a-1)x_n^2 + x_{n+1}^2 \\
&\quad - (a-1)x_{n+2}^2 - x_n x_{n+1} + x_n x_{n+2} - x_{n+1}x_{n+2}]x_{n+3} \\
&\quad - (a-1)x_n x_{n+1}(x_n - x_{n+1}) - x_n x_{n+2}^2 \\
&\quad - ax_{n+1}x_{n+2}(x_{n+1} - x_{n+2}) + x_n x_{n+1} x_{n+2}, \\
Q_2(n) &= (x_{n+2} - x_{n+1})(x_n - x_{n+1}) \\
&\quad \times [x_{n+3}^2 - (x_n + x_{n+2})x_{n+3} + x_n x_{n+2}], \\
Q_3(n) &= [(4-5a)x_n x_{n+1} + (3a-2)x_n x_{n+2} \\
&\quad - ax_{n+1}x_{n+2} + (3a-2)x_{n+1}^2]x_{n+3}^2 \\
&\quad + [(3a-4)x_n^2 x_{n+1} + (2-a)x_n^2 x_{n+2} \\
&\quad + (2-a)x_n x_{n+1}^2 + (4-5a)x_n x_{n+2}^2 \\
&\quad + 6(a-1)x_n x_{n+1} x_{n+2} + (4-5a)x_{n+1}^2 x_{n+2} \\
&\quad + (3a-2)x_{n+1}x_{n+2}^2]x_{n+3} + (3a-4)x_n^2 x_{n+2}^2 \\
&\quad + (6-5a)x_n^2 x_{n+1} x_{n+2} + (3a-4)x_n x_{n+1}^2 x_{n+2} \\
&\quad + (2-a)x_n x_{n+1} x_{n+2}^2.
\end{aligned}
$$

5. Summary

In this paper, we have shown how to derive integrable mappings for a given integrable nonlinear ordinary differential, differential-difference and lattice equations. More specifically, we explain how to derive integrable mappings through four different techniques: (i) discretization technique, (ii) Lax pair approach, (iii) periodic reduction of integrable nonlinear partial difference equations and (iv) construction of sufficient number of integrals of motion. The applicability of these methods has been illustrated through Ricatti equation, a scalar second-order nonlinear ordinary differential equation with cubic nonlinearity, 2- and 3-coupled second-order nonlinear ordinary differential equations with cubic nonlinearity, lattice equations of Korteweg–de Vries, modified Korteweg–de Vries and sine-Gordon types. Since each of the third-order autonomous mapping listed in (29)–(34) (i) admits two functionally independent integrals, (ii) is measure preserving and (iii) has Lax representation, they are integrable.

Acknowledgement

This work forms part of the start-up grant given by UGC and DST PURSE grant given by University of Madras.

Appendix A

The entries of $M_1(n,\lambda)$, $M_2(n,\lambda)$, $M_3(n,\lambda)$, $M_4(n,\lambda)$, $L_4(n)$ for mappings (29)–(34) are listed below.

Mapping (29)

$$M_1(n,\lambda) = \frac{a_1x_{n+1} + a_4 + (a_1x_{n+1} + a_1)[x_n + x_{n+2}] + a_1x_nx_{n+1}x_{n+2}}{x_nx_{n+2}},$$

$$M_2(n,\lambda) = \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1},$$

$$M_3(n,\lambda) = (\lambda + M_1(n,\lambda))x_n,$$

$$M_4(n,\lambda) = \frac{a_1x_nx_{n+2} + a_1x_n + a_1x_{n+2} + a_1 + [a_1x_{n+2} + a_5x_nx_{n+2} + a_1x_n + 1]x_{n+1}}{x_{n+1}}.$$

Mapping (30)

$$M_1(n,\lambda) = \frac{a_4x_{n+1} + a_2x_{n+1}^2 + a_2x_{n+1}[x_n + x_{n+2}]}{x_nx_{n+2}},$$

$$M_2(n,\lambda) = \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1},$$

$$M_3(n,\lambda) = (\lambda + M_1(n,\lambda))x_n,$$

$$M_4(n,\lambda) = \frac{[a_5x_nx_{n+2} + a_2x_{n+2} + a_2x_n]x_{n+1} + a_2x_nx_{n+2} + x_{n+1}^2}{x_{n+1}^2}.$$

Mapping (31)

$$\begin{aligned} M_1(n,\lambda) &= \frac{1}{2}t[a_4[x_n + x_{n+1} + x_{n+2}] + a_2[x_n + x_{n+2}]x_{n+1} \\ &\quad + a_3x_nx_{n+2} + a_1x_nx_{n+1}x_{n+2}], \\ M_2(n,\lambda) &= \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1}, \\ M_3(n,\lambda) &= (\lambda + M_1(n,\lambda))x_n, \quad M_4(n,\lambda) = M_1(n,\lambda) + 1. \end{aligned}$$

Mapping (32)

$$\begin{aligned} M_1(n,\lambda) &= \frac{1}{2}\left[a_4x_nx_{n+1} + a_4x_{n+1}x_{n+2} + a_3x_nx_{n+1}x_{n+2} + a_1x_nx_{n+1}^2x_{n+2}\right] \\ M_2(n,\lambda) &= \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1}, \\ M_3(n,\lambda) &= (\lambda + M_1(n,\lambda))x_n, \quad M_4(n,\lambda) = M_1(n,\lambda) + 1. \end{aligned}$$

Mapping (33)

$$\begin{aligned} M_1(n,\lambda) &= \frac{1}{2}[a_1[(x_n + x_{n+1})x_{n+2}^2 + x_n^2(x_{n+2} + x_{n+1}) \\ &\quad + 2x_nx_{n+1}x_{n+2} - (x_{n+2} + x_n)x_{n+1}^2] \\ &\quad + a_2[(x_n + x_{n+1})x_{n+2} + (x_n - x_{n+1})x_{n+1}) \\ &\quad + a_3(x_n + x_{n+2}] + a_4[(x_{n+2} + x_n - x_{n+1}]x_nx_{n+1}x_{n+2}) \\ &\quad + a_5[x_n^2 + x_{n+2}^2 + x_{n+1}^2 + x_nx_{n+2} - x_{n+1}x_{n+2} - x_nx_{n+1}]], \\ M_2(n,\lambda) &= \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1}, \\ M_3(n,\lambda) &= (\lambda + M_1(n,\lambda))x_n, \quad M_4(n,\lambda) = M_1(n,\lambda) + 1. \end{aligned}$$

Mapping (34)

$$\begin{aligned} M_1(n,\lambda) &= \frac{1}{2}[a_1[(x_n + x_{n+1})x_{n+2}^2 + x_{n+1}^2(x_{n+2} + x_n) \\ &\quad + x_n^2(x_{n+1} + x_{n+2}) + 2x_nx_{n+1}x_{n+2}] \\ &\quad + a_2[(x_n + x_{n+1})x_{n+2} + x_nx_{n+1}] \\ &\quad + a_3[x_n + x_{n+1} + x_{n+2}] + a_4[x_n^2 + x_{n+1}^2 + x_{n+2}^2]], \\ M_2(n,\lambda) &= \frac{1}{x_n}, \quad L_4(n) = -M_1(n,\lambda)x_nx_{n+1}, \\ M_3(n,\lambda) &= (\lambda + M_1(n,\lambda))x_n, \quad M_4(n,\lambda) = M_1(n,\lambda) + 1. \end{aligned}$$

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